Complex Interpolation and Entropy Explosion

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Thanks to the organizers!

Outline

Work in Progress

- This is work in progress.
- "Work in progress" is a euphemism for "things did not work out as I hoped they would, but I am still not giving up hope."
- The first part is a slight refinement of the talk I gave in Madrid in 2010, based on ideas inspired by the work of Cwikel and Janson.
- "Inspired by" could be a euphemism for "stolen from."
- In the second part, I want to mention some very old work with Pedro Fernndez, that could have some relation to the phenomenon of the misbehavior of entropy under interpolation.

The Perennial Question

- If $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ are compatible couples of Banach spaces, if $T : \bar{A} \to \bar{B}$ is bounded, if $T : A_0 \to B_0$ is compact, is $T : \bar{A}_\theta \to \bar{B}_\theta$ compact?, where $\theta \in (0, 1)$ and $\bar{A}_\theta$ is the complex interpolate of $\bar{A}$.

A reduction

- Work by Cobos, Peetre, Cwikel shows that if the result is true for $\bar{A} = \ell^1(FL^1, FL^1_1)$, $\bar{B} = \ell^\infty(FL^\infty, FL^\infty_1)$, then it is true.
- Definitions of these spaces will be given, even though everybody knows what they are.

The Obligatory Definitions/Notation Section

- $S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$.
- If $\bar{A} = (A_0, A_1)$ be a Banach couple, then $\mathcal{F}(\bar{A}) = \mathcal{F}(A_0, A_1)$ is the family of all continuous $f : S \to A_0 + A_1$ such that:
  - $f|_{S^0}$ is analytic (as a $A_0 + A_1$-valued function).
  - $f(j + i\tau) \in A_j$ for all $\tau \in \mathbb{R}$ and $\tau \mapsto f(j + i\tau) : \mathbb{R} \to A_j$ is continuous; $j = 0, 1$.
  - f is periodic of period $2\pi i$; $f(z + 2\pi i) = f(z)$ for all $z \in S$.

  $\mathcal{F}(\bar{A})$ becomes a Banach space with $\|f\|_{\mathcal{F}(\bar{A})} = \max_{j=0,1} \sup_{\tau\in[-\pi,\pi]} \|f(j + i\tau)\|_{A_j}$.
- If $0 \leq \theta \leq 1$, the interpolation space $\bar{A}_\theta$ is defined by

$$\begin{aligned} \bar{A}_\theta &= \{f(\theta) : f \in \mathcal{F}(\bar{A})\}; \\ \|a\|_{\bar{A}_\theta} &= \inf\{\|f\|_{\mathcal{F}(\bar{A})} : f \in \mathcal{F}(\bar{A}), f(\theta) = a\}. \end{aligned}$$

Definitions Continued

- If (A_0, A_1) is a Banach couple, if $K_0 \subset A_0$, if $K_1 \subset A_1$, then

  $$\mathcal{F}(K_0, K_1) = \{f \in \mathcal{F}(A_0, A_1) \,:\, f(j+it) \in K_j, j = 0, 1, t \in \mathbb{R}\}.$$

- $L^\infty([-\pi, \pi])$ consists of all measurable, bounded, 2π-periodic functions from $\mathbb{R} \to \mathbb{C}$; normed as usual.
- $C_p(\mathbb{R}, L^\infty([-\pi, \pi]))$ is the space of all continuous 2π-periodic functions from $\mathbb{R}$ to $L^\infty([-\pi, \pi])$.

And Continued

- If $1 \leq p \leq \infty$, $0 \leq \theta \leq 1$, we say a sequence $a = \{a_n\} \in FL^p_\theta$ iff there exists a function $f \in L^p([-\pi, \pi])$ of which the sequence $\{e^{n\theta} a_n\}$ is the sequence of Fourier coefficients; i.e.,

$$a_n = \frac{e^{-n\theta}}{2\pi} \int_{-\pi}^{\pi} e^{-int} f(t)\, dt, \quad n = 0, \pm 1, \pm 2, \ldots . \qquad (1)$$

- FL^p_θ becomes a Banach space defining $\|a\|_{FL^p_\theta} = \|f\|_{L^p([-\pi,\pi])}$, if $a = \{a_n\}$ and f are related as in (1).
- CF_θ is the set of all sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that there is $f \in C_p(\mathbb{R})$ and (1) holds. It is a closed subspace of FL^∞_θ.

Returning to the Perennial Question

A stronger result that,if true, would imply a positive answer is:

- Let $\bar{B} = (B_0, B_1)$ be a Banach couple, let K_0 be a compact subset of B_0 and K_1 a bounded subset of B_1. Then

$$K_\theta = \{f(\theta) \,:\, f \in \mathcal{F}(K_0, K_1)\}$$

is a compact subset of $\bar{B}_\theta$ for $\theta \in (0,1)$. I call a couple for which this result is true a Calderón-Cwikel couple, CC-couple for short. The work of Cobos, Peetre, Cwikel (and possibly et al.) now proves:

Theorem

If $\ell^\infty(FL^\infty, FL_1^\infty)$ is a CC-couple, then all Banach couples are CC-couples, and the perennial question gets a positive answer and ceases to be perennial.

A result of Cwikel and Janson

- About two years ago or so, Michael Cwikel and Svante Janson proved that $(FL^{\infty}, FL^{\infty}_1)$ is a CC-couple.
- This is very close to what one needs, or perhaps still very far.
- It is very easy to prove: If (B_0, B_1) is a CC-couple, then so is $\ell^p(B_0, B_1)$ for all p, $1 \leq p < \infty$.
- But ℓ^{∞} is a very different animal.

More Definitions

- If K is a bounded subset of a Banach space B, the k-th entropy number of K will be defined as the infimum of all ρ such that K can be covered by no more than 2^{k-1} balls of radius ρ. In symbols,

$$e_k(K) = inf\{\rho : \exists b_1, \dots, b_{2^{k-1}} \in B \text{ such that } K \subset \bigcup_{i=1}^{2^{k-1}} B_B(b_i, \rho)\}.$$

- K is relatively compact if and only if $\beta(K) = \lim_{k\to\infty} e_k(K) = 0$. The number $\beta(K)$ is the measure of non-compactness of K.
- If $T : A \to B$ is a linear operator from the Banach space A to the Banach space B, the k-th entropy number of T is defined by $e_k(T) = e_k(T(B_A))$.
- If X is a Banach space, $x \in X$, $r > 0$, $B_X(x, r)$ is the open ball of center x, radius r; $B_X = B_X(0, 1)$.

The Main Result of This Part

Theorem

Let K_0, K_1 be bounded subsets of FL^∞, FL_1^∞, respectively. Let $0 < \theta < 1$. There exists a constant C such that if $k_0, k_1 \in \mathbb{N}$, then there is $\tilde{k} \in \mathbb{N}$ satisfying:

$$e_k(K_\theta) \leq C e_{k_0}^{1-\theta} e_{k_1}^{\theta}$$

for $k \geq \tilde{k}$. In particular,

$$\beta(K_\theta) \leq C \beta(K_0)^{1-\theta} \beta(K_1)^{\theta}$$

so that if K_0 or K_1 is relatively compact, so is K_θ in $(FL^\infty, FL_1^\infty)_\theta$.

This theorem is a mild extension of the Cwikel-Janson result mentioned before. In the next frames I show how their ideas can be used to prove it.

An Observation

- It is not hard to prove that if $f \in C(\mathbb{R}, L^\infty([-\pi, \pi]))$, in particular if $f \in C_p(\mathbb{R}, L^\infty([-\pi, \pi]))$, then there exists a measurable function $\mathbb{R} \times \mathbb{R} \to \mathbb{C}$, also denoted by f such that $\{s \in \mathbb{R} : f(t)(s) \neq f(t, s)\}$ is a null set for all $t \in \mathbb{R}$. We identify the two f's.
- This observation plays a role in seeing that some of the objects dealt with here are well defined.

A corollary to Lusin's Theorem

- The following consequence of Lusin's Theorem is more or less at the beginning of the work of Cwikel and Janson on (FL^∞, FL_1^∞) and the complex method.

Lemma

Let X be a compact metric space and let μ be a regular Borel measure in X. If K is a countable subset of $L^\infty(\mu)$, then for every $\epsilon > 0$ there exists a compact subset S of X such that $\mu(X \backslash S) < \epsilon$ and $f|_S \in C(S)$ for all $f \in K$. If K is relatively compact in $L^\infty(\mu)$, then $\{f|_S : f \in K\}$ is relatively compact in $C(S)$.

A Corollary to the Corollary

Theorem

(Cwikel-Janson) Let $\mu \in L^1(\mathbb{R})$, $\mu \geq 0$. *If* $h \in C_p(\mathbb{R}, L^\infty([-\pi, \pi]))$, *we define*

$$L_\mu h(s) = \int_{-\infty}^{\infty} h(t, t+s)\mu(t)\, dt.$$

Then $L_\mu h$ *(defined a.e. by Fubini) is (is equal a.e. to) a continuous* 2π*-periodic function on* $\mathbb{R}$.

Entropy and Equicontinuity; a Simple Lemma

Lemma

Let $K \subset C([-\pi, \pi])$, assume K is bounded, and assume there exist $\rho > 0$, $\delta > 0$ such that $|f(t) - f(s)| < \rho$ whenever $s, t \in [-\pi, \pi]$ and $|s - t| < \delta$. There exists then $k_0 \in \mathbb{N}$ such that $e_k(K) \leq 4\rho$ if $k \geq k_0$. Moreover one can take $k_0 \sim |\log \rho|/\delta$.

The Proof

Let $M = \sup\{\|f\|_{L^\infty([-\pi,\pi])} : f \in K\}$. We partition the interval $[-\pi, \pi]$ into subintervals $t_0 = -\pi < t_0 < \cdots < t_n = \pi$ with $t_i - t_{i-1} < \delta$; $n \sim 2\pi/d$. We cover $\{z \in \mathbb{C} : |z| \leq M\}$ with N discs $D_1, \ldots, D_N$ of radius $< \rho$, thus $N \sim M^2/\rho^2$. Let τ_k be the center of D_k for $k = 1, \ldots, N$. Let A be the set of all $(n+1)$-tuples $(k_0, \ldots, k_n)$ such that $k_i \in \{1, \ldots, N\}$ for $i = 0, 1, \ldots, n$ and such that there exists $f \in K$ with $f(t_i) \in D_{k_i}$ for $i = 0, \ldots, n$. For each $(n+1)$-tuple in A select a function $f \in K$ such that $f(t_i) \in D_{k_i}$ for $i = 0, \ldots, n$, and let $\mathcal{T}$ be the set of all such $f \in K$. The set $\mathcal{T}$ has at most N^n elements.

The Proof Continued

Let $g \in K$. For each i, $0 \leq i \leq n$, there is k_i such that $g(t_i) \in D_{k_i}$; by definition of A, $(k_0, \ldots, k_n) \in A$ and if $f \in \mathcal{T}$ corresponds to this $(n+1)$-tuple, then $|g(t_i) - f(t_i)| < 2\rho$ for $i = 0, \ldots, n$. Thus, for $t \in [-\pi, \pi]$, letting i be such that $t_{i-1} \leq t \leq t_i$,

$$|g(t) - f(t)| \leq |g(t) - g(t_i) + |g(t_i) - f(t_i)| + |f(t_i) - f(t)| < 4\rho.$$

This implies that if k is such that $2^{k-1} \sim N^n \sim (M^2/\rho^2)^{2\pi/\delta}$, then $e_k < 4\rho$. The result follows. $\square$

What you Know Matters

- One of the difficulties in dealing with the complex interpolation spaces at a general level is, in my opinion, that it is hard to know exactly what is in, and what is not in, the space $\bar{A}_\theta$, where $\bar{A} = (A_0, A_1)$ is a Banach couple. In the case at hand it is useful, essential, that one knows:

Theorem

$(FL^\infty, FL_1^\infty)_\theta = CF_\theta$ *with equality of norms.*

Some Comments on the Proof of this Result, and Related Matters

- It is fairly easy to see that $(FL^\infty, FL^\infty_1)_\theta \subset FL^\infty_\theta$.
- The details get a bit messy, perhaps, but if $f \in \mathcal{F}(FL^\infty, FL^\infty_1)$, then $f(z) = \{f_n(z)\}$ where each $f_n : S \to \mathbb{C}$ is continuous, and analytic on S^0.
- FORMALLY, one can define $F(z,s) = \sum_{n=-\infty}^{\infty} f_n(z) e^{n(z+is)}$; then $F(z)$ defined by $F(z)(s) = F(z,s)$ satisfies $F(\theta + it) \in L^\infty([-\pi,\pi])$ for $\theta = 0, 1$, hence also for $\theta \in (0,1)$.
- To se that $(FL^\infty, FL^\infty_1)_\theta \subset FC_\theta$ requires a bit more work, but one should keep in mind that it is obvious that functions in $FL^\infty \cap FL^\infty_1$ are continuous, and that $FL^\infty \cap FL^\infty_1$ is dense in $(FL^\infty, FL^\infty_1)_\theta$ for all $\theta \in (0,1)$.

Comments Continued

- One might conjecture that $(\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))_\theta = \ell^\infty(FC_\theta)$.
- This conjecture is almost certainly false, and probably easy to prove false. It would be true if ∞ gets replaced by $p \in [1, \infty)$. In fact, in general, if $\bar{B} = (B_0, B_1)$ is Banach couple, then $(\ell^p(B_0), \ell^p(B_1))_\theta = \ell^p(\bar{B}_\theta)$, if $1 \leq p < \infty$. A proof can be found, for example, in Professor Triebel's *Interpolation Theory, Function Spaces, Differential Operators*.
- If the conjecture were true, it would be easy to put an end to the perennial question.
- As my friend Julio Bastida used to say, if my grandmother had wheels, she would be a bicycle.

Proving the main Theorem

- The Theorem repeated:

Theorem

Let K_0, K_1 be bounded subsets of FL^∞, FL_1^∞, respectively. Let $0 < \theta < 1$. There exists a constant C such that if $k_0, k_1 \in \mathbb{N}$, then there is $\tilde{k} \in \mathbb{N}$ satisfying:

$$e_k(K_\theta) \le C e_{k_0}^{1-\theta} e_{k_1}^{\theta}$$

for $k \ge \tilde{k}$. In particular,

$$\beta(K_\theta) \le C \beta(K_0)^{1-\theta} \beta(K_1)^{\theta}$$

so that if K_0 or K_1 is relatively compact, so is K_θ in $(FL^\infty, FL_1^\infty)_\theta$.

Proving ...

- Let $k_0, k_1 \in \mathbb{N}$, $\theta \in (0,1)$ and let $\rho_j > e_k(K_j)$ for $j = 0, 1$. If $a = \{a_n\}_{n\in\mathbb{Z}}$ is a sequence in K_j, let $h_a^{(j)} \in L^\infty([-\pi,\pi])$ be the function whose Fourier coefficients are given by $\{e^{jn} a_n\}$, $j = 0, 1$.
- There exist $\{\phi_i^j\}_{i=1}^{2^{k_j-1}} \subset L^\infty([-\pi,\pi])$ for $j = 0, 1$, such that if $a = \{a_n\}_{n\in\mathbb{Z}} \in K_j$, then there is i such that

$$\|h_a^{(j)} - \phi_i^j\|_{L^\infty([-\pi,\pi])} \leq \rho_j,$$

  $j = 0, 1$.
- μ_0, μ_1 are the Poisson kernels for the strip, so that if $f : S \to \mathbb{C}$ is continuous with analytic restriction to S°, then

$$f(z) = \sum_{j=0}^{1} \int_{-\infty}^{\infty} \mu_j(z,t) f(j+it)\, dt.$$

Proving . . .

- To simplify, we drop the subscript or superscript j for a while, so that k refers to either k_0 or k_1, etc.
- Let $\eta > 0$. By Lusin's Theorem there is a compact subset A of $[-\pi, \pi]$ such that if $\tilde{A} = \bigcup_{j \in \mathbb{Z}} (2j\pi + A)$ is the periodic extension of A to all of $\mathbb{R}$, then

$$\int_{(\mathbb{R} \setminus \tilde{A}) + s} \mu(\theta, t)\, dt < \eta \tag{2}$$

  for all $s \in \mathbb{R}$ and $\phi_i \in C(A)$ for $i = 1, \ldots, 2^{k-1}$.
- There is then $\delta > 0$ such that if $s_1, s_2 \in A$ and $|s_1 - s_2| < \delta$, then $|\phi_i(s_1) - \phi_i(s_2)| < \eta$ for $i = 1, \ldots, 2^{k-1}$.

And We Keep on Proving

- Let $f = \{f_n\} \in \mathcal{F}(K_0, K_1)$, let $h(t)(s) = h^{(j)}_{f(j+it)}(s)$; $j = 0$ or 1 (which we don't mention).
- Let $s_1, s_2 \in \mathbb{R}$, $|s_1 - s_2| < \delta$. Let $t \in \mathbb{R}$. Assume first $t \in (\tilde{A} - s_1) \cap (\tilde{A} - s_2)$. There is i_t such that $\|h(t) - \phi_{i_t}\|_{L^\infty([-\pi,\pi])} < \beta$.
- Since $t + s_1, t + s_2 \in \tilde{A}$,

$$\begin{aligned} |h(t, t+s_1) - h(t, t+s_2)| \leq \\ |h(t, t+s_1) - \phi_{i_t}(t, t+s_1)| + |\phi_{i_t}(t, t+s_1) - \phi_{i_t}(t, t+s_2)| \\ + |\phi_{i_t}(t, t+s_2) - h(t, t+s_2)| < 2\rho + \eta. \end{aligned}$$

Proof Goes On

- Because of (2), this implies that if $s_1, s_2 \in [-\pi, \pi]$ and $|s_1 - s_2| < \delta$, then

$$\int_{-\infty}^{\infty} |h(t, t+s_1) - h(t, t+s_2)| \mu(\theta, t)\, dt < 3\rho \|\mu(\theta)\|_{L^1([-\pi,\pi])}. \tag{3}$$

More of the proof, will there ever be an end to it?

- Putting back the subscript and superscript j we proved
There is $\delta > 0$ such that if $s_1, s_2 \in [-\pi, \pi]$ and $|s_1 - s_2| < \delta$, then

$$\int_{-\infty}^{\infty} |h_j(t, t+s_1) - h_j(t, t+s_2)| \mu_j(\theta, t)\, dt < \begin{cases} 3(1-\theta)\rho_0 & \text{if } j = 0, \\ 3\theta\rho_1 & \text{if } j = 1, \end{cases}$$

where $h_j(t, t+s) = h^{(j)}_{f(j+it)}(t+s)$, $j = 0, 1$.

Almost the End

- Let $g_f(\theta)$ be the function with Fourier coefficients $\{e^{n\theta} f_n(\theta)\}$. Then $g_f \in C([-\pi,\pi])$ and $\|g_f\|_{L^\infty([-\pi,\pi])} = \|f(\theta)\|_\theta$.
- The inequality of the previous frame implies: If $|s_1 - s_2| < \delta$, then

$$
\begin{aligned}
&|g_f(\theta)(s_1) - g_f(s_2)| \\
&\leq \left(\frac{1}{1-\theta} \int_{-\infty}^{\infty} |h_0(t, t+s_1) - h_0(t, t+s_2)| \mu_0(\theta, t)\, dt \right)^{1-\theta} \\
&\quad \times \left(\frac{1}{\theta} \int_{-\infty}^{\infty} |h_1(t, t+s_1) - h_1(t, t+s_2)| \mu_1(\theta, t)\, dt \right)^{\theta} \\
&\leq 3\rho_0^{1-\theta} \rho_1^{\theta}.
\end{aligned}
$$

The End of the Proof

- By the "entropy-equicontinuity lemma," we are done: There exists $\tilde{k}$ such that if $k \geq \tilde{k}$ then

$$e_k(K_\theta) \leq 12\rho_0^{1-\theta}\rho_1^{\theta}.$$

The entropy explosion?

- I expected to have something more positive in this direction, but time ran out. Or my brain ran out.
- The k such that
$$e_k(K_\theta) \leq C e_{k_0}^{1-\theta} e_{k_1}^{\theta}$$
seems to have no relation whatsoever with k_0, k_1. It is of order $1/\delta$, and δ could be arbitrarily small.
- What I mean by "entropy explosion" is that during interpolation, real or complex, the entropy numbers seem to multiply without control. The "interpolated compactness," if it exists, is very weak. (Although technically strong).

Introduction

- Eleven years ago, Pedro and I set out to prove a formula for the entropy numbers of an operator interpolated by the real method.
- We failed.
- But some interesting observations, never published (perhaps for a good reason), were reached.

Very Sketchy Outline

- Given Banach spaces X, Y a bounded linear operator $T : X \to Y$ we wanted to get a formula for the entropy numbers of the operator $\tilde{T} : \ell_N^p(w_n X) \to \ell_N^\infty(Y)$, in terms of the entropy numbers of T; where $N = 1, 2, \ldots$, $0 < p < \infty$ and $w_n > 0$ for $n \in \mathbb{N}$, and $\tilde{T}(a_n) = (Ta_n)$.
- It seemed at the time that if one could control the entropy numbers of $\tilde{T}$, independently of N, one could control what seemed to be an explosion of entropy numbers when applying the real interpolation method.
- In view of the nice paper by Professors Edmunds and Netrusov, I began thinking of reviving our work.

A Result

- We had the following result, which may or may not be useful. I won't bore you with interpreting the terms. Roughly,

$$e_{[\log \lambda(\rho)]}(T) \sim \frac{1}{\rho}, \qquad e_{[\log K_N(\rho)]}(\tilde{T}) \sim \frac{1}{\rho}.$$

Lemma

Let $\rho > 0$ and let $0 = \rho_0 < \rho_1 < \cdots < \rho_n = \rho$ be a partition of the interval $[0, \rho]$. Then

$$\begin{aligned} K_{N+1}(\rho) \quad &\leq \quad \lambda(\frac{\rho_1}{w_{N+1}})K_N(\rho) \\ &+ \quad \sum_{k=2}^{n} \left(\lambda(\frac{\rho_k}{w_{N+1}}) - \lambda(\frac{\rho_{k-1}}{w_{N+1}}) \right) K_N\left((\rho^p - \rho_{k-1}^p)^{1/p} \right). \end{aligned}$$

- **Thanks for listening!**