Optimal embeddings of spaces of generalized smoothness in the critical case

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[1]joint work with Susana Moura and Júlio Neves

Outline

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Spaces of generalized smoothness

History:

- **Several authors**
  (Gol'dman, Kalyabin, Lizorkin, Merucci, Cobos, Fernandez, Leopold, Edmunds, Triebel,...)

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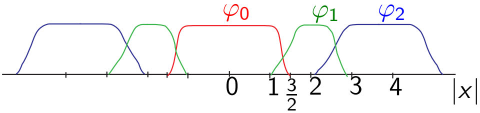
↷ Consider general Fourier analytical approach
(Edmunds & Triebel, 1998), (Moura 2001)

$$B^{(s,\Psi)}_{p,q} \quad \text{and} \quad F^{(s,\Psi)}_{p,q} \quad \rightsquigarrow \quad \Psi \ \text{ slowly varying}$$

Besov spaces

$\{\varphi_k\}_{k=0}^{\infty}$ . . . smooth dyadic resolution of unity, i.e.,

- $\sum_{k=0}^{\infty} \varphi_k = 1,$



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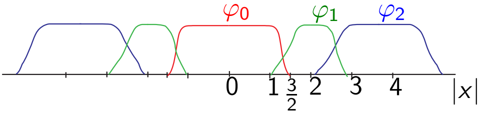

- $\operatorname{supp} \varphi_k$ compact

▶ Besov spaces: $0 < p, q \leq \infty$, $s \in \mathbb{R}$

$$\|f|B^s_{pq}\| = \left(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^{\vee} | L_p\|^q \right)^{1/q}$$

Generalized Besov spaces

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$$\Psi(x) = (1 + |\log x|)^a \left(1 + \log\left(1 + |\log x|\right)\right)^b, \quad x \in (0,1],\ a, b \in \mathbb{R},$$
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Definition

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, Ψ slowly varying function. The space $B^{(s,\Psi)}_{pq}$ consists of all $f \in S'$ such that

$$\|f|B^{(s,\Psi)}_{pq}\| = \left(\sum_{k=0}^{\infty} 2^{ksq}\Psi(2^{-k})^q \|(\varphi_k \hat{f})^\vee|L_p\|^q\right)^{1/q} < \infty.$$

Generalized Hölder spaces

$C_B \ldots$ space of bounded, continuous functions

- Modulus of continuity:

$$\omega(f,t) := \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)|, \quad t>0$$

- $\mathcal{L}_r$, $0<r\le\infty$: class of all continuous functions $\mu : (0,1] \to (0,\infty)$ with

$$\left(\int_0^1 \frac{1}{\mu(t)^r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} = \infty \qquad \text{and} \qquad \left(\int_0^1 \frac{t^r}{\mu(t)^r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty.$$

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Definition

Let $0 < r \le \infty$, $\mu \in \mathcal{L}_r$. The space $\Lambda^{\mu(\cdot)}_{\infty,r}$ consists of all $f \in C_B$ for which

$$||f|\Lambda^{\mu(\cdot)}_{\infty,r}|| := \|f|L_\infty\| + \left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty$$

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Motivation

Continuity envelopes (Triebel & Haroske, 2001)

↷ Continuity envelope of a function space X

$$\mathfrak{E}_{\mathsf{C}}(X) := \left(\mathcal{E}_{\mathsf{C}}^{X}(t), u_{X}\right)$$

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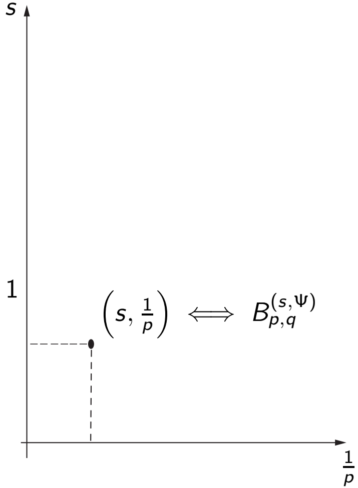
⇢ Of interest: $X \hookrightarrow C_B$ and $X \not\hookrightarrow \mathrm{Lip}^1$

The critical case



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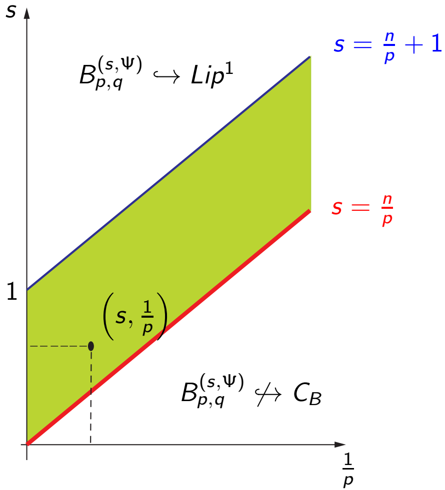
Embedding in C_B?



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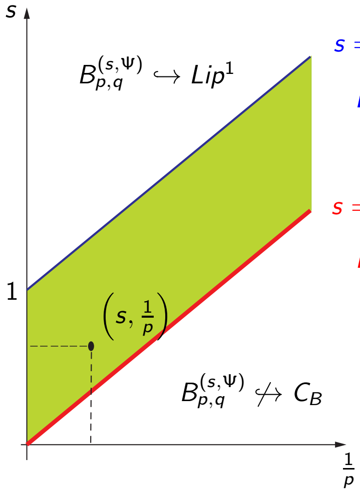

$s = \frac{n}{p} + 1$

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$$\left(\int_0^{\varepsilon} \left[\frac{\omega(f,t)}{t\mathcal{E}_{\mathsf{C}}^{X}(t)}\right]^{\nu} \mu_H(\mathrm{d}t)\right)^{\frac{1}{\nu}} \leq c\|f|X\|$$

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⇢ in B-space context yields (optimal?) embeddings

$$B_{p,q}^{(s,\Psi)} \hookrightarrow \Lambda_{\infty,\nu}^{\mu(\cdot)}$$

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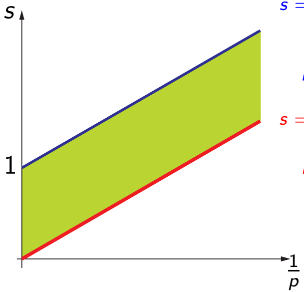

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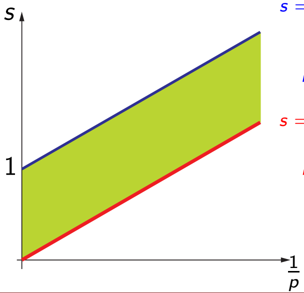

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$$\mathfrak{E}_{\mathsf{C}}\left(B^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_t^1 \left[\Psi(y)\, y^{-s}\right]^{-q'} y^{-(1+\frac{n}{p})q'} \frac{dy}{y}\right)^{1/q'}, q\right)$$

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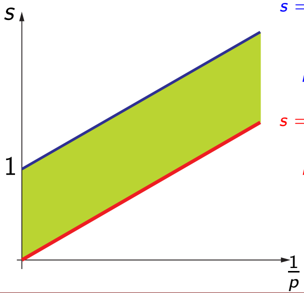

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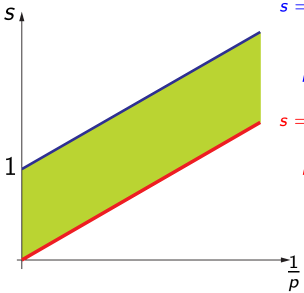

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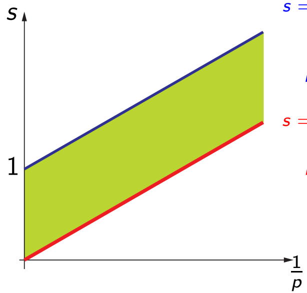

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$\rightsquigarrow$ improve (Moura, Neves & Piotrowski, 2009)

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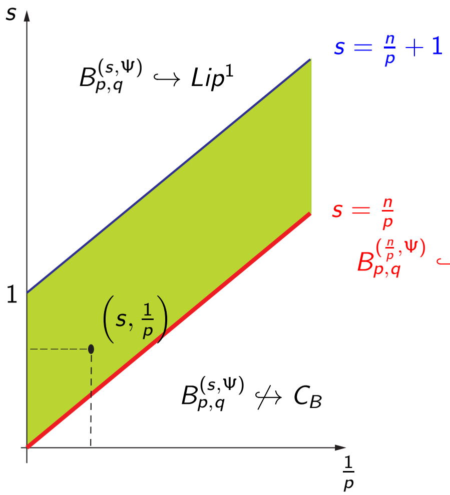

$$B^{(\frac{n}{p},\Psi)}_{p,q} \hookrightarrow C_B \quad \Longleftrightarrow \quad \left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{q'}$$

Embedding results

Theorem (Moura, Neves, and S., 2010)

Let $0 < p \leq \infty$, $0 < q, r \leq \infty$, $\mu \in \mathcal{L}_r$, Ψ *a slowly varying function with*

$$\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}_0} \in \ell_{q'}.$$

(i) *If* $0 < q \leq r \leq \infty$, *then*

$$B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n),$$

if, and only if,

$$(*) \qquad \sup_{N\geq 0}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}\left(\sum_{k=N}^{\infty}\Psi(2^{-k})^{-q'}\right)^{\frac{1}{q'}} < \infty.$$

(ii) *If* $0 < r < q \leq \infty$, *then ...*

Embedding results

Theorem (Moura, Neves, and S., 2010)

(ii) *If* $0 < r < q \leq \infty$*, then* $B^{(\frac{n}{p},\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ *if, and only if,*

$$\left\{ \sum_{N=0}^{\infty} \left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} \cdot \left(\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right) \cdot \left(\sum_{k=N}^{\infty} \Psi(2^{-k})^{-q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}} < \infty$$

and

$$\left\{ \sum_{N=0}^{\infty} \left(\sum_{j=N}^{\infty} 2^{-jr} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{u}{q}} \cdot 2^{-Nr} \left(\int_{2^{-(N+1)}}^{2^{-N}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right) \cdot \left(\sum_{k=0}^{N} 2^{kq'} \Psi(2^{-k})^{-q'} \right)^{\frac{u}{q'}} \right\}^{\frac{1}{u}} < \infty,$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

Ideas of Proof:

- use equivalent characterization of Besov spaces via
  $\curvearrowright$ Peetre's maximal function
  $\curvearrowright$ atomic decompositions
- construct extremal functions $f^{\mathbf{a}} \longrightarrow$ estimates for $\omega(f^{\mathbf{a}}, 2^{-k})$
- Hardy inequalities for non-negative sequences

... long & technical calculations unavoidable in limiting case

Hardy inequalities

Theorem (Gol'dman, 1998)

Let $(b_m)_{m\in\mathbb{N}_0}$, $(d_m)_{m\in\mathbb{N}_0}$ be non-negative sequences.

(i) Let $0 < q \leq r \leq \infty$. Then

$$\left(\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} a_k d_k\right)^r {b_j}^r\right)^{\frac{1}{r}} \lesssim \left(\sum_{m=0}^{\infty} a_m^q\right)^{\frac{1}{q}}$$

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Sketch of the proof of Main Theorem (i)

- "if part" Assume $0 < q \leq r \leq \infty$ and that $(*)$ holds.

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$$\boxed{\omega(f,2^{-j}) \leq c\sum_{k=0}^{j} 2^k 2^{-j}\|(\varphi_k^* f)_a|L_\infty\| + \sum_{k=j+1}^{\infty}\|(\varphi_k^* f)_a|L_\infty\|}$$

$\rightsquigarrow$ (Triebel, FS I)

(cont.)

$$\left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

(cont.)

$$\left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

$$\sim \left(\sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \omega(f,t)^r \mu(t)^{-r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

(cont.)

$$\left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}}$$

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(cont.)

$$
\begin{aligned}
&\left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \\
&\sim \left(\sum_{j=0}^{\infty} \int_{2^{-(j+1)}}^{2^{-j}} \omega(f,t)^r \mu(t)^{-r} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \\
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&\lesssim \underbrace{\left(\sum_{j=0}^{\infty} 2^{-jr} m_j \left[\sum_{k=0}^{j} 2^k \|(\varphi_k^* f)_a | L_\infty\|\right]^r\right)^{\frac{1}{r}}}_{=(I)} + \underbrace{\left(\sum_{j=0}^{\infty} m_j \left[\sum_{k=j}^{\infty} \|(\varphi_k^* f)_a | L_\infty\|\right]^r\right)^{\frac{1}{r}}}_{=(II)}.
\end{aligned}
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$(*)$ & Hardy inequalities $\curvearrowright$

(cont.)

$$\left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} \lesssim \left(\sum_{l=0}^{\infty} \Psi(2^{-l})^q \, \|(\varphi_l^* f)_a | L_\infty\|^q\right)^{\frac{1}{q}}$$

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characterization by Peetre maximal functions

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$$\therefore \quad B_{p,q}^{(n/p,\Psi)} \hookrightarrow B_{\infty,q}^{(0,\Psi)} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}.$$

Sketch of the proof of Main Theorem (i)

- "only if part" Contruct extremal functions $f^{\mathbf{a}}$,

$$f^{\mathbf{a}}(x) := \sum_{j=1}^{\infty} a_j \, \Psi(2^{-j})^{-1} \, \phi(2^j x), \qquad x \in \mathbb{R}^n, \quad \phi \in C_0^{\infty}(\mathbb{R}^n),$$

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Outline

Optimal weights

Corollary (Moura, Neves, and S., 2010)

Let $\boxed{1 < q \le \infty}$ *and define weights*

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}} \left(\int_0^t \Psi(s)^{-q'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q'}+\frac{1}{r}}, \qquad t \in (0,1].$$

Among the embeddings

$$B_{p,q}^{(\frac{n}{p},\Psi)} \hookrightarrow \Lambda_{\infty,r}^{\mu(\cdot)}, \qquad 1 < q \le r \le \infty,$$

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Furthermore, the embedding with $\mu = \lambda_{qq}$ *and* $r = q$*, i.e.,*

$$B_{p,q}^{(\frac{n}{p},\Psi)} \hookrightarrow \Lambda_{\infty,q}^{\lambda_{qq}(\cdot)}$$

is optimal.

Outline

Spaces of generalized smoothness

↷ Consider (more) general Fourier analytical approach (Farkas & Leopold, 2006)

$$B^{\boldsymbol{\sigma},\mathbf{N}}_{p,q} \quad \text{and} \quad F^{\boldsymbol{\sigma},\mathbf{N}}_{p,q} \quad \rightsquigarrow \quad \boldsymbol{\sigma},\mathbf{N} \ \text{admissible sequences}$$

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Definition (Admissible sequence)

A sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is an **admissible sequence** if

$$c_0\,\gamma_j \leq \gamma_{j+1} \leq c_1\,\gamma_j, \qquad j \in \mathbb{N}_0.$$

Spaces of generalized smoothness

Let $\mathbf{N} = (N_j)_{j \in \mathbb{N}_0}$ be an admissible, strongly increasing sequence

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Definition (Farkas & Leopold, 2006)

Let $0 < p, q \leq \infty$ and $\boldsymbol{\sigma} = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence. The space $B^{\boldsymbol{\sigma},\mathbf{N}}_{p,q}$ consists of all $f \in S'$ such that

$$\|f \mid B^{\boldsymbol{\sigma},\mathbf{N}}_{p,q}\| := \Big(\sum_{j=0}^{\infty} \sigma_j^q \, \|(\varphi_j^{\mathbf{N}}\hat{f})^\vee \big| L_p\|^q\Big)^{\frac{1}{q}} < \infty.$$

Spaces of generalized smoothness

Approach recovers well-known function spaces:

- $\boldsymbol{\sigma} = (2^{sj})_{j\in\mathbb{N}_0}$ $\mathbf{N} = (2^{j})_{j\in\mathbb{N}_0}$

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- also possible: $\quad \underline{s}(\boldsymbol{\sigma}) < \overline{s}(\boldsymbol{\sigma})$

Main Theorem (Moura, Neves & S., 2010)

Let $0 < p \leq \infty$, $0 < q, r \leq \infty$, $\mu \in \mathcal{L}_r$. Let $\boldsymbol{\sigma}$ and $\mathbf{N}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$. Put

$$\boldsymbol{\tau} = \boldsymbol{\sigma}\mathbf{N}^{-n/p}$$

and assume that

$$\overline{s}(\boldsymbol{\tau}) = 0 \quad \text{and} \quad \boldsymbol{\tau}^{-1} \in \ell_{q'}.$$

(i) If $0 < q \leq r \leq \infty$, then

$$B^{\boldsymbol{\sigma},\mathbf{N}}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}$$

if, and only if,

$$(*) \quad \sup_{M \geq 0} \left(\sum_{j=0}^{M} \int_{N_{j+1}^{-1}}^{N_j^{-1}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \left(\sum_{k=M}^{\infty} \tau_k{}^{-q'} \right)^{\frac{1}{q'}} < \infty.$$

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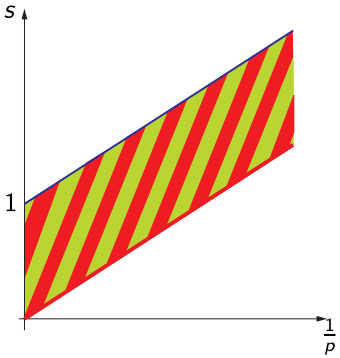

Generalization of previous results

⇢ cover new spaces in limiting case

$$\underline{s}(\boldsymbol{\tau}) = 0$$

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Example: (Kühn, Leopold, Sickel, Skrzypczak, 2006)

↷ construct new sequence $\boldsymbol{\tau}$ with

$$0 = \underline{s}(\boldsymbol{\tau}) < \overline{s}(\boldsymbol{\tau}) < 1 = \underline{s}(\mathbf{N}) \qquad \text{and} \qquad \boldsymbol{\tau}^{-1} \in \ell_{q'}$$

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Thank you!