Structure of the set of hypercyclic functions for some classical hypercyclic operators

Juan Benigno Seoane Sepúlveda

Departamento de Análisis Matemático
Universidad Complutense de Madrid (Spain)

July 2011

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\text{Orb}(T, x) := \{T^n x \, : \, n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Hypercyclic Operators

Let X be a separable infinite-dimensional Fréchet space.

Definition

A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists $x \in X$ such that its orbit under T, $\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X.

Examples

- Birkhoff (1929): The translation operator is hypercyclic on $\mathcal{H}(\mathbb{C})$.
- MacLane (1952): The derivative operator is also hypercyclic on $\mathcal{H}(\mathbb{C})$.

The construction and properties of these hypercyclic entire functions has been studied by several authors:

Seidel & Walsh'41, Blair & Rubel'83, Duyos Ruiz'84, Grosse-Erdmann'90, Chan & Shapiro'91, Luh, Martirosian & Muller'98, Bernal & Bonilla'02.

Transitive Operators

Definition

A bounded linear operator $T : X \to X$ is said to be *transitive* if for every pair of non-void open sets U, V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

- Hypercyclicity $\Leftrightarrow$ Transitivity (Baire Category Theorem).
- In fact, every hypercyclic operator has a G_δ dense set of hypercyclic vectors.
- Besides, the translation and the derivative operator share a G_δ dense set of such hypercyclic vectors.
- Godefroy & Shapiro'91: The *translation* and the *derivative* operator share a dense hypercyclic manifold.

Transitive Operators

Definition

A bounded linear operator $T : X \to X$ is said to be *transitive* if for every pair of non-void open sets U, V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

- Hypercyclicity $\Leftrightarrow$ Transitivity (Baire Category Theorem).
- In fact, every hypercyclic operator has a G_δ dense set of hypercyclic vectors.
- Besides, the translation and the derivative operator share a G_δ dense set of such hypercyclic vectors.
- Godefroy & Shapiro'91: The *translation* and the *derivative* operator share a dense hypercyclic manifold.

Transitive Operators

Definition

A bounded linear operator $T : X \to X$ is said to be *transitive* if for every pair of non-void open sets U, V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

- Hypercyclicity $\Leftrightarrow$ Transitivity (Baire Category Theorem).
- In fact, every hypercyclic operator has a G_δ dense set of hypercyclic vectors.
- Besides, the translation and the derivative operator share a G_δ dense set of such hypercyclic vectors.
- Godefroy & Shapiro'91: The *translation* and the *derivative* operator share a dense hypercyclic manifold.

Transitive Operators

Definition

A bounded linear operator $T : X \to X$ is said to be *transitive* if for every pair of non-void open sets U, V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

- Hypercyclicity $\Leftrightarrow$ Transitivity (Baire Category Theorem).
- In fact, every hypercyclic operator has a G_δ dense set of hypercyclic vectors.
- Besides, the translation and the derivative operator share a G_δ dense set of such hypercyclic vectors.
- Godefroy & Shapiro'91: The *translation* and the *derivative* operator share a dense hypercyclic manifold.

Transitive Operators

Definition

A bounded linear operator $T : X \to X$ is said to be *transitive* if for every pair of non-void open sets U, V there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$.

- Hypercyclicity $\Leftrightarrow$ Transitivity (Baire Category Theorem).
- In fact, every hypercyclic operator has a G_δ dense set of hypercyclic vectors.
- Besides, the translation and the derivative operator share a G_δ dense set of such hypercyclic vectors.
- Godefroy & Shapiro'91: The *translation* and the *derivative* operator share a dense hypercyclic manifold.

Algebrability

Definition (Aron, Gurariy, S.)

A set A is *algebrable* if there is an algebra $\mathcal{B} \subset A \cup \{0\}$ so that $\mathcal{B}$ has an infinite minimal system of generators.
We say that $S = \{z_\alpha\}_\alpha$ *is a minimal system of generators of an algebra* $\mathcal{A}(S)$ *if for every* α_0, $z_{\alpha_0} \notin \mathcal{A}(S \setminus \{z_{\alpha_0}\})$.

Aim

Study the behaviour of the powers of the hypercyclic functions of each one of these classical operators: Birkhoff's and MacLane's.

Algebrability

Definition (Aron, Gurariy, S.)

A set A is *algebrable* if there is an algebra $\mathcal{B} \subset A \cup \{0\}$ so that $\mathcal{B}$ has an infinite minimal system of generators.

We say that $S = \{z_\alpha\}_\alpha$ is a minimal system of generators of an algebra $\mathcal{A}(S)$ if for every α_0, $z_{\alpha_0} \notin \mathcal{A}(S \setminus \{z_{\alpha_0}\})$.

Aim

Study the behaviour of the powers of the hypercyclic functions of each one of these classical operators: Birkhoff's and MacLane's.

Algebrability

Definition (Aron, Gurariy, S.)

A set A is *algebrable* if there is an algebra $\mathcal{B} \subset A \cup \{0\}$ so that $\mathcal{B}$ has an infinite minimal system of generators.
We say that $S = \{z_\alpha\}_\alpha$ *is a minimal system of generators of an algebra* $\mathcal{A}(S)$ *if for every* α_0, $z_{\alpha_0} \notin \mathcal{A}(S \setminus \{z_{\alpha_0}\})$.

Aim

Study the behaviour of the powers of the hypercyclic functions of each one of these classical operators: Birkhoff's and MacLane's.

Algebrability

Definition (Aron, Gurariy, S.)

A set A is *algebrable* if there is an algebra $\mathcal{B} \subset A \cup \{0\}$ so that $\mathcal{B}$ has an infinite minimal system of generators.
We say that $S = \{z_\alpha\}_\alpha$ *is a minimal system of generators of an algebra* $\mathcal{A}(S)$ *if for every* α_0, $z_{\alpha_0} \notin \mathcal{A}(S \setminus \{z_{\alpha_0}\})$.

Aim

Study the behaviour of the powers of the hypercyclic functions of each one of these classical operators: Birkhoff's and MacLane's.

Birkhoff Operator

Theorem (Birkhoff, 1929)

The translation operator

$$\begin{array}{cccc} \tau_1 & : & \mathcal{H}(\mathbb{C}) & \longrightarrow & \mathcal{H}(\mathbb{C}) \\ & & f(z) & \mapsto & f(z+1) \end{array}$$

is hypercyclic.

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f \in HC(\tau_1)$, and $g \in \mathcal{H}(\mathbb{C})$.
If the order of each zero of g is a multiple of p, then $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Birkhoff Operator

Theorem (Birkhoff, 1929)

The translation operator

$$\begin{array}{rcccc} \tau_1 & : & \mathcal{H}(\mathbb{C}) & \longrightarrow & \mathcal{H}(\mathbb{C}) \\ & & f(z) & \mapsto & f(z+1) \end{array}$$

is hypercyclic.

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f \in HC(\tau_1)$, and $g \in \mathcal{H}(\mathbb{C})$.
If the order of each zero of g is a multiple of p, then $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0,$ and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.

By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1-z)\exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1 - z) \exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z + n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z + n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Proof. Let $(a_i)_i$ be the non-null zeros of g with multiplicity pm_i with $m_i \in \mathbb{N}_0$, and let 0 be a zero of g with multiplicity pm, for some $m \in \mathbb{N}_0$.
By Weierstrass' Theorem, there is $(p_i)_i \subset \mathbb{N}_0$, and $\varphi \in \mathcal{H}(\mathbb{C})$, such that

$$g(z) = z^{pm} e^{\varphi(z)} \prod_{i=1}^{\infty} E_{p_i}(z/a_i)^{pm_i},$$

with $E_0(z) := 1 - z$, $E_p(z) := (1 - z) \exp\left(z + z^2/2 + \ldots + z^p/p\right)$, for $p \geq 1$.
Define

$$\tilde{g}(z) = z^m e^{\varphi(z)/p} \prod_{i=1}^{\infty} E_{p_i}\left(z/a_i\right)^{m_i}.$$

Next, since $f \in HC(\tau_1)$, for any compact $K \subset \mathbb{C}$, there is $(n_j)_j \in \mathbb{N}$ with

$$\|f(z+n_j) - \tilde{g}(z)\|_K \longrightarrow 0 \quad \text{as} \quad j \to \infty$$

It follows that $\|f^p(z+n_j) - g(z)\|_K \longrightarrow 0$, and $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$.

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\text{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\text{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Theorem (Aron, Conejero, Peris and S., 2007)

Let $1 < p \in \mathbb{N}$, $f, g \in \mathcal{H}(\mathbb{C})$. If $g \in \overline{\mathrm{Orb}(\tau_1, f^p)}$, then the order of each zero of g is a multiple of p.

Proof. Suppose z_0 is a zero of order q of g with $q/p \notin \mathbb{N}$. In fact z_0 is the unique zero in some bounded region D. Suppose that $(n_j)_j \subset \mathbb{N}$ verifies

$$f^p(z + n_j) \to g(z) \quad \text{when} \quad j \to \infty, \text{ uniformly on } D.$$

By Hurwitz's theorem, there is $n_j \in \mathbb{N}$ such that

$$\dot{p} = \#Z(f^p(z + n_j)) = \#Z(g(z)).$$

Corollary

Let $1 < p, q \in \mathbb{N}$, and $f \in HC(\tau_1)$. Then

$$z^q \in \overline{\mathrm{Orb}(\tau_1, f^p)} \iff q/p \in \mathbb{N}.$$

Corollary

Let

$$B_k := \{f \in \mathcal{H}(\mathbb{C}) \, : \, f^k \in HC(\tau_1)\}.$$

We have that $B_k = \varnothing$ for every $k > 1$.
Thus, $HC(\tau_1)$ is not *algebrable*.

Of course, all the previous results also hold for any translation operator on $\mathcal{H}(\mathbb{C})$.

Corollary

Let $1 < p, q \in \mathbb{N}$, and $f \in HC(\tau_1)$. Then

$$z^q \in \overline{\mathrm{Orb}(\tau_1, f^p)} \Longleftrightarrow q/p \in \mathbb{N}.$$

Corollary

Let

$$B_k := \{f \in \mathcal{H}(\mathbb{C}) \,:\, f^k \in HC(\tau_1)\}.$$

We have that $B_k = \varnothing$ for every $k > 1$.
Thus, $HC(\tau_1)$ is not *algebrable*.

Of course, all the previous results also hold for any translation operator on $\mathcal{H}(\mathbb{C})$.

Corollary

Let $1 < p, q \in \mathbb{N}$, and $f \in HC(\tau_1)$. Then

$$z^q \in \overline{\mathrm{Orb}(\tau_1, f^p)} \Longleftrightarrow q/p \in \mathbb{N}.$$

Corollary

Let

$$B_k := \{f \in \mathcal{H}(\mathbb{C}) \, : \, f^k \in HC(\tau_1)\}.$$

We have that $B_k = \varnothing$ for every $k > 1$.
Thus, $HC(\tau_1)$ is not *algebrable*.

Of course, all the previous results also hold for any translation operator on $\mathcal{H}(\mathbb{C})$.

Corollary

Let $1 < p, q \in \mathbb{N}$, and $f \in HC(\tau_1)$. Then

$$z^q \in \overline{\mathrm{Orb}(\tau_1, f^p)} \iff q/p \in \mathbb{N}.$$

Corollary

Let

$$B_k := \{f \in \mathcal{H}(\mathbb{C}) \,:\, f^k \in HC(\tau_1)\}.$$

We have that $B_k = \varnothing$ for every $k > 1$.
Thus, $HC(\tau_1)$ is not *algebrable*.

Of course, all the previous results also hold for any translation operator on $\mathcal{H}(\mathbb{C})$.

MacLane Operator

Theorem (MacLane, 1952)

The derivative operator

$$\begin{array}{rccc} D & : & \mathcal{H}(\mathbb{C}) & \longrightarrow & \mathcal{H}(\mathbb{C}) \\ & & f(z) & \mapsto & f'(z) \end{array}$$

is hypercyclic.

Theorem (Aron, Conejero, Peris and S., 2007)

For every $k \in \mathbb{N}$, $M_k := \{f \in \mathcal{H}(\mathbb{C}) \,:\, f^k \in HC(D)\}$ is a G_δ dense set.

MacLane Operator

Theorem (MacLane, 1952)

The derivative operator

$$\begin{array}{rcccc} D & : & \mathcal{H}(\mathbb{C}) & \longrightarrow & \mathcal{H}(\mathbb{C}) \\ & & f(z) & \mapsto & f'(z) \end{array}$$

is hypercyclic.

Theorem (Aron, Conejero, Peris and S., 2007)

For every $k \in \mathbb{N}$, $M_k := \{f \in \mathcal{H}(\mathbb{C}) : f^k \in HC(D)\}$ is a G_δ dense set.

Corollary

There exists $f \in \mathcal{H}(\mathbb{C})$ such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$.
Moreover, the following set is residual

$$\{f \in \mathcal{H}(\mathbb{C}) : f^k \in HC(D) \text{ for every } k \in \mathbb{N}\}.$$

Corollary

Notice that $B_1 \cap \left(\cap_{j=1}^{\infty} M_k\right)$ is a G_δ dense set as well.

Corollary

There exists $f \in \mathcal{H}(\mathbb{C})$ such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$. Moreover, the following set is residual

$$\{f \in \mathcal{H}(\mathbb{C}) : f^k \in HC(D) \text{ for every } k \in \mathbb{N}\}.$$

Corollary

Notice that $B_1 \cap \left(\cap_{j=1}^{\infty} M_k\right)$ is a G_δ dense set as well.

Corollary

There exists $f \in \mathcal{H}(\mathbb{C})$ such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$.
Moreover, the following set is residual

$$\{f \in \mathcal{H}(\mathbb{C}) : f^k \in HC(D) \text{ for every } k \in \mathbb{N}\}.$$

Corollary

Notice that $B_1 \cap \left(\cap_{j=1}^{\infty} M_k\right)$ is a G_δ dense set as well.

Corollary

There exists $f \in \mathcal{H}(\mathbb{C})$ such that $f^k \in HC(D)$ for every $k \in \mathbb{N}$.
Moreover, the following set is residual

$$\{f \in \mathcal{H}(\mathbb{C}) : f^k \in HC(D) \text{ for every } k \in \mathbb{N}\}.$$

Corollary

Notice that $B_1 \cap \left(\cap_{j=1}^{\infty} M_k\right)$ is a G_δ dense set as well.

Question

Is $HC(D)$ algebrable? spaceable?

Theorem (Shkarin, 2010)

$HC(D)$ is spaceable.

Theorem (Aron, Conejero, Peris, Seoane, 2007)

$HC(D)$ is algebrable.

Question

Is $HC(D)$ algebrable? spaceable?

Theorem (Shkarin, 2010)

$HC(D)$ is spaceable.

Theorem (Aron, Conejero, Peris, Seoane, 2007)

$HC(D)$ is algebrable.

Question

Is $HC(D)$ algebrable? spaceable?

Theorem (Shkarin, 2010)

$HC(D)$ is spaceable.

Theorem (Aron, Conejero, Peris, Seoane, 2007)

$HC(D)$ is algebrable.

Algebrability and related topics

Theorem (Aron, Conejero, Peris and S., 2007)

The set of everywhere surjective functions on $\mathbb{C}$ contains an uncountably generated algebra $\mathcal{A}$.

Theorem (Aron, Pérez–García and S., 2006)

Given a set $E \subset \mathbb{T}$ of measure zero, the set of continuous functions whose Fourier series expansion is divergent at any point $t \in E$ is *dense–algebrable*, i.e. there exists an infinite dimensional, infinitely generated dense subalgebra of $\mathcal{C}(\mathbb{T})$ every non-zero element of which has a Fourier series expansion divergent in E.

Algebrability and related topics

Theorem (Aron, Conejero, Peris and S., 2007)

The set of everywhere surjective functions on $\mathbb{C}$ contains an uncountably generated algebra $\mathcal{A}$.

Theorem (Aron, Pérez–García and S., 2006)

Given a set $E \subset \mathbb{T}$ of measure zero, the set of continuous functions whose Fourier series expansion is divergent at any point $t \in E$ is *dense–algebrable*, i.e. there exists an infinite dimensional, infinitely generated dense subalgebra of $\mathcal{C}(\mathbb{T})$ every non-zero element of which has a Fourier series expansion divergent in E.

Algebrability and related topics

Theorem (Aron, Conejero, Peris and S., 2007)

The set of everywhere surjective functions on $\mathbb{C}$ contains an uncountably generated algebra $\mathcal{A}$.

Theorem (Aron, Pérez–García and S., 2006)

Given a set $E \subset \mathbb{T}$ of measure zero, the set of continuous functions whose Fourier series expansion is divergent at any point $t \in E$ is *dense–algebrable*, i.e. there exists an infinite dimensional, infinitely generated dense subalgebra of $\mathcal{C}(\mathbb{T})$ every non-zero element of which has a Fourier series expansion divergent in E.

Bounded linear (non)-absolutely summing operators

A Banach space E is said to have the "*two series property*" provided there exist unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in E^* and $\sum_{i=1}^{\infty} x_i$ in E such that

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\frac{|f_j(x_i)|^2}{\|f_j\|}\right]^{\frac{1}{2}} = +\infty.$$

Theorem (Puglisi and Seoane, 2008)

Let E be a Banach space with the two series property. Then

$$\mathcal{L}(E,\ell_2)\setminus\Pi_1(E,\ell_2)$$

is lineable.

Bounded linear (non)-absolutely summing operators

A Banach space E is said to have the "*two series property*" provided there exist unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in E^* and $\sum_{i=1}^{\infty} x_i$ in E such that

$$\sum_{i=1}^{\infty} \left[\sum_{j=1}^{\infty} \frac{|f_j(x_i)|^2}{\|f_j\|} \right]^{\frac{1}{2}} = +\infty.$$

Theorem (Puglisi and Seoane, 2008)

Let E be a Banach space with the two series property. Then

$$\mathcal{L}(E, \ell_2) \setminus \Pi_1(E, \ell_2)$$

is lineable.

Bounded linear (non)-absolutely summing operators

A Banach space E is said to have the "*two series property*" provided there exist unconditionally convergent series $\sum_{i=1}^{\infty} f_i$ in E^* and $\sum_{i=1}^{\infty} x_i$ in E such that

$$\sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\frac{|f_j(x_i)|^2}{\|f_j\|}\right]^{\frac{1}{2}}=+\infty.$$

Theorem (Puglisi and Seoane, 2008)

Let E be a Banach space with the two series property. Then

$$\mathcal{L}(E,\ell_2)\setminus\Pi_1(E,\ell_2)$$

is lineable.

In 2008, Puglisi and Seoane posed the following question:

Question

If E is a superreflexive Banach space and $p \geq 1$, is it true that the set

$$\mathcal{L}(E; F) \setminus \Pi_p(E; F)$$

is lineable for every Banach space F?

- In 2009, Botelho, Diniz, and Pellegrino gave a positive answer to the above question for large families of Banach spaces (they considered E to be a superreflexive Banach space containing a complemented infinite dimensional subspace with unconditional basis, and F a Banach space having an infinite unconditional basic sequence.)
- In 2010, Kitson and Timoney also studied lineability and (even!) spaceability of these types of subsets of operators.

In 2008, Puglisi and Seoane posed the following question:

Question

If E is a superreflexive Banach space and $p \geq 1$, is it true that the set

$$\mathcal{L}(E;F) \setminus \Pi_p(E;F)$$

is lineable for every Banach space F?

- In 2009, Botelho, Diniz, and Pellegrino gave a positive answer to the above question for large families of Banach spaces (they considered E to be a superreflexive Banach space containing a complemented infinite dimensional subspace with unconditional basis, and F a Banach space having an infinite unconditional basic sequence.)
- In 2010, Kitson and Timoney also studied lineability and (even!) spaceability of these types of subsets of operators.

In 2008, Puglisi and Seoane posed the following question:

Question

If E is a superreflexive Banach space and $p \geq 1$, is it true that the set

$$\mathcal{L}(E;F) \setminus \Pi_p(E;F)$$

is lineable for every Banach space F?

- In 2009, Botelho, Diniz, and Pellegrino gave a positive answer to the above question for large families of Banach spaces (they considered E to be a superreflexive Banach space containing a complemented infinite dimensional subspace with unconditional basis, and F a Banach space having an infinite unconditional basic sequence.)
- In 2010, Kitson and Timoney also studied lineability and (even!) spaceability of these types of subsets of operators.

In 2008, Puglisi and Seoane posed the following question:

Question

If E is a superreflexive Banach space and $p \geq 1$, is it true that the set

$$\mathcal{L}(E; F) \setminus \Pi_p(E; F)$$

is lineable for every Banach space F?

- In 2009, Botelho, Diniz, and Pellegrino gave a positive answer to the above question for large families of Banach spaces (they considered E to be a superreflexive Banach space containing a complemented infinite dimensional subspace with unconditional basis, and F a Banach space having an infinite unconditional basic sequence.)
- In 2010, Kitson and Timoney also studied lineability and (even!) spaceability of these types of subsets of operators.

In 2008, Puglisi and Seoane posed the following question:

Question

If E is a superreflexive Banach space and $p \geq 1$, is it true that the set

$$\mathcal{L}(E;F) \setminus \Pi_p(E;F)$$

is lineable for every Banach space F?

- In 2009, Botelho, Diniz, and Pellegrino gave a positive answer to the above question for large families of Banach spaces (they considered E to be a superreflexive Banach space containing a complemented infinite dimensional subspace with unconditional basis, and F a Banach space having an infinite unconditional basic sequence.)
- In 2010, Kitson and Timoney also studied lineability and (even!) spaceability of these types of subsets of operators.

THANK YOU

FOR YOUR ATTENTION!!!