Embeddings and the growth envelope of Besov spaces involving only slowly varying smoothness

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Basic notation

$\mathcal{A} \lesssim \mathcal{B} \ldots \exists c \in (0, +\infty)$ s.t. $\mathcal{A} \leq c\mathcal{B}$

$\mathcal{A} \approx \mathcal{B} \ldots \mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$

$\Omega \subset \mathbb{R}^n$ a domain, $f \in \mathcal{M}(\Omega)$

$f^*(t) := \inf \{\lambda \geq 0 : |\{x \in \Omega : |f(x)| > \lambda\}|_n \leq t\}$

$f^{**}(t) := t^{-1} \int_0^t f^*(\tau)\, d\tau,\ t > 0$

$X, Y \ldots$ (quasi-) Banach spaces $X \hookrightarrow Y$

$f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, $h \in \mathbb{R}^n$

$\Delta_h f(x) := f(x+h) - f(x)$, $x \in \mathbb{R}^n$

$\omega_1(f, t)_p := \sup_{x \in \mathbb{R}^n, |h| \leq t} \|\Delta_h f(x)\|_p$, $t \geq 0$

Slowly varying functions

Definition

$b \in SV = SV(0,1) \iff \quad b \in \mathcal{M}^+(0,1)$, $0 \not\equiv b \not\equiv \infty$, and
$\forall \varepsilon > 0 \quad \exists\ h_\varepsilon \in \mathcal{M}^+(0,1;\uparrow), \quad \exists\ h_{-\varepsilon} \in \mathcal{M}^+((0,1;\downarrow)$ such that

$$t^{\varepsilon} b(t) \approx h_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx h_{-\varepsilon}(t) \quad \forall\ t \in (0,1)$$

- $b \in SV(0,1) \ \Rightarrow\ \exists\ \tilde{b} \in SV(0,1) \cap C((0,1)),\ \tilde{b} \approx b$ on $(0,1)$

Examples

$$\ell(t) := 1 + |\log t|, \quad t \in (0,1)$$
$$\ell_1 := \ell, \qquad \ell_i := \ell_1 \circ \ell_{i-1}, \quad i > 1$$

1. $b(t) = \boldsymbol{\ell}^{\boldsymbol{\alpha}}(t) := \prod_{i=1}^{m} \ell_i^{\alpha_i}(t), \quad t \in (0,1), \quad m \in \mathbb{N}$,
   $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$
2. $b(t) = \exp\left(|\log t|^{\beta}\right), \quad t \in (0,1), \quad \beta \in (0,1)$

The Besov space $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$

Definition

Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,1)$ be such that

$$\|t^{-1/r} b(t)\|_{r,(0,1)} = \infty. \tag{1}$$

The Besov space $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ consists of those functions $f \in L_p(\mathbb{R}^n)$ for which the norm

$$\|f\|_{B^{0,b}_{p,r}} := \|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,1)} \tag{2}$$

is finite.

Theorem 1

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Assume that* ω *is a non-negative measurable function on* $(0,1)$. *Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3}$$

for all $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \left\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u))^p \, du \Big)^{1/p} \right\|_{r,(0,1)} \tag{4}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$.

Main ingredients of the proof

Theorem (Kolyada, (1988))

If $1 \le p < \infty$*, then*

$$t\Big(\int_{t^n}^{\infty} s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s}\Big)^{1/p} \lesssim \omega_1(f,t)_p$$

for all $t > 0$ *and* $f \in L_p(\mathbb{R}^n)$.

Theorem (Caetano, Gogatishvili, Opic, (2008))

Let $f \in L_p(\mathbb{R}^n)$ *and*

$$F(x)) := \begin{cases} f^*(V_n|x|^n), & x \in \mathbb{R}^n, \quad \text{if } p = 1, \\ f^{**}(V_n|x|^n), & x \in \mathbb{R}^n, \quad \text{if } p \in (1, \infty), \end{cases}$$

where $V_n := |B_n(0,1)|_n$.
Then

$$\omega_1(F,t)_p \lesssim t\Big(\int_{t^n}^{\infty} s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p \, du \, \frac{ds}{s}\Big)^{1/p}$$

for all $t > 0$ *and* $f \in L_p(\mathbb{R}^n)$.

The classical Lorentz space $\Lambda_q^{loc}(\omega)$

Definition

Given $q \in (0, \infty]$ and $\omega \in \mathcal{M}^+(0,1)$, the *classical Lorentz space* $\Lambda_q^{loc}(\omega)$ is defined to be the set of all measurable functions $f \in \mathbb{R}^n$ such that

$$\|\omega f^*\|_{q;(0,1)} < \infty.$$

In particular, putting $\omega(t) := t^{1/p-1/q}\ b(t)$, $t \in (0,1)$, where $b \in SV(0,1)$, we obtain the *Lorentz-Karamata space* $L_{p,q;b}^{loc}$.

Note that Lorentz-Karamata spaces involve as particular cases the generalized Lorentz-Zygmund spaces, the Lorentz spaces, the Zygmund classes and Lebesgue spaces.

Some more notation

$1 \le p < \infty, \quad 1 \le r \le \infty, \quad 0 < q \le \infty$

If $b \in SV(0,1)$, we put $b(t) := 1$ for $t \in [1,2]$,

$$b_r(t) := \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)}, \quad t \in (0,1),$$

$$b_\infty^{**}(t) := t^{-1} \int_0^t b_\infty(\tau)\, d\tau, \quad t \in (0,1).$$

If $\omega \in \mathcal{M}^+(0,1)$, we put

$$\Omega_q(t) := \|\omega(s)\|_{q,(0,t)}, \quad t \in (0,1].$$

$\rho := \infty$ if $p \le q$ and $\frac{1}{\rho} := \frac{1}{q} - \frac{1}{p}$ if $q < p$

Theorem 2

Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1).

(i) *Let* $1 \le r \le q \le \infty$. *Then inequality* (3) *holds for all* $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if*

$$\Omega_q(1) + \|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \lesssim b_r(t) \quad \text{for all} \quad t \in (0,1). \qquad (5)$$

(ii) *Let* $0 < q < r < \infty$. *Then inequality* (3) *holds for all* $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if*

$$\Omega_q(1) + \int_0^1 \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^{\frac{qr}{r-q}} b_r(t)^{\frac{r^2}{q-r}} b(t^{\frac{1}{n}})^r \frac{dt}{t} < \infty. \qquad (6)$$

(iii) *Let* $0 < q < r = \infty$. *Then inequality* (3) *holds for all* $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if*

$$\Omega_q(1) + \int_{(0,1)} \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^q d(b^{**}_\infty(t)^{-q}) < \infty. \qquad (7)$$

Theorem 3

Let $1 \leq p < \infty$, $1 \leq r \leq \infty$, $0 < q \leq \infty$. *Let* $b \in SV(0,1)$ *satisfy* (1). *Define, for all* $t \in (0,1)$,

$$\tilde{b}(t) := \begin{cases} b_r(t)^{1-r/q+r/\max\{p,q\}} b(t^{1/n})^{r/q-r/\max\{p,q\}} & \text{if } r \neq \infty \\ b_\infty(t) & \text{if } r = \infty \end{cases} . \quad (8)$$

Then the inequality

$$\|t^{1/p-1/q}\tilde{b}(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \quad (9)$$

holds for all $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if* $q \geq r$.

Theorem 4

Let $1 \le p < \infty$, $1 \le r \le q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1).
(i) *Let* $\kappa \in \mathcal{M}_0^+(0,1;\downarrow)$. *Then the inequality*

$$\|t^{1/p-1/q}\tilde{b}(t)\kappa(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{10}$$

holds for all $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if* κ *is bounded.*
(ii) *Let* $\kappa \in \mathcal{M}_0^+(0,1)$ *and* $q = \infty$. *Then inequality* (10) *holds for all* $f \in B^{0,b}_{p,r}(\mathbb{R}^n)$ *if and only if* $\|\kappa\|_{\infty,(0,1)} < \infty$.

Definition (GE)

Let $(A, \|\cdot\|_A) \subset \mathcal{M}(\mathbb{R}^n)$ *be a quasi-normed space s.t.* $A \not\hookrightarrow L_\infty$.
A function $h \in C^+((0,\varepsilon];\downarrow)$, $\varepsilon \in (0,1)$, *is called the* (local) growth envelope function *of the space* A *provided that*

$$h(t) \approx \sup_{\|f\|_A \le 1} f^*(t) \quad \text{for all } \; t \in (0,\varepsilon]. \tag{11}$$

Given a growth envelope function h *of the space* A *and a number* $u \in (0,\infty]$, *the pair* (h,u) *is called the* (local) growth envelope *of the space* A *when the inequality*

$$\Big(\int_{(0,\varepsilon)} \Big(\frac{f^*(t)}{h(t)} \Big)^q \, d\mu_H(t) \Big)^{1/q} \lesssim \|f\|_A$$

(*with the usual modification when* $q = \infty$) *holds for all* $f \in A$ *if and only if the exponent* q *satisfies* $q \ge u$. *Here* μ_H *is the Borel measure associated with the non-decreasing function* $H(t) := -\ln h(t)$, $t \in (0,\varepsilon)$. *The component* u *in the growth envelope pair is called the* fine index.

Theorem 5

Let $1 \le p < \infty$, $1 \le r \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1)*. Then the growth envelope of* $B^{0,b}_{p,r}(\mathbb{R}^n)$ *is the pair*

$$(t^{-1/p}\, b_r(t)^{-1}, \max\{p, r\}).$$

Note that

$$t^{-1/p}\, b_r(t)^{-1} \approx \int_t^2 s^{-1/p-1}\, b_r(s)^{-1}\, ds =: h(t) \quad \forall t \in (0,1)$$

and that the function h is non-increasing and absolutely continuous on the whole interval $(0, 1)$.

Remark

Put $H(t) := -\ln h(t)$ for $t \in (0,\varepsilon)$, where $\varepsilon \in (0,1)$ is small enough. Since $H'(t) \approx \frac{1}{t}$ for a.e. $t \in (0,\varepsilon)$, the measure μ_H associated with the function H satisfies $d\mu_H(t) \approx \frac{dt}{t}$. Thus, by Definition (GE) and Theorem 5,

$$\|t^{1/p-1/q} b_r(t) f^*(t)\|_{q,(0,\varepsilon)} \lesssim \|f\|_{B^{0,b}_{p,r}} \quad \text{for all } f \in B^{0,b}_{p,r}(\mathbb{R}^n) \tag{12}$$

if and only if

$$q \geq \max\{p, r\}. \tag{13}$$

Hence, if (13) holds, then inequality (12) gives the same result as inequality (9) of Theorem 3 (since (13) implies that $\tilde{b} = b_r$). However, if $r \leq q < p$, then inequality (12) does not hold, while inequality (9) does. This means that the embeddings of Besov spaces $B^{0,b}_{p,r}(\mathbb{R}^n)$ given by Theorem 3 cannot be described in terms of growth envelopes when $1 \leq r \leq q < p < \infty$.

Theorems 3 and 4 imply that if $1 \le p < \infty$, $1 \le r \le q \le \infty$, then

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow L_{p,q;\tilde{b}}(\Omega),$$

where Ω is a domain in $\mathbb{R}^n$ of finite Lebesgue measure, and that this embedding is optimal within the scale of Lorentz-Karamata spaces.

Compact embeddings

Theorem 6

Let $1 \le p < \infty$, $1 \le r \le q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (1). *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and let* $0 < P \le p$. *Assume that* $\bar{b} \in SV(0,1)$ *and, if* $P = p > q$, *that* $\bar{b}/\tilde{b}$ *is a non-decreasing function on the interval* $(0, \delta)$ *for some* $\delta \in (0,1)$. *Then*

$$B^{0,b}_{p,r}(\mathbb{R}^n) \hookrightarrow\hookrightarrow L_{P,q;\bar{b}}(\Omega) \quad {}^1)$$

if and only if

$$\lim_{t \to 0+} \frac{t^{1/P}\bar{b}(t)}{t^{1/p}\tilde{b}(t)} = 0.$$

) This means that the mapping $u \mapsto u|_\Omega$ from $B^{0,b}_{p,r}(\mathbb{R}^n)$ into $L_{P,q;\bar{b}}(\Omega)$ is compact.

Thank you for your attention