Optimality and iteration

Luboš Pick (Charles University, Prague)

Santiago de Compostela, July 21, 2011

This is a joint work with Andrea Cianchi

Main goal

Main goal

We study the following basic question:

Main goal

We study the following basic question:

Can optimal results be iterated?

What is meant by optimality

What is meant by optimality

DEFINITION.

What is meant by optimality

DEFINITION. X, Y are function spaces,

What is meant by optimality

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X,

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X, $\mathfrak{M}$ is some category of (quasi-)normed function spaces.

What is meant by optimality

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X, $\mathfrak{M}$ is some category of (quasi-)normed function spaces.

We say that *Y is the optimal range partner for X with respect to T within $\mathfrak{M}$* if

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X, $\mathfrak{M}$ is some category of (quasi-)normed function spaces.

We say that *Y is the optimal range partner for X with respect to T within $\mathfrak{M}$* if

- $Y \in \mathfrak{M}$;

What is meant by optimality

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X, $\mathfrak{M}$ is some category of (quasi-)normed function spaces.

We say that *Y is the optimal range partner for X with respect to T within $\mathfrak{M}$* if

- $Y \in \mathfrak{M}$;
- T is bounded from X to Y (notation $T : X \to Y$);

DEFINITION. X, Y are function spaces, T is an operator, defined at least on X, $\mathfrak{M}$ is some category of (quasi-)normed function spaces.

We say that *Y is the optimal range partner for X with respect to T within $\mathfrak{M}$* if

- $Y \in \mathfrak{M}$;
- T is bounded from X to Y (notation $T : X \to Y$);
- Y is the smallest such space in $\mathfrak{M}$, that is, if $Z \in \mathfrak{M}$ is such that $T : X \to Z$, then $Y \hookrightarrow Z$.

Sobolev spaces

Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Given $u : \Omega \to \mathbb{R}$ and $m \in \mathbb{N}$, we denote the *m-th gradient* of u by $D^m u$.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Given $u : \Omega \to \mathbb{R}$ and $m \in \mathbb{N}$, we denote the *m-th gradient* of u by $D^m u$. That is,

$$D^m u := \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right)_{|\alpha| \leq m}.$$

Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Given $u : \Omega \to \mathbb{R}$ and $m \in \mathbb{N}$, we denote the *m-th gradient* of u by $D^m u$. That is,

$$D^m u := \left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)_{|\alpha| \leq m}.$$

The *Euclidean–Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions u which together with $|D^m u|$ belong to $L^p(\Omega)$.

Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Given $u : \Omega \to \mathbb{R}$ and $m \in \mathbb{N}$, we denote the *m-th gradient* of u by $D^m u$. That is,

$$D^m u := \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right)_{|\alpha| \leq m}.$$

The *Euclidean–Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions u which together with $|D^m u|$ belong to $L^p(\Omega)$.

The *general Euclidean–Sobolev space* $W^m X(\Omega)$ is the set of all functions u which together with $|D^m u|$ belong to X, where X is a function space on Ω.

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain of measure 1, $n \geq 2$.

Given $u : \Omega \to \mathbb{R}$ and $m \in \mathbb{N}$, we denote the *m-th gradient* of u by $D^m u$. That is,

$$D^m u := \left(\frac{\partial^\alpha u}{\partial x^\alpha} \right)_{|\alpha| \leq m}.$$

The *Euclidean–Sobolev space* $W^{m,p}(\Omega)$ is the set of all functions u which together with $|D^m u|$ belong to $L^p(\Omega)$.

The *general Euclidean–Sobolev space* $W^m X(\Omega)$ is the set of all functions u which together with $|D^m u|$ belong to X, where X is a function space on Ω.

NOTE: $W^{m,p}(\Omega) = W^m L^p(\Omega)$.

Order of an embedding

Order of an embedding

A very important ingredient in a Sobolev embedding is the **order** m.

Order of an embedding

A very important ingredient in a Sobolev embedding is the **order** m.

We will carefully distinguish *first-order embeddings*

$$W^1X \hookrightarrow Y$$

Order of an embedding

A very important ingredient in a Sobolev embedding is the **order** m.

We will carefully distinguish *first-order embeddings*

$$W^1X \hookrightarrow Y$$

from the *higher-order ones*

Order of an embedding

A very important ingredient in a Sobolev embedding is the **order** m.

We will carefully distinguish *first-order embeddings*

$$W^1X \hookrightarrow Y$$

from the *higher-order ones*

$$W^mX \hookrightarrow Y \qquad (m > 1).$$

Iteration

Iteration

Suppose that

$$W^1X \hookrightarrow Y$$

Suppose that

$$W^1X \hookrightarrow Y \qquad \text{and} \qquad W^1Y \hookrightarrow Z.$$

Iteration

Suppose that

$$W^1X \hookrightarrow Y \qquad \text{and} \qquad W^1Y \hookrightarrow Z.$$

Then, of course,

Iteration

Suppose that

$$W^1X \hookrightarrow Y \qquad \text{and} \qquad W^1Y \hookrightarrow Z.$$

Then, of course,

$$W^2X \hookrightarrow W^1Y \hookrightarrow Z$$

Suppose that

$$W^1X \hookrightarrow Y \qquad \text{and} \qquad W^1Y \hookrightarrow Z.$$

Then, of course,

$$W^2X \hookrightarrow W^1Y \hookrightarrow Z, \qquad \text{hence} \qquad W^2X \hookrightarrow Z.$$

Iteration

Suppose that

$$W^1 X \hookrightarrow Y \qquad \text{and} \qquad W^1 Y \hookrightarrow Z.$$

Then, of course,

$$W^2 X \hookrightarrow W^1 Y \hookrightarrow Z, \qquad \text{hence} \qquad W^2 X \hookrightarrow Z.$$

QUESTION REVISITED:

Suppose that

$$W^1 X \hookrightarrow Y \qquad \text{and} \qquad W^1 Y \hookrightarrow Z.$$

Then, of course,

$$W^2 X \hookrightarrow W^1 Y \hookrightarrow Z, \qquad \text{hence} \qquad W^2 X \hookrightarrow Z.$$

QUESTION REVISITED: *Is there any loss of information in the iteration process or not?*

Let $\mathfrak{M}$ be the class of *Lebesgue spaces* on Ω.

Let $\mathfrak{M}$ be the class of *Lebesgue spaces* on Ω.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}, \quad 1 \leq p < n.$$

Non-limiting embeddings for Lebesgue spaces

Let $\mathfrak{M}$ be the class of *Lebesgue spaces* on Ω.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}, \quad 1 \leq p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

Let $\mathfrak{M}$ be the class of *Lebesgue spaces* on Ω.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}, \quad 1 \leq p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

(ii) Non-limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,p} \hookrightarrow L^{\frac{np}{n-mp}}, \quad 1 \leq p < \frac{n}{m}.$$

Non-limiting embeddings for Lebesgue spaces

Let $\mathfrak{M}$ be the class of *Lebesgue spaces* on Ω.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}, \quad 1 \le p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

(ii) Non-limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,p} \hookrightarrow L^{\frac{np}{n-mp}}, \quad 1 \le p < \frac{n}{m}.$$

Then *the range space is optimal within* $\mathfrak{M}$.

Question formulated for this example

Question formulated for this example

QUESTION: Is the higher-order embedding preserved under iteration of the first-order ones?

Answer

Answer

Let $p < \frac{n}{2}$.

Answer

Let $p < \frac{n}{2}$. As we have seen, we have

Answer

Let $p < \frac{n}{2}$. As we have seen, we have

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}.$$

Answer

Let $p < \frac{n}{2}$. As we have seen, we have

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}.$$

Iterating this, we get

Answer

Let $p < \frac{n}{2}$. As we have seen, we have

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}.$$

Iterating this, we get

$$W^{2,p} \hookrightarrow W^{1,\frac{np}{n-p}} \hookrightarrow L^{\frac{n\frac{np}{n-p}}{n-\frac{np}{n-p}}} = L^{\frac{np}{n-2p}}.$$

Answer

Let $p < \frac{n}{2}$. As we have seen, we have

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p}}.$$

Iterating this, we get

$$W^{2,p} \hookrightarrow W^{1,\frac{np}{n-p}} \hookrightarrow L^{\frac{n\frac{np}{n-p}}{n-\frac{np}{n-p}}} = L^{\frac{np}{n-2p}}.$$

So, in this case, the range space obtained by iteration is optimal, hence **no information is lost**.

Limiting embeddings for Lebesgue spaces

Limiting embeddings for Lebesgue spaces

Let $\mathfrak{M}$ be still the class of *Lebesgue spaces* on Ω.

Let $\mathfrak{M}$ be still the class of *Lebesgue spaces* on Ω.

Limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,\frac{n}{m}} \hookrightarrow \begin{cases} L^q, \ q \in [1,\infty), & \text{if } 1 \le m \le n-1; \\ L^\infty & \text{if } m = n. \end{cases}$$

Let $\mathfrak{M}$ be still the class of *Lebesgue spaces* on Ω.

Limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,\frac{n}{m}} \hookrightarrow \begin{cases} L^q, \ \ q \in [1,\infty), & \text{if } 1 \le m \le n-1; \\ L^\infty & \text{if } m = n. \end{cases}$$

In the latter case, the range space is optimal within $\mathfrak{M}$.

Let $\mathfrak{M}$ be still the class of *Lebesgue spaces* on Ω.

Limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,\frac{n}{m}} \hookrightarrow \begin{cases} L^q, \ \ q \in [1,\infty), & \text{if } 1 \leq m \leq n-1; \\ L^\infty & \text{if } m = n. \end{cases}$$

In the latter case, **the range space is optimal within $\mathfrak{M}$**.

QUESTION: Can the latter embedding be obtained by iteration of first-order ones?

Answer

Answer

Let $n > 1$.

Answer

Let $n > 1$. Then

$$W^{n,1} \hookrightarrow W^{n-1,\frac{n}{n-1}} \hookrightarrow W^{n-2,\frac{n}{n-2}} \hookrightarrow \ldots \hookrightarrow W^{1,n} \hookrightarrow L^q \supsetneq L^\infty.$$

Answer

Let $n > 1$. Then

$$W^{n,1} \hookrightarrow W^{n-1,\frac{n}{n-1}} \hookrightarrow W^{n-2,\frac{n}{n-2}} \hookrightarrow \ldots \hookrightarrow W^{1,n} \hookrightarrow L^q \supsetneq L^\infty.$$

So, in the limiting case, **there is a loss of information**.

Orlicz spaces

Let $n > 2$ and let $\mathfrak{M}$ be the category of *Orlicz spaces* on Ω.

Orlicz spaces

Let $n > 2$ and let $\mathfrak{M}$ be the category of *Orlicz spaces* on Ω.

Then

$$W^{2,\frac{n}{2}}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega)$$

Let $n > 2$ and let $\mathfrak{M}$ be the category of *Orlicz spaces* on Ω.

Then

$$W^{2,\frac{n}{2}}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega)$$

and *the range at each step is optimal within* $\mathfrak{M}$.

Orlicz spaces

Let $n > 2$ and let $\mathfrak{M}$ be the category of *Orlicz spaces* on Ω.

Then

$$W^{2,\frac{n}{2}}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-1}}(\Omega)$$

and *the range at each step is optimal within* $\mathfrak{M}$.

However, it is known that

$$W^{2,\frac{n}{2}}(\Omega) \hookrightarrow \exp L^{\frac{n}{n-2}}(\Omega) \subsetneq \exp L^{\frac{n}{n-1}}(\Omega),$$

hence, again, **there is a loss of information in the iteration process**.

An attempt for a remedy

An attempt for a remedy

QUESTION What could have been done in order to avoid the loss?

An attempt for a remedy

QUESTION What could have been done in order to avoid the loss?

A POSSIBILITY: Use *Lorentz spaces* instead of Lebesgue and Orlicz ones!

Lorentz spaces

Lorentz spaces

Let $\mathfrak{M}$ be the category of *Lorentz spaces*.

Let $\mathfrak{M}$ be the category of *Lorentz spaces*.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p},p}, \quad 1 \leq p < n.$$

Let $\mathfrak{M}$ be the category of *Lorentz spaces*.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p},p}, \quad 1 \leq p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

Let $\mathfrak{M}$ be the category of *Lorentz spaces*.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p},p}, \quad 1 \leq p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

(ii) Non-limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,p} \hookrightarrow L^{\frac{np}{n-mp},p}, \quad 1 \leq p < \frac{n}{m}.$$

Let $\mathfrak{M}$ be the category of *Lorentz spaces*.

(i) Non-limiting *first-order* Euclidean–Sobolev embedding:

$$W^{1,p} \hookrightarrow L^{\frac{np}{n-p},p}, \quad 1 \leq p < n.$$

Then *the range space is optimal within* $\mathfrak{M}$.

(ii) Non-limiting *higher-order* Euclidean–Sobolev embedding:

$$W^{m,p} \hookrightarrow L^{\frac{np}{n-mp},p}, \quad 1 \leq p < \frac{n}{m}.$$

Then *the range space is optimal within* $\mathfrak{M}$.

Limiting embeddings for Lorentz spaces

Using *Lorentz spaces*, we get

$$W^{n,1} = W^n L^1 \hookrightarrow W^{n-1} L^{\frac{n}{n-1},1} \hookrightarrow W^{n-2} L^{\frac{n}{n-2},1} \hookrightarrow$$
$$\hookrightarrow \ldots \hookrightarrow W^1 L^{n,1} \hookrightarrow L^\infty.$$

Limiting embeddings for Lorentz spaces

Using *Lorentz spaces*, we get

$$W^{n,1} = W^n L^1 \hookrightarrow W^{n-1} L^{\frac{n}{n-1},1} \hookrightarrow W^{n-2} L^{\frac{n}{n-2},1} \hookrightarrow$$
$$\hookrightarrow \dots \hookrightarrow W^1 L^{n,1} \hookrightarrow L^\infty.$$

So now the range space is optimal, hence **there is no loss of information in the iteration process**.

A summary

A summary

We believe that there is a general principle behind the scene.

We believe that there is a general principle behind the scene.

The fact that optimality survived iteration in the last example is not caused by the fact that the spaces used were in particular Lorentz spaces, but because, in this case, the Lorentz spaces happen to coincide with the *optimal rearrangement-invariant spaces*.

A conjecture

A WORKING CONJECTURE:

A conjecture

A WORKING CONJECTURE: Let $\mathfrak{M}$ be the category of *rearrangement-invariant (r.i.) spaces*.

A conjecture

A WORKING CONJECTURE: Let $\mathfrak{M}$ be the category of *rearrangement-invariant (r.i.) spaces*. Let $m \in \mathbb{N}$ and let $X_0, X_1, \dots, X_m \in \mathfrak{M}$.

A WORKING CONJECTURE: Let $\mathfrak{M}$ be the category of *rearrangement-invariant (r.i.) spaces*. Let $m \in \mathbb{N}$ and let $X_0, X_1, \ldots, X_m \in \mathfrak{M}$. Assume that

$$W^1 X_j \hookrightarrow X_{j+1}, \quad j = 0, \ldots, m-1,$$

and *the range is optimal within $\mathfrak{M}$ at each step*.

A conjecture

A WORKING CONJECTURE: Let $\mathfrak{M}$ be the category of *rearrangement-invariant (r.i.) spaces*. Let $m \in \mathbb{N}$ and let $X_0, X_1, \dots, X_m \in \mathfrak{M}$. Assume that

$$W^1 X_j \hookrightarrow X_{j+1}, \quad j = 0, \dots, m-1,$$

and *the range is optimal within $\mathfrak{M}$ at each step*. Then

$$W^m X_0 \hookrightarrow X_m$$

A WORKING CONJECTURE: Let $\mathfrak{M}$ be the category of *rearrangement-invariant (r.i.) spaces*. Let $m \in \mathbb{N}$ and let $X_0, X_1, \ldots, X_m \in \mathfrak{M}$. Assume that

$$W^1 X_j \hookrightarrow X_{j+1}, \quad j = 0, \ldots, m-1,$$

and *the range is optimal within $\mathfrak{M}$ at each step*. Then

$$W^m X_0 \hookrightarrow X_m$$

and *the range is optimal within $\mathfrak{M}$*.

Description of our approach

Description of our approach

The key ingredient is the **first-order reduction theorem**.

The key ingredient is the **first-order reduction theorem**.

A *reduction theorem* is a principle that reduces a Sobolev-type embedding $W^1X(\Omega) \hookrightarrow Y(\Omega)$ to a one-dimensional problem involving boundedness of certain integral operator T between *representation spaces* $\overline{X}(0,1)$ and $\overline{Y}(0,1)$ of $X(\Omega)$ and $Y(\Omega)$.

The key ingredient is the **first-order reduction theorem**.

A *reduction theorem* is a principle that reduces a Sobolev-type embedding $W^1X(\Omega) \hookrightarrow Y(\Omega)$ to a one-dimensional problem involving boundedness of certain integral operator T between *representation spaces* $\overline{X}(0,1)$ and $\overline{Y}(0,1)$ of $X(\Omega)$ and $Y(\Omega)$.

In other words, reduction theorem asserts the equivalence of

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

Description of our approach

The key ingredient is the **first-order reduction theorem**.

A *reduction theorem* is a principle that reduces a Sobolev-type embedding $W^1X(\Omega) \hookrightarrow Y(\Omega)$ to a one-dimensional problem involving boundedness of certain integral operator T between *representation spaces* $\overline{X}(0,1)$ and $\overline{Y}(0,1)$ of $X(\Omega)$ and $Y(\Omega)$.

In other words, reduction theorem asserts the equivalence of

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

to

Description of our approach

The key ingredient is the **first-order reduction theorem**.

A *reduction theorem* is a principle that reduces a Sobolev-type embedding $W^1X(\Omega) \hookrightarrow Y(\Omega)$ to a one-dimensional problem involving boundedness of certain integral operator T between *representation spaces* $\overline{X}(0,1)$ and $\overline{Y}(0,1)$ of $X(\Omega)$ and $Y(\Omega)$.

In other words, reduction theorem asserts the equivalence of

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

to

$$T : \overline{X}(0,1) \to \overline{Y}(0,1)$$

for an appropriate T.

An iteration principle for integral operators

An iteration principle for integral operators

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

An iteration principle for integral operators

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\, ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$.

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$.

An iteration principle for integral operators

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$. Let $m \in \mathbb{N}$ and let $\overline{X}_0, \ldots, \overline{X}_m \in \mathfrak{M}$.

An iteration principle for integral operators

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$. Let $m \in \mathbb{N}$ and let $\overline{X}_0, \ldots, \overline{X}_m \in \mathfrak{M}$. Assume that

$$T : \overline{X}_j(0,1) \to \overline{X}_{j+1}(0,1), \quad j = 0, \ldots, m-1,$$

An iteration principle for integral operators

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$. Let $m \in \mathbb{N}$ and let $\overline{X}_0, \dots, \overline{X}_m \in \mathfrak{M}$. Assume that

$$T : \overline{X}_j(0,1) \to \overline{X}_{j+1}(0,1), \quad j = 0, \dots, m-1,$$

and let *the range space be optimal within $\mathfrak{M}$ at each step*.

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^1 g(s)\frac{\varphi(s)}{s}\, ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$. Let $m \in \mathbb{N}$ and let $\overline{X}_0, \dots, \overline{X}_m \in \mathfrak{M}$. Assume that

$$T : \overline{X}_j(0,1) \to \overline{X}_{j+1}(0,1), \quad j = 0, \dots, m-1,$$

and let *the range space be optimal within $\mathfrak{M}$ at each step*. Then

$$T^m : \overline{X}_0(0,1) \to \overline{X}_m(0,1)$$

THEOREM. Let

$$Tg(t) := \int_{t^\alpha}^{1} g(s)\frac{\varphi(s)}{s}\,ds,$$

where $\alpha > 0$ and φ is increasing and bounded on $(0,1)$. Let $\mathfrak{M}$ be the category of *r.i. spaces* on $(0,1)$. Let $m \in \mathbb{N}$ and let $\overline{X}_0, \dots, \overline{X}_m \in \mathfrak{M}$. Assume that

$$T : \overline{X}_j(0,1) \to \overline{X}_{j+1}(0,1), \quad j = 0, \dots, m-1,$$

and let *the range space be optimal within $\mathfrak{M}$ at each step*. Then

$$T^m : \overline{X}_0(0,1) \to \overline{X}_m(0,1)$$

and *the range space is optimal within $\mathfrak{M}$*.

Consequences for Sobolev embeddings

COROLLARY. The working conjecture is true for any type of Sobolev embedding for which a reduction theorem is known involving an appropriate T.

A closer look on the one-dimensional operators

QUESTION: What is T in concrete examples of embeddings?

A closer look on the one-dimensional operators

QUESTION: What is T in concrete examples of embeddings?

We shall illustrate this on various examples.

First example: Euclidean–Sobolev embeddings

THEOREM (reduction theorem for Euclidean–Sobolev embeddings).

THEOREM (reduction theorem for Euclidean–Sobolev embeddings).
The embedding

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

First example: Euclidean–Sobolev embeddings

THEOREM (reduction theorem for Euclidean–Sobolev embeddings).
The embedding

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

First example: Euclidean–Sobolev embeddings

THEOREM (reduction theorem for Euclidean–Sobolev embeddings).
The embedding

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$Tg(t) := \int_t^1 s^{\frac{1}{n}-1} g(s)\, ds.$$

First example: Euclidean–Sobolev embeddings

THEOREM (reduction theorem for Euclidean–Sobolev embeddings).
The embedding

$$W^1X(\Omega) \hookrightarrow Y(\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$Tg(t) := \int_t^1 s^{\frac{1}{n}-1} g(s)\, ds.$$

(Edmunds–Kerman–Pick, JFA 2000).

Second example: boundary trace embeddings

Second example: boundary trace embeddings

THEOREM (reduction theorem for boundary trace embeddings).

Second example: boundary trace embeddings

THEOREM (reduction theorem for boundary trace embeddings).
The boundary trace embedding

$$\mathrm{Tr} : W^1X(\Omega) \to Y(\partial\Omega)$$

Second example: boundary trace embeddings

THEOREM (reduction theorem for boundary trace embeddings).
The boundary trace embedding

$$\mathrm{Tr} : W^1X(\Omega) \to Y(\partial\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

Second example: boundary trace embeddings

THEOREM (reduction theorem for boundary trace embeddings). The boundary trace embedding

$$\mathrm{Tr} : W^1X(\Omega) \to Y(\partial\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$Tg(t) := \int_{t^{\frac{n}{n-1}}}^{1} s^{\frac{1}{n}-1} g(s)\, ds.$$

THEOREM (reduction theorem for boundary trace embeddings). The boundary trace embedding

$$\mathrm{Tr} : W^1 X(\Omega) \to Y(\partial\Omega)$$

holds if and only if

$$T : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$Tg(t) := \int_{t^{\frac{n}{n-1}}}^{1} s^{\frac{1}{n}-1} g(s)\, ds.$$

(Cianchi–Kerman–Pick, J. Anal. Math. 2008).

Third example: Gaussian–Sobolev embeddings

THEOREM (reduction theorem for Gaussian–Sobolev embeddings).

Third example: Gaussian–Sobolev embeddings

THEOREM (reduction theorem for Gaussian–Sobolev embeddings).
The Gaussian–Sobolev embedding

$$W^1X(\mathbb{R}^n,\gamma_n) \hookrightarrow Y(\mathbb{R}^n,\gamma_n)$$

Third example: Gaussian–Sobolev embeddings

THEOREM (reduction theorem for Gaussian–Sobolev embeddings).
The Gaussian–Sobolev embedding

$$W^1X(\mathbb{R}^n,\gamma_n) \hookrightarrow Y(\mathbb{R}^n,\gamma_n)$$

holds if and only if

$$T:\overline{X}(0,1) \to \overline{Y}(0,1),$$

THEOREM (reduction theorem for Gaussian–Sobolev embeddings).
The Gaussian–Sobolev embedding

$$W^1X(\mathbb{R}^n,\gamma_n)\hookrightarrow Y(\mathbb{R}^n,\gamma_n)$$

holds if and only if

$$T:\overline{X}(0,1)\to\overline{Y}(0,1),$$

where

$$Tg(t):=\int_t^1\frac{g(s)}{s\sqrt{1+\log(\frac{1}{s})}}\,ds.$$

Third example: Gaussian–Sobolev embeddings

THEOREM (reduction theorem for Gaussian–Sobolev embeddings).
The Gaussian–Sobolev embedding

$$W^1X(\mathbb{R}^n,\gamma_n)\hookrightarrow Y(\mathbb{R}^n,\gamma_n)$$

holds if and only if

$$T:\overline{X}(0,1)\to\overline{Y}(0,1),$$

where

$$Tg(t):=\int_t^1\frac{g(s)}{s\sqrt{1+\log(\frac{1}{s})}}\,ds.$$

(Cianchi–Pick, JFA 2009).

Further examples

Further examples

The method can be further applied for example to

Further examples

The method can be further applied for example to

- Euclidean–Sobolev embeddings on irregular domains that have known isoperimetric exponent $\alpha \in [\frac{n-1}{n}, 1)$;

Further examples

The method can be further applied for example to

- Euclidean–Sobolev embeddings on irregular domains that have known isoperimetric exponent $\alpha \in [\frac{n-1}{n}, 1)$;
- trace embeddings to subsets of lower dimensions than $n-1$;

Further examples

The method can be further applied for example to

- Euclidean–Sobolev embeddings on irregular domains that have known isoperimetric exponent $\alpha \in [\frac{n-1}{n}, 1)$;
- trace embeddings to subsets of lower dimensions than $n - 1$;
- product probability measure spaces between exponential and Gaussian with known isoperimetric profile;

The method can be further applied for example to

- Euclidean–Sobolev embeddings on irregular domains that have known isoperimetric exponent $\alpha \in [\frac{n-1}{n}, 1)$;
- trace embeddings to subsets of lower dimensions than $n-1$;
- product probability measure spaces between exponential and Gaussian with known isoperimetric profile;
- Heisenberg chain;

Further examples

The method can be further applied for example to

- Euclidean–Sobolev embeddings on irregular domains that have known isoperimetric exponent $\alpha \in [\frac{n-1}{n}, 1)$;
- trace embeddings to subsets of lower dimensions than $n - 1$;
- product probability measure spaces between exponential and Gaussian with known isoperimetric profile;
- Heisenberg chain;
- and probably lot more.

Description of results for Sobolev embeddings

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the "if" part of a higher-order reduction theorem;

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the "if" part of a higher-order reduction theorem;
- a construction of the optimal r.i. range space if the domain space is prescribed;

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the "if" part of a higher-order reduction theorem;
- a construction of the optimal r.i. range space if the domain space is prescribed;
- a reiteration (stability) theorem for iterated embeddings.

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the “if” part of a higher-order reduction theorem;
- a construction of the optimal r.i. range space if the domain space is prescribed;
- a reiteration (stability) theorem for iterated embeddings.

NOTE: The remaining “only if” part must be proved in each case.

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the "if" part of a higher-order reduction theorem;
- a construction of the optimal r.i. range space if the domain space is prescribed;
- a reiteration (stability) theorem for iterated embeddings.

NOTE: The remaining "only if" part must be proved in each case. There is no general method for that.

Description of results for Sobolev embeddings

Suppose we consider a Sobolev-type embedding for which a suitable first-order reduction theorem is known.

Then our general iteration machinery yields basically three types of results:

- the "if" part of a higher-order reduction theorem;
- a construction of the optimal r.i. range space if the domain space is prescribed;
- a reiteration (stability) theorem for iterated embeddings.

NOTE: The remaining "only if" part must be proved in each case. There is no general method for that. Usually we test the embedding on certain suitable type of special functions.

Euclidean–Sobolev embeddings – higher-order reduction theorem

Euclidean–Sobolev embeddings – higher-order reduction theorem

THEOREM (Higher-order reduction theorem.) Let $m \in \mathbb{N}$, $\alpha \in [\frac{n-1}{n}, 1)$. Then

$$W^m X(\Omega) \hookrightarrow Y(\Omega)$$

holds for every domain $\Omega \subset \mathbb{R}^n$ having the isoperimetric exponent α if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

Euclidean–Sobolev embeddings – higher-order reduction theorem

THEOREM (Higher-order reduction theorem.) Let $m \in \mathbb{N}$, $\alpha \in [\frac{n-1}{n}, 1)$. Then

$$W^m X(\Omega) \hookrightarrow Y(\Omega)$$

holds for every domain $\Omega \subset \mathbb{R}^n$ having the isoperimetric exponent α if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$H_m g(t) := \int_t^1 s^{-1+m(1-\alpha)} g(s)\, ds.$$

Euclidean–Sobolev embeddings – higher-order optimal range construction

THEOREM. (Higher-order optimal range construction.) Let $m \in \mathbb{N}$, $\alpha \in [\frac{n-1}{n}, 1)$ and let $X(0,1)$ be an r.i. space.

THEOREM. (Higher-order optimal range construction.) Let $m \in \mathbb{N}$, $\alpha \in [\frac{n-1}{n}, 1)$ and let $X(0,1)$ be an r.i. space. Then **the space $X_{m,\alpha}(0,1)$**, whose associate norm is given by

$$\|g\|_{(X_{m,\alpha}(0,1))'} := \|s^{m(1-\alpha)} g^{**}(s)\|_{\overline{X}'(0,1)},$$

is the optimal r.i. space such that

$$W^m X(\Omega) \hookrightarrow X_{m,\alpha}(\Omega)$$

for every domain $\Omega \subset \mathbb{R}^n$ with isoperimetric exponent α.

Euclidean–Sobolev embeddings – reiteration theorem

THEOREM. (reiteration theorem.) Under the assumptions and the notation of the preceding two theorems, one has

$$(X_{k,\alpha})_{h,\alpha} = X_{k+h,\alpha}$$

for every $k, h \in \mathbb{N}$.

Gaussian–Sobolev embeddings – higher-order reduction theorem

Gaussian–Sobolev embeddings – higher-order reduction theorem

THEOREM. (Higher-order reduction theorem for Gaussian embeddings). Let $m \in \mathbb{N}$. Then

$$W^m X(\mathbb{R}^n, \gamma_n) \hookrightarrow Y(\mathbb{R}^n, \gamma_n)$$

holds for every $n \in \mathbb{N}$ with constant independent of n if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

THEOREM. (Higher-order reduction theorem for Gaussian embeddings). Let $m \in \mathbb{N}$. Then

$$W^m X(\mathbb{R}^n, \gamma_n) \hookrightarrow Y(\mathbb{R}^n, \gamma_n)$$

holds for every $n \in \mathbb{N}$ with constant independent of n if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$H_m g(t) := \frac{1}{(1+\log\frac{1}{t})^{\frac{m-1}{2}}} \int_t^1 \frac{(\log\frac{s}{t})^{m-1}}{s(1+\log\frac{1}{t})^{\frac{1}{2}}}\, g(s)\, ds.$$

Traces – higher-order reduction theorem

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$,

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition.

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \leq d \leq n$ and $n - d \leq m$.

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \leq d \leq n$ and $n - d \leq m$. Let H_d be an affine d-dimensional subspace of $\mathbb{R}^n$ such that $\Omega \cap H_d \neq 0$.

Traces – higher-order reduction theorem

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \leq d \leq n$ and $n - d \leq m$. Let H_d be an affine d-dimensional subspace of $\mathbb{R}^n$ such that $\Omega \cap H_d \neq 0$. Then

$$Tr : W^m X(\Omega) \hookrightarrow Y(\Omega \cap H_d)$$

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \le d \le n$ and $n - d \le m$. Let H_d be an affine d-dimensional subspace of $\mathbb{R}^n$ such that $\Omega \cap H_d \neq 0$. Then

$$Tr : W^m X(\Omega) \hookrightarrow Y(\Omega \cap H_d)$$

holds if and only if

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \leq d \leq n$ and $n - d \leq m$. Let H_d be an affine d-dimensional subspace of $\mathbb{R}^n$ such that $\Omega \cap H_d \neq 0$. Then

$$Tr : W^m X(\Omega) \hookrightarrow Y(\Omega \cap H_d)$$

holds if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

THEOREM. (Higher-order reduction theorem for traces). Let $m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ connected bounded open set, satisfying the inner cone condition. Let $d \in \mathbb{N}$, $1 \leq d \leq n$ and $n - d \leq m$. Let H_d be an affine d-dimensional subspace of $\mathbb{R}^n$ such that $\Omega \cap H_d \neq 0$. Then

$$Tr : W^m X(\Omega) \hookrightarrow Y(\Omega \cap H_d)$$

holds if and only if

$$H_m : \overline{X}(0,1) \to \overline{Y}(0,1),$$

where

$$H_m g(t) := \int_{t^{\frac{n}{d}}}^{1} s^{-1+\frac{m}{n}} g(s)\, ds.$$