On generalized K - Functionals and moduli of smoothness related to trigonometric approximation

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1. Introduction and basic concepts

1.1 Trigonometric approximation in L_p, $0 < p < 1$

Let $f \in L_p(\mathbb{T}^d)$ if $0 < p < \infty$ or $f \in C(\mathbb{T}^d)$ if $p = \infty$, $n \in \mathbb{N}$.

$$E_n(f)_p := \inf\{\|f - g\|_p \ : \ g \in \mathcal{T}_n\}$$

Best approximation by trigonometric polynomials of spherical order less than n

Let $k \in \mathbb{N}$ and $\delta > 0$.

$$\omega_k(f, \delta)_p := \sup_{|h| \leq \delta} \|\Delta_h^k f(\cdot)\|_p$$

Modulus of smoothness of order k

Good news: V. I. Ivanov 1975 ($d = 1$), Storozhenko, Krotov, Oswald (1975, 1977, 1978)

- Direct (Jackson) theorem: $0 < p < 1$, $f \in L_p(\mathbb{T}^d)$

$$E_n(f)_p \leq c_{p,k}\, \omega_k(f,\, n^{-1})_p$$

- Inverse (Bernstein) theorem: $0 < p < 1$, $f \in L_p(\mathbb{T}^d)$

$$\omega_k(f,\, n^{-1})_p \leq c_{p,k} n^{-k} \Big(\sum_{\nu=1}^{n+1} \nu^{kp-1} E_\nu(f)_p \Big)^{1/p}$$

- Proof of Jackson Theorem:

  Step 1: Approximation by piecewise constant (polynomial) functions

  Step 2: Approximation of piecewise constant (polynomial) functions by linear means generated by appropriate kernels (depending on p)

- Key inequalities for trigonometric polynomials (Bernstein, Nikol'skij, ...) hold true for $0 < p < 1$!

Bad news:

- In contrast to the case $1 \le p \le \infty$ - no constructive method for approximation by trigonometric polynomials
- Classical summability methods (Fourier means)

  Let $f \in L_p(\mathbb{T}^d)$ $1 \le p < \infty$, or $f \in C(\mathbb{T}^d)$ (if $p = \infty$).

  $\varphi \in \mathcal{K}$: continuous, $\operatorname{supp} \varphi \in \{\xi \, : \, |\xi| \le 1\}$, $\varphi(-\xi) = \overline{\varphi(\xi)}$ for each $\xi \in \mathbb{R}^d$ and $\varphi(0) = 1$.

$$\mathcal{M}_n^\varphi f(x) := (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(y) W_n^\varphi(x-y)\, dy \ , \quad n \in \mathbb{N}$$

  where the kernel

$$W_n^\varphi(y) := \sum_{k \in \mathbb{Z}^d} \varphi\Big(\frac{k}{n}\Big)\, e^{iky} \ , \quad n \in \mathbb{N} \ , \ y \in \mathbb{T}^d$$

  is a trigonometric polynomial of order less than n.
  Do not make sense if $f \in L_p(\mathbb{T}^d)$, $0 < p < 1$!

- Classical K - functionals

  If $1 \le p \le \infty$ then $(d = 1)$

$$\omega_k(f,\ t)_p \asymp \inf\{\|f - g\|_p + t^k \|D^k g\|_p :\ g \in W_p^k\}$$
$$=: \mathcal{K}(f, t^k; L_p, W_p^k)$$

  are trivial if $0 < p < 1$!

- Ditzian, Hristov, Ivanov 1995: If $0 < p < 1$ then

$$\inf\{\|f - g\|_p + t^k \|D^k g\|_p :\ g \in \mathcal{T}\} = 0$$

1.2 Families of linear polynomial operators

Alternative approach: K.V. Runovski 1993/94

- $g \in \mathcal{T}_n \implies$

$$\mathcal{M}_n^{\varphi} g(x) = \sum_{|k|<n} \varphi\Big(\frac{k}{n}\Big) \widehat{g}(k) e^{ikx}$$
$$= (2n+1)^{-d} \sum_{\nu_1=0}^{2n} \cdots \sum_{\nu_d=0}^{2n} g\Big(\frac{2\pi\nu}{2n+1}\Big) W_n^{\varphi}\Big(x - \frac{2\pi\nu}{2n+1}\Big)$$
$$=: \mathcal{S}_n^{\varphi} g(x)$$

- $T_\lambda g := g(\cdot + \lambda)$ translation operator $\Longrightarrow$

$$\mathcal{S}_n^{\varphi} g = \mathcal{M}_n^{\varphi} g$$
$$= [T_{-\lambda} \circ \mathcal{M}_n^{\varphi} \circ T_\lambda] g = [T_{-\lambda} \circ \mathcal{S}_n^{\varphi} \circ T_\lambda] g$$

if $g \in \mathcal{T}_n$

Definition. $f \in L_p(\mathbb{T}^d)$, $0 < p < \infty$, or $f \in C(\mathbb{T}^d)$ (if $p = \infty$)
$\varphi \in \mathcal{K}$, $\lambda \in \mathbb{T}^d$, $n \in \mathbb{N}$

$$\mathcal{L}^{\varphi}_{n;\lambda} f(x) = (2n+1)^{-d} \sum_{\nu_1=0}^{2n} \cdots \sum_{\nu_d=0}^{2n} f\left(\frac{2\pi\nu}{2n+1} + \lambda\right) W^{\varphi}_n\left(x - \lambda - \frac{2\pi\nu}{2n+1}\right)$$

- well-defined for almost all λ for all $f \in L_p(\mathbb{T}^d)$, $0 < p < \infty$
- trigonometric polynomial of order less than n for almost all λ
- used as a constructive approximation method in L_p, $0 < p < 1$
- alternative proofs of direct and inverse theorems

1.3 Problems and intentions

- Convergence - Criteria

  $\mathcal{M}_n^\varphi f \to f$ in $L_p(\mathbb{T}^d)$, $1 \le p < \infty$ and in $C(\mathbb{T}^d)$

  $\mathcal{S}_n^\varphi f \to f$ in $C(\mathbb{T}^d)$

  $\mathcal{L}_{n;\lambda}^\varphi f \to f$ in $L_p(\mathbb{T}^d \times \mathbb{T}^d)$, $0 < p < \infty$

  for all f, in dependence on the generator φ
- unified (universal) approach to trigonometric approximation in L_p for all $0 < p \le \infty$ via families of operators $(\mathcal{L}_{n;\lambda}^\varphi)$

  in particular extension to $0 < p < 1$
- approximation error and smoothness
  direct and inverse theorems (equivalence),
  appropriate K -functionals and moduli of smoothness
- Example: approximation by Bochner-Riesz families and means

2. Convergence - a unified approach

2.1 General results

$0 < p < \infty$, $g : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$ measurable, $g = g(x, \lambda)$

$$\|g\|_{\overline{p}} = \left((2\pi)^{-d} \int\limits_{\mathbb{T}^d} \int\limits_{\mathbb{T}^d} |g(x,\lambda)|^p \, dx \, d\lambda \right)^{1/p}$$

$$= \left((2\pi)^{-d} \int\limits_{\mathbb{T}^d} \|g(\cdot,\lambda)\|_p^p \, d\lambda \right)^{1/p}$$

average with respect to the parameter λ

Theorem 1. Let $\varphi \in \mathcal{K}$ and let $F\varphi \in L_1(\mathbb{R}^d)$.

(1) Let $0 < p \le \infty$. Then

$$\|\mathcal{L}_{n;\lambda}^{\varphi} f(x) - f(x)\|_{\overline{p}} \to 0 \quad \text{for all} \quad f \in L_p(\mathbb{T}^d) \quad \text{iff} \quad F\varphi \in L_p(\mathbb{R}^d)$$

(2) If $1 \le p \le \infty$ then

$$\|\mathcal{L}_{n;\lambda}^{\varphi} f(x) - f(x)\|_{\overline{p}} \asymp \|\mathcal{M}_n^{\varphi} f(x) - f(x)\|_p$$

on $L_p(\mathbb{T}^d)$ (if $p < \infty$ and $C(\mathbb{T}^d)$ (if $p = \infty$).

(3) We have

$$\|\mathcal{L}_{n;\lambda}^{\varphi} f(x) - f(x)\|_{\overline{\infty}} \asymp \|\mathcal{S}_n^{\varphi} f(x) - f(x)\|_\infty$$

on $C(\mathbb{T}^d)$.

- Runovski, S. 2004, Lasser, Runovski 2003 (Preprint), Rukasov, Runovski, S. 2009
- $1 \leq p \leq \infty$ classical approximation processes

  well-known in various (and more general) cases, many examples

  books by Zygmund, Timan, Butzer, Nessel (1971), Stein Weiss (1971), DeVore, Lorentz(1993), Trigub, Belinsky (2004)
- $1 \leq p \leq \infty$ general approach, necessary and sufficient conditions for convergence

  Feichtinger, Weisz 2006

2.2 Example

- Bochner-Riesz kernels

$$\varphi_\alpha(\xi) = \left(1 - |\,\xi\,|^2\right)_+^\alpha \Rightarrow F\varphi_\alpha(\xi) \;=\; \pi^{-\alpha}\,\Gamma(\alpha+1)\,|\,\xi\,|^{-\alpha-d/2}\,J_{\alpha+d/2}(|\,\xi\,|)$$

$$\Rightarrow \|\,F\varphi_\alpha\,\|_{L_p(\mathbb{R}^d)}^p \;\asymp\; \int_1^\infty r^{-p(\alpha+d/2+1/2)+d-1}\,dr$$

$$\Rightarrow F\varphi_\alpha \in L_p(\mathbb{R}^d) \;\text{ iff }\; p > \frac{2d}{d+2\alpha+1}$$

Hence $F\varphi_\alpha \in L_1(\mathbb{R}^d)$ iff $\alpha > \frac{d-1}{2}$ (Bochner's critical index).

If $\alpha > \dfrac{d-1}{2}$, then convergence iff $\dfrac{2d}{d+2\alpha+1} < p \le \infty$.

3. Approximation processes and smoothness

3.1 How to describe smoothness related to an approximation process

We discuss some examples and known results.

- Bochner-Riesz means: $\alpha > \frac{d-1}{2}$, $1 \le p \le \infty$

$$\mathcal{B}_n^\alpha f(x) := \sum_{|k|<n} \left(1 - \left|\frac{k}{n}\right|^2\right)^\alpha \hat{f}(k) e^{ikx} , \quad n \in \mathbb{N}$$

- Ditzian 2005, Belinsky, Trigub 2004

$$\mathcal{K}_\Delta(f,\delta)_p := \inf\{\|f-g\|_p + \delta^2 \|\Delta g\|_p \ : \ g \in W_p^2(\mathbb{T}^d)\}$$

$$\|f - \mathcal{B}_n^\alpha f\|_p \asymp \mathcal{K}_\Delta(f, 1/n)_p$$

- Ditzian 2005, Belinsky, Trigub 2004

$$\mathcal{K}_\Delta(f,\delta)_p \asymp \omega_2(f,\delta)_p \quad \text{if} \quad 1 < p < \infty$$

$$\mathcal{K}_\Delta(f,\delta)_p \asymp \sup_{|h|\le\delta} \left\| 2d\, f(x) - \sum_{j=1}^{d} [f(x+he_j) + f(x-he_j)] \right\|_p$$

if $1 \le p \le \infty$

- approximation by Bochner-Riesz is related to smoothness generated by the Laplacian if $1 \le p \le \infty$

  adapted modulus of smoothness if $p = 1$ or $p = \infty$

  Is there a counterpart if $p < 1$?

- Hristov, Ivanov 1990 (d=1), Ditzian, Chen, Hristov, Ivanov 1995 (d>1)

$$\mathcal{R}_\Delta(f,\delta)_p := \inf\{\|f-g\|_p + \delta^2\|\Delta g\|_p \ : \ g \in \mathcal{T}_{1/\delta}\}$$

$$\mathcal{R}_\Delta(f,\delta)_p \asymp \mathcal{K}_\Delta(f,\delta)_p \ \text{ if } \ 1 \le p \le \infty$$

  Realization of the $\mathcal{K}$-functional on $\mathcal{T}_{1/\delta}$

- Ditzian, Hristov, Ivanov 1995

$$\mathcal{K}_\Delta(f,\delta)_p = 0 \ \text{ if } \ 0 < p < 1\,.$$

- Ditzian, Hristov, Ivanov 1995 (1990), $d = 1$

$$\mathcal{P}_k(f,1/n)_p := \inf\{\|f-g\|_p + n^{-k}\|g^{(k)}\|_p \ : \ g \in \mathcal{T}_n\}$$

$$\mathcal{P}_k(f,1/n)_p \asymp \omega_k(f,1/n)_p \ \text{ if } \ 0 < p \le \infty$$

  "Polynomial K - functionals" are appropriate to characterize smoothness if $0 < p < 1$?

3.2 Polynomial K - functionals related to smoothness

Generators of smoothness and operators: We denote by $\mathcal{S}$ the class of complex-valued functions ψ defined on $\mathbb{R}^d$ satisfying the following conditions:

1) ψ is continuous on $\mathbb{R}^d$ and C^∞ on $\mathbb{R}^d \setminus \{0\}$;
2) $\psi(-\xi) = \overline{\psi(\xi)}$ for each $\xi \in \mathbb{R}^d$;
3) $\psi(0) = 0$, $\psi(\xi) \neq 0$ for $\xi \in \mathbb{R}^d \setminus \{0\}$.

Each function $\psi \in \mathcal{S}$ generates a family of linear operators $\mathcal{D}_\delta(\psi)$, $\delta \geq 0$, defined on the space $\mathcal{T}$ by

$$\mathcal{D}_\delta(\psi) g(x) := \sum_{k \in \mathbb{Z}^d} \psi(\delta k) \hat{g}(k) e^{ikx}$$

Important subclasses:

$$\mathcal{H}_\beta := \{\psi \in \mathcal{S} \; : \; \psi(\tau\xi) = \tau^\beta \psi(\xi) \} \, , \quad \beta > 0$$

Then $\mathcal{D}_\delta(\psi) = \delta^\beta \mathcal{D}(\psi)$.

Examples:

- $\psi(\xi) = (i\xi)^k \ , \ k \in \mathbb{N} \quad \Rightarrow \quad \mathcal{D}(\psi)g \ = \ g^{(k)} \ \ (d = 1)$
- $\psi(\xi) = |\xi| \quad \Rightarrow \quad \mathcal{D}(\psi)$ Riesz derivative $(d = 1)$
- $\psi(\xi) = -|\xi|^2 \quad \Rightarrow \quad \mathcal{D}(\psi)g \ = \ \Delta g \ , \quad (d \geq 1)$
- $\psi(\xi) = |\xi|^\alpha \quad \Rightarrow \quad \mathcal{D}(\psi)g \ = \ (-\Delta)^{\alpha/2} g \ , \quad (\alpha > 0)$

Polynomial K - functionals:

Let $\psi \in \mathcal{S}$, $\delta > 0$, $f \in L_p(\mathbb{T}^d)$, $0 < p \leq \infty$.

$$\mathcal{P}_\psi(f,\delta)_p := \inf\{\|f-g\|_p + \|\mathcal{D}_\delta(\psi)g\|_p \ : \ g \in \mathcal{T}_{1/\delta} \ \}$$

If $\psi \in \mathcal{H}_\beta$ then

$$\mathcal{P}_\psi(f,\delta)_p := \inf\{\|f-g\|_p + \delta^\beta\|\mathcal{D}(\psi)g\|_p \ : \ g \in \mathcal{T}_{1/\delta} \ \}$$

where $\mathcal{T}_{1/\delta}$ trigonometric polynomials of spherical order less than $1/\delta$

3.3 Direct and inverse theorems

Let $\widetilde{p} = \min(1, p)$ and let $\eta \in C_0^\infty(\mathbb{R}^d)$ such that

$$\eta(\xi) = 1 \quad \text{if} \quad |\xi| < \varrho < 1/2$$

$$\eta(\xi) = 0 \quad \text{if} \quad |\xi| > 2\varrho$$

Theorem 2. η as above, $0 < p \le \infty$, $\varphi \in \mathcal{K}$, $\psi \in \mathcal{S}$. If

$$F\varphi \in L_{\widetilde{p}}(\mathbb{R}^d) \quad \text{and} \quad F\left[\frac{(1-\varphi)\eta}{\psi}\right] \in L_{\widetilde{p}}(\mathbb{R}^d)$$

then

$$\|\mathcal{L}_{n;\lambda}^{\varphi} f - f\|_{\overline{p}} \le c\,\mathcal{P}_\psi(f, 1/n)_p$$

for all $n \in \mathbb{N}$ and $f \in L_p(\mathbb{T}^d)$ $(f \in C(\mathbb{T}^d)$ if $p = \infty)$.

Let η be as above and let $\theta \in C_0^\infty$ such that

$$\theta(\xi) = 1 \ \text{ if } \ 2\varrho \le |\xi| \le 1 \ ,$$

$$\eta(\xi) + \theta(\xi) = 1 \ \text{ if } \ |\xi| \le 1 \ .$$

Theorem 3. $\eta\,,\theta$ as above, $0 < p \le \infty$, $\psi \in \mathcal{S}$. Let $\varphi \in \mathcal{K}$, such that $\varphi(\xi) \ne 1$ if $\xi \ne 0$. If

$$F\left[\frac{\psi\eta}{1-\varphi}\right] \in L_{\widetilde{p}}(\mathbb{R}^d)$$

and if for some $m \in \mathbb{N}$

$$F\left[\frac{\varphi^m\theta}{1-\varphi}\right] \in L_{\widetilde{p}}(\mathbb{R}^d)$$

then

$$\mathcal{P}_\psi(f, 1/n)_p \ \le \ c\, \|\mathcal{L}^{\varphi}_{n;\lambda} f - f\|_{\overline{p}}$$

for all $n \in \mathbb{N}$ and $f \in L_p(\mathbb{T}^d)$ $(f \in C(\mathbb{T}^d)$ if $p = \infty)$.

4. Approximation by Bochner-Riesz families

4.1 Convergence

- Generator: $\varphi_\alpha(\xi) = \left(1 - |\,\xi\,|^2\right)_+^\alpha$, $\alpha > 0$.

  Kernels: $W_n^\alpha = \sum\limits_{|k|<n} (1 - |k/n|^2)^\alpha \, e^{iky}$

  Bochner-Riesz means:

$$\mathcal{B}_n^\alpha f(x) :== \sum_{|k|<n} \left(1 - \left|\frac{k}{n}\right|^2\right)^\alpha \hat{f}(k) e^{ikx} \ , \quad n \in \mathbb{N}$$

  Bochner-Riesz families:

$$\mathcal{B}_{n;\lambda}^\alpha f(x) := f(x) = (2n+1)^{-d} \sum_{\nu=0}^{2n} f\left(\frac{2\pi\nu}{2n+1} + \lambda\right) W_n^\alpha\left(x - \lambda - \frac{2\pi\nu}{2n+1}\right)$$

Theorem 4.

(1) We have

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \to 0 \ \ (n \to \infty)$$

on C, L_1 or in L_p for all $1 \leq p \leq +\infty$ if and only if $\alpha > (d-1)/2$.

(2) Let $0 < p < 1$. We have

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \to 0 \ \ (n \to \infty)$$

on L_p if and only if $\alpha > d(1/p - 1/2) - 1/2$.

In (1) we can replace $\mathcal{B}^{\alpha}_{n;\lambda} f$ and $\overline{p}$ by $\mathcal{B}^{\alpha}_{n} f$ and p, respectively.

4.2 Characterizations by (polynomial) $\mathcal{K}$-functionals

Generator of smoothness: $\psi(\xi) = -|\xi|^2$

Differential operator: Δ Laplace - operator

Corresponding $\mathcal{P}$ functional: $\delta > 0$, $0 < p \leq \infty$

$$\mathcal{P}_{\Delta}(f,\delta)_p = \inf\{\|f-g\|_p + \delta^2\|\Delta g\|_p : g \in \mathcal{T}_{1/\delta}\}$$

Corresponding $\mathcal{K}$ functional: $\delta > 0$, $1 \leq p \leq \infty$

$$\mathcal{K}_{\Delta}(f,\delta)_p = \inf\{\|f-g\|_p + \delta^2\|\Delta g\|_p : g \in W_p^2(\mathbb{T}^d)\}$$

Theorem 5.

(1) Let $\frac{2d}{d+2\alpha+1} < p < 1$. Then

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \asymp \mathcal{P}_{\Delta}(f, 1/n)_p \ , \quad n \in \mathbb{N}, , \ f \in L_p(\mathbb{T}^d)$$

(2) Let $\alpha > \frac{d-1}{2}$ and let $1 \leq p \leq \infty$. Then

$$\begin{aligned} \|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} &\asymp \|f - \mathcal{B}^{\alpha}_{n} f\|_p \\ &\asymp \mathcal{P}_{\Delta}(f, 1/n)_p \\ &\asymp \mathcal{K}_{\Delta}(f, 1/n)_p \end{aligned}$$

4.3 Characterizations by moduli of smoothness

- $0 < p \leq \infty\ ,\ \delta > 0.$

$$\omega_2(f,\delta)_p := \sup_{|h|\leq\delta} \|\Delta_h^2 f(x)\|_p$$

- $0 < p \leq \infty\ ,\ \delta > 0.$

$$\widetilde{\omega}_2(f,\delta)_p := \sup_{|h|\leq\delta} \left\| 2d\, f(x) - \sum_{j=1}^{d} (f(x+he_j) + f(x-he_j)) \right\|_p$$

Theorem 6.

(1) Let $\alpha > \frac{d-1}{2}$ and let $1 \leq p \leq \infty$. Then

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \asymp \|f - \mathcal{B}^{\alpha}_{n} f\|_{p} \asymp \widetilde{\omega}_2(f, 1/n)_p$$

(2) Let $d = 1$, $0 < p < 1$, $\alpha > 1/p - 1$. Then

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \asymp \omega_2(f, 1/n)_p$$

(3) Let $d > 1$, $0 < p < 1$, $\alpha > d(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2}$. If $\frac{d}{d+2} < p < 1$, then

$$\|f - \mathcal{B}^{\alpha}_{n;\lambda} f\|_{\overline{p}} \asymp \widetilde{\omega}_2(f, 1/n)_p$$

If $0 < p \leq \frac{d}{d+2}$ then the upper estimate holds and the estimate from below is not true.

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- Work in progress: General moduli of smoothness adapted to approximation processes and (polynomial) K-Functionals
  partial results: $d = 1$, $1 \leq p \leq \infty$, Riesz modulus related to Riesz derivative and Fejér means

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