The initial purpose of my studies was to find condition on a r.i. space A such that the set of functions

$$G = \{f \in L_0 : \ g(t) = t^{-\alpha}[f^{**}(t) - f^*(t)] \in A, \quad f^*(\infty) = 0\},$$

with arbitrary fixed $\alpha \in (0, 1)$, becomes r.i. space too with the norm

$$\|f\|_G \sim \|t^{-\alpha}[f^{**}(t) - f^*(t)]\|_A.$$

Such sets G attracted my attention due to their important role in the theory of Sobolev type embedding. Namely, in the paper by M.Milman and E.Pustylnik (2004) it was shown that, for any $f \in C^m(\mathbb{R}^n)$ vanishing at infinity together with all its derivatives up to the order $m - 1$, one has that

$$\|t^{-m/n}[f^{**}(t) - f^*(t)]\|_A \lesssim \| \, |D^m f| \, \|_A$$

for any $m = 1, 2, \ldots , n - 1$, provided that $\pi_A \geq m/n$.

Here I denote by π_A the lower Boyd index and ρ_A, in what follows, will mean the upper index:

$$\pi_A = \lim_{s \to 0} \frac{\ln d_A(s)}{\ln s}, \quad \rho_A = \lim_{s \to \infty} \frac{\ln d_A(s)}{\ln s},$$

where

$$d_A(s) = \sup_{f \in A} \frac{\|f(t/s)\|_A}{\|f(t)\|_A}.$$

It was also shown in the same paper that the set G with $\alpha = m/n$ is smaller than any r.i. space B such that $\|f\|_B \lesssim \| \, |D^m f| \, \|_A$. This immediately implies that if G itself is a r.i. space, then it gives an optimal Sobolev type embedding $W^m_A \hookrightarrow G$ among all r.i. spaces.

The problem of linearity and normability of the set

$$G = \{f : \ g(t) = t^{-\alpha}[f^{**}(t) - f^*(t)] \in A, \quad f^*(\infty) = 0\}$$

is rather simple if $\pi_A > \alpha$, since in this case

$$\|t^{-\alpha}[f^{**}(t) - f^*(t)]\|_A \sim \|t^{-\alpha} f^{**}(t)\|_A$$

and the right-hand term here is obviously a norm. Thus we may restrict the problem to the *limiting case* $\pi_A = \alpha$ alone. Remark that, in general, the Boyd indices π_A and ρ_A can be rather different and some conditions on ρ_A may be needed even for the fixed $\pi_A = \alpha$.

Unfortunately, all my attempts of a direct solution of the problem in the remaining limiting case $\pi_A = \alpha$ were unsuccessful. Moreover, the set G turned out to be nonlinear even for the simplest classical space L_p with $p = 1/\alpha$. A bit later I proved that G is nonlinear for any r.i. space A with the fundamental function $\varphi_A(t) \sim t^\alpha$, except for the Lorentz space $L_{p1}, p = 1/\alpha$ that gives $G = L_\infty$, and for a long time I conjectured that this space is unique. (By the way, another extreme Lorentz space, $L_{p\infty}$, gives $G =$ weak-L^∞, the famous r.i. hull of BMO, introduced by Bennet, De Vore and Sharpley.)

The first indirect method, that I applied to obtaining the desired spaces A, was the use of some other criteria of optimal Sobolev type embeddings. For example, it is known that

$$W_A^m \hookrightarrow B \quad \Longleftrightarrow \quad Q_{m/n}A \hookrightarrow B,$$

where

$$Q_\alpha g(t) = \int_t^\infty s^\alpha g(s)\,\frac{ds}{s}\,, \qquad 0 \le \alpha < 1.$$

The set G satisfies this condition: if $g = t^{-\alpha}(f^{**} - f^*)$, then

$$Q_\alpha g = \int_t^\infty s^\alpha s^{-\alpha}(f^{**} - f^*)\,\frac{ds}{s} = \int_t^\infty (-f^{**})' ds = f^{**}(t) \in B$$

for any r.i. space B with $\rho_B < 1$. Thus if we define $\|g\|_A = \|Q_\alpha g\|_B$, we obtain that $G = B$.

Unfortunately, the space A, thus obtained, is not r.i. And the replacement of $\|Q_\alpha g\|_B$ by a r.i. counterpart $\|Q_\alpha g^*\|_B$ gives essentially **smaller** space, cancelling the proof of coincidence $G = B$.

The proof, but not the fact!

The norms $\|Q_\alpha g\|_B$ and $\|Q_\alpha g^*\|_B$, that for arbitrary functions satisfy only inequality $\|Q_\alpha g\|_B \lesssim \|Q_\alpha g^*\|_B$, appeared to be equivalent on the set of functions $g = t^{-\alpha}(f^{**} - f^*)$, which alone is used in the definition of the set G. This follows from a rather subtle property of rearrangements, stated in the monograph by Krein, Semenov and Petunin:

If a function $x(t)$ is non-negative and increasing, $\gamma > -1$, $\gamma + \delta < -1$, then

$$\int_0^\infty t^\gamma [t^\delta x(t)]^*\,dt \le C \int_0^\infty t^{\gamma+\delta} x(t)\,dt.$$

This inequality can be easily changed to the form

$$\int_t^\infty s^\gamma [s^\delta x(s)]^*\,ds \le C \int_t^\infty s^{\gamma+\delta} x(s)\,ds\,,$$

and after taking $\gamma = \alpha - 1,\ \delta = -\alpha - 1,$

$$x = t(f^{**}(t) - f^*(t)), \quad g = t^{-\alpha}(f^{**}(t) - f^*(t)),$$

we obtain that

$$\int_t^\infty s^{\alpha-1} g^*(s)\,ds \lesssim \int_t^\infty s^{\alpha-1} g(s)\,ds.$$

All this discussion may be considered as an introduction, allowing us to replace the initial problem by the equivalent one: to describe all r.i. spaces A such that $\|g\|_A \sim \|Q_\alpha g^*\|_B$ for some r.i. space B. Just at this place we will need the K-monotonicity of couples of Lorentz spaces that are close (and even equal) to the space L_1. In particular, we will use the couple L_1, L_{p1} with $p > 1$.

At the beginning I was sure that that this fact is known. Moreover, I met a paper, where the K-monotonicity of the couple L_1, L_{p1} was explicitly used with the reference to the monograph by Bergh and Löfström. But Bergh and Löfström only mentioned this fact in some remark without proof and with rather vague explanation. I proceeded to seek the proof in various papers and monographs, but vainly. Moreover, the biggest experts in this topic M.Cwikel and Yu.Brudnyi said me that the K-monotonicity of such couples of Lorentz spaces does not follow from their results and apparently is not proved.

The Lorentz space Λ_α with fundamental function $\alpha(t)$ is defined by the norm

$$\|f\|_{\Lambda_\alpha} = \int_0^\infty |f^*(t)| d\alpha(t).$$

Passing to equivalent function, we may always suppose that $\alpha(t)$ is differentiable with positive derivative $\alpha'(t)$. If $\alpha(t) = t^{1/p}$, we obtain that $\Lambda_\alpha = L_{p1}$. For $\alpha(t) = t$ we get the space L_1. Obviously $\|f\|_{\Lambda_\alpha} = \|f^*\|_{L_1(\alpha')}$, where $L_1(\alpha')$ means the space L_1 with the weight $\alpha'(t)$.

In our problem it is enough to consider only Lorentz spaces with positive Boyd indices, so that $\alpha'(t) \sim \alpha(t)/t$. In this case the similarity between Lorentz and L_1 spaces extends to their sums:

$$\|f\|_{\Lambda_{\alpha+\beta}} \sim \|f^*\|_{L_1(\alpha')+L_1(\beta')}\,,$$

and moreover,

$$K(t, f, \Lambda_\alpha, \Lambda_\beta) \sim K(t, f^*, L_1(\alpha'), L_1(\beta'))\,.$$

Theorem 1. *Let $\Lambda_\alpha, \Lambda_\beta$ be Lorentz spaces with positive Boyd indices. Then interpolation in this couple can be described only by the real method.*

Proof. Let $f, g \in \Lambda_\alpha + \Lambda_\beta$ be such that

$$K(t, f, \Lambda_\alpha, \Lambda_\beta) \leq K(t, g, \Lambda_\alpha, \Lambda_\beta),$$

then

$$K\big(t, f^*, L_1(\alpha'), L_1(\beta')\big) \lesssim K\big(t, g^*, L_1(\alpha'), L_1(\beta')\big).$$

But the couple $L_1(\alpha'), L_1(\beta')$ is well known as a Calderón one, hence there exists a linear operator T bounded on the spaces $L_1(\alpha')$ and $L_1(\beta')$ and such that $T(g^*) = f^*$. Moreover, $T:\ \Lambda_\alpha \to L_1(\alpha')$ and $T:\ \Lambda_\beta \to L_1(\beta')$, since $\|h\|_{L_1(\alpha')} \leq \|h\|_{\Lambda_\alpha}$ and $\|h\|_{L_1(\beta')} \leq \|h\|_{\Lambda_\beta}$ for any function h.

Consider now the Hardy operator

$$Qh(t) = \int_t^\infty h(s)\,\frac{ds}{s}\,.$$

If $h(t) \geq 0$ then $(Qh)^* = Qh$ and thus

$$\|Qh\|_{\Lambda_\alpha} = \|Qh\|_{L_1(\alpha')}, \qquad \|Qh\|_{\Lambda_\beta} = \|Qh\|_{L_1(\beta')}.$$

Since Q is bounded on $L_1(\alpha')$ and $L_1(\beta')$, we obtain that

$$\|QTh\|_{\Lambda_\alpha} \leq \|Q|Th|\,\|_{\Lambda_\alpha} = \|Q|Th|\,\|_{L_1(\alpha')} \leq \|Th\|_{L_1(\alpha')} \lesssim \|h\|_{\Lambda_\alpha}$$

and similarly $\|QTh\|_{\Lambda_\beta} \lesssim \|h\|_{\Lambda_\beta}$ for any $h(t)$. This implies that $\|QTh\|_A \lesssim \|h\|_A$ for any space A which is interpolation in the couple $\Lambda_\alpha, \Lambda_\beta$. For our initial functions f, g this means that $\|Q(f^*)\|_A \lesssim \|g\|_A$.

But

$$Q(f^*)(t) = \int_t^\infty f^*(s)\,\frac{ds}{s} \geq \int_t^{2t} f^*(s)\,\frac{ds}{s} \geq f^*(2t)\,\ln 2,$$

thus

$$\|f\|_A \sim \|f^*(2t)\|_A \lesssim \|Q(f^*)\|_A \lesssim \|g\|_A,$$

and the theorem is proved. $\square$

Theorem 2. *Let A be a r.i. space interpolation in the couple $L_1, L_{p1},\ p = 1/\alpha$. Then the set*

$$G = \{f \in L_0 :\ g(t) = t^{-\alpha}[f^{**}(t) - f^*(t)] \in A, \quad f^*(\infty) = 0\},$$

is r.i. space with the norm equivalent to $\|t^{-\alpha}[f^{**}(t) - f^*(t)]\,\|_A$.

Proof. As we already know, it is enough to show that $\|g\|_A \sim \|Q_\alpha g^*\|_B$ for some r.i. space B. From Theorem 1 we have that the couple L_1, L_{p1} is a Calderón couple, hence

$$A = (L_1, L_{p1})^K_\Phi \qquad \text{for some parameter space } \ \Phi.$$

Let us define

$$B = (L_{q1}, L_\infty)^K_\Phi \qquad \text{with } \ q = p/(p-1) \quad \text{and the same } \ \Phi.$$

The norms in spaces A, B can be written more explicitly if we use the known formulas for K-functional in considered couples:

$$K(t, f, L_1, L_{p1}) \sim \int_0^{t^{p/(p-1)}} f^*(s)\,ds + t\int_{t^{p/(p-1)}}^\infty s^{1/p} f^*(s)\,\frac{ds}{s},$$

$$K(t, f, L_{q1}, L_\infty) \sim \int_0^{t^q} s^{1/q} f^*(s)\,\frac{ds}{s}.$$

For our p, q, we obtain

$$K(t, Q_\alpha(g^*), L_{q1}, L_\infty) \sim \int_0^{t^q} s^{1/q} \left(\int_s^\infty \tau^{1/p} g^*(\tau)\,\frac{d\tau}{\tau}\right) \frac{ds}{s}$$

$$= \int_0^{t^q} \tau^{1/p} g^*(\tau) \left(\int_0^\tau s^{1/q}\frac{ds}{s}\right) \frac{d\tau}{\tau} + \int_{t^q}^\infty \tau^{1/p} g^*(\tau) \left(\int_0^{t^q} s^{1/q}\frac{ds}{s}\right) \frac{d\tau}{\tau}$$

$$\sim \int_0^{t^q} g^*(\tau)\,d\tau + t\int_{t^q}^\infty \tau^{1/p} g^*(\tau)\,\frac{d\tau}{\tau} \sim K(t, g, L_1, L_{p,1}).$$

Consequently,

$$\|g\|_A = \|K(t, g, L_1, L_{p1})\|_\Phi \sim \|K(t, Q_\alpha(g^*), L_{q1}, L_\infty)\|_\Phi = \|Q_\alpha(g^*)\|_B,$$

which gives the desired representation for the space A. $\square$

Theorem 2 gives full description of all spaces A solving our problem. For example, we may take any space with Boyd indices less than α. But at the very beginning I explained that the only important case is just when $\pi_A = \alpha$. Examples of such spaces A are not simple for constructing. I can propose the space A of functions $f :\ (0, 1) \mapsto \mathbb{R}$ with the norm

$$\|g\|_A = \left\| \frac{t^{-\alpha}}{\ln(e/t)} \int_t^1 s^{\alpha-1} g^{**}(s) ds \right\|_{L_p}, \quad p = \frac{1}{\alpha}.$$

This space is interesting, because defines as B the famous Hansson-Brézis-Wainger space with the norm

$$\|f\|_B = \left\| \left(\frac{1}{t} \ln \frac{e}{t}\right)^{\alpha} f^{**}(t) \right\|_{L_p}, \quad p = \frac{1}{\alpha}.$$

Remark. An assertion, converse to Theorem 2, is also true, namely, linearity and normability of the space

$$G = \{f \in L_0 : \; g(t) = t^{-\alpha}[f^{**}(t) - f^*(t)] \in A, \quad f^*(\infty) = 0\}$$

implies interpolation property of the space A in the couple L_1, L_{p1}. The proof is rather standard, using maximal Calderón operator for the couples L_1, L_{p1} and L_{q1}, L_∞. Unfortunately, we obtain thus an additional condition on A, namely, $\rho_A < 1$.

Having direct and converse assertions together, we get a possibility to construct the minimal r.i. space, containing the given set G, when it itself is not such. As an example I recall the space $A = L_{p\infty}$, which gives a nonlinear set $G =$ weak-L^∞. Using some results on optimal interpolation in ultrasymmetric spaces, we immediately obtain that the minimal r.i. extension of the set weak-L^∞ (and thus of the space BMO) is the Zygmund space $\exp L$.

The set of functions $g(t) = t^{-\alpha}[f^{**}(t) - f^*(t)]$ forms a rather special part of any r.i. space, even not a cone. Nevertheless, the totality of their norms defines any r.i. space up to equivalence of norms. More precisely, if the norms of two r.i. spaces B_1, B_2 are equivalent on these functions, then $B_1 = B_2$. This follows from a rather surprising fact that for any non-increasing nonnegative function $h(t)$, there exists a function $g(t)$ of the above mentioned form, such that

$$\int_0^t h(s)ds \sim \int_0^t g^*(s)ds$$

with the equivalent constant independent of h.

Indeed, passing (if needed) to equivalent function, we may consider only the functions $h(t)$ of the form

$$h(t) = \sum_{k=-\infty}^{\infty} c_k \chi_{(0,\lambda_k)}(t), \qquad \forall\, c_k \geq 0, \qquad \sum_{k=0}^{\infty} c_k \leq 1,$$

where a two-sided monotone sequence $\{\lambda_k\}$ is such that

$$\lim_{k\to-\infty} \lambda_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty.$$

Then the proclaimed $g(t) = t^{-\alpha}(f^{**} - f^*)$ will be obtained if we take

$$f(t) = \sum_{k=-\infty}^{\infty} c_k(\lambda_k^{\alpha} - t^{\alpha})\chi_{(0,\lambda_k)}(t).$$