# On the Interplay of Regularity and Decay in Case of Radial Functions

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joint work with L. Skrzypczak (Poznan) and J. Vybiral (Linz)


• W. S. and L. Skrzypczak, *Radial subspaces of Besov and Lizorkin-Triebel classes: extended Strauss lemma and compactness of embeddings*, JFAA **6** (2000), 639-662.

• W. S., L. Skrzypczak and J. Vybiral, *On the interplay of regularity and decay in case of radial functions I. Inhomogeneous spaces*, Comm. in Cont. Math. (accepted).

• W. S. and L. Skrzypczak, *On the Interplay of regularity and decay in case of radial functions II. Homogeneous spaces*, Jena, Poznan, 2010.

• W. S., L. Skrzypczak and J. Vybiral, *Radial subspaces of Besov- and Lizorkin-Triebel spaces - complex interpolation and characterization by differences*, (in preparation), Jena, Poznan, Linz, 2011.

# 1. Introduction

**The Radial Lemma** (Strauss 1977).
Let $d \geq 2$. Every radial function $f \in H^1(\mathbb{R}^d)$ is almost everywhere equal to a function $\widetilde{f}$, continuous for $x \neq 0$, such that

$$|\widetilde{f}(x)| \leq c\,|x|^{\frac{1-d}{2}}\,\|\,f\,|H^1(\mathbb{R}^d)\|\,, \tag{1}$$

where $c$ depends only on $d$.

This lemma contains three different assertions:

(a) the existence of a special representative of $f$, which is continuous outside the origin;
(b) the decay of $f$ near infinity;
(c) the controlled unboundedness of $f$ near the origin.

## Compactness of embeddings

$$H^1(\mathbb{R}^d) \hookrightarrow L_q(\mathbb{R}^d)\,, \qquad 2 \le q \begin{cases} \le \frac{2d}{d-2} & d > 2\,, \\ < \infty & d = 2\,. \end{cases}$$

Strauss (1977), Coleman, Glazer, Martin (1978) and Berestycki, P.L. Lions (1979):

$$RH^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L_q(\mathbb{R}^d)\,, \qquad 2 < q < \begin{cases} \frac{2d}{d-2} & d > 2\,, \\ \infty & d = 2\,. \end{cases}$$

Necessity of these restrictions for the compactness of the embedding: Ebihara, Schonbeck (1986).

# 3. The regularity of radial distributions

### 3.1 Radial distributions

**Definition**

*A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ is called radial, if it is invariant under rotations around the origin, i.e.,*

$$f(\phi(\Phi)) = f(\phi) \qquad \text{für alle} \quad \phi \in \mathcal{S}(\mathbb{R}^d)$$

*and all rotations $\Phi$.*

**Examples**: 1) The Dirac distribution $\delta : \varphi \mapsto \varphi(0)$ is radial and

$$\delta \in B^{d(\frac{1}{p}-1)}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad p.$$

2) The spherical mean $f : \varphi \mapsto \int_{|x|=1} \varphi(x)\, dx$ is radial and

$$f \in B^{\frac{1}{p}-1}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad p.$$

## 3.2 The regularity of radial distributions outside the origin

**Theorem 1**

*Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Furthermore, we assume*

$$s > \max\left(0, \frac{1}{p} - 1\right).$$

*If $f \in RB^s_{p,q}(\mathbb{R}^d)$ s.t.* $\quad 0 \notin \operatorname{supp} f$, *then $f \in L_1(\mathbb{R}^d)$.*

$$\mathrm{tr}\; f(t) := f(t, 0, \ldots, 0), \qquad t \in \mathbb{R}.$$

**Theorem 2**

*Under the restrictions of Thm. 1 we have* $\quad \mathrm{tr}\; f \in B^s_{p,q}(\mathbb{R})$.

## 3.3 The continuity outside of the origin

Let $s > 1/p$ und $f \in RB^s_{p,q}(\mathbb{R}^d)$. Let $\varphi$ be a smooth radial function s.t. $0 \notin \operatorname{supp} \varphi$ and

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} |D^\alpha \varphi(x)| < \infty \, .$$

Then the pointwise product $\varphi \cdot f \in RB^s_{p,q}(\mathbb{R}^d)$ and

$$\operatorname{tr}(\varphi \cdot f) \in B^s_{p,q}(\mathbb{R}) \hookrightarrow B^{s-1/p}_{\infty,\infty}(\mathbb{R}) = \mathcal{Z}^{s-1/p}(\mathbb{R}) \, .$$

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s-d/p}_{\infty,\infty}(\mathbb{R}^d) = \mathcal{Z}^{s-d/p}(\mathbb{R}^d)$$

### Corollary 1

*Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$, and $s > 1/p$. Let $\varphi$ be as above. If $f \in RB^s_{p,q}(\mathbb{R}^d)$, then $\varphi f \in \mathcal{Z}^{s-1/p}(\mathbb{R}^d)$ follows.*

### Corollary 2

*Let $\tau > 0$. Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. If either $s > 1/p$ (and $q$ arbitrary) or $s = 1/p$ and $q \leq 1$, then $f \in RB^s_{p,q}(\mathbb{R}^d)$ is uniformly continuous on $|x| > \tau$.*

- P.L. Lions (1982) (Sobolev spaces), S. and Skrzypczak (2000);
- $U := \{(s,p,q) : \quad (s,p,q)$ as in Cor. 2$\}$.

# 4. Decay of radial functions near infinity. I

**Theorem 3**

*Let* $d \ge 2$, $0 < p < \infty$, *and* $0 < q \le \infty$. *Furthermore we assume* $(s, p, q) \in U$. *Then there exists a constant* $c$ *s.t.*

$$|x|^{(d-1)/p} \, |f(x)| \le c \, \| \, f \, |B^s_{p,q}(\mathbb{R}^d)\|$$

*holds for all* $|x| \ge 1$ *and all* $f \in RB^s_{p,q}(\mathbb{R}^d)$. *Moreover*

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{p}} \, |f(x)| = 0$$

*holds for all* $f \in RB^s_{p,q}(\mathbb{R}^d)$.

## Theorem 4

(i) *Let* $(s, p, q) \in U$. *Then there exists a constant* $c > 0$ *s.t. for all* $x$, $|x| > 1$ *there exists a smooth radial function* $f \in RB^s_{p,q}(\mathbb{R}^d)$ *with* $\| f \, |RB^s_{p,q}(\mathbb{R}^d)\| = 1$, *and satisfying*

$$|x|^{\frac{d-1}{p}} \, |f(x)| \ge c \, . \qquad (2)$$

(ii) *Let* $(s, p, q) \notin U$ *and* $\frac{1}{p} > \sigma_p(d)$. *For all sequences* $(x^j)_{j=1}^\infty \subset \mathbb{R}^d \setminus \{0\}$ *with* $\lim_{j \to \infty} |x^j| = \infty$ *there exists a radial function* $f \in RB^s_{p,q}(\mathbb{R}^d)$, $\| f \, |RB^s_{p,q}(\mathbb{R}^d)\| = 1$, *s.t.* $f$ *is unbounded in any neighborhood of* $x^j$, $j \in \mathbb{N}$.

## Remark

Let $p$ be fixed. Then increasing the smoothness $s$ will not result in an improved decay rate.

# 5. Controlled unboundedness of radial functions near the origin

### Theorem 5

*Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$.*
(i) *Let $(s, p, q) \in U$ and $s < \frac{d}{p}$. Then there exists a constant $c > 0$ s.t.*

$$|x|^{\frac{d}{p}-s}\,|f(x)| \leq c\,\| f\,|RB^s_{p,q}(\mathbb{R}^d)\| \qquad (3)$$

*holds for all $0 < |x| \leq 1$ and all $f \in RB^s_{p,q}(\mathbb{R}^d)$.*
(ii) *Let $\sigma_p(d) < s < d/p$. Then there exists a constant $c > 0$ s.t. for all $x$, $0 < |x| < 1$, there is a smooth radial function $f \in RB^s_{p,q}(\mathbb{R}^d)$, $\| f\,|RB^s_{p,q}(\mathbb{R}^d)\| = 1$, satisfying*

$$|x|^{\frac{d}{p}-s}\,|f(x)| \geq c\,. \qquad (4)$$

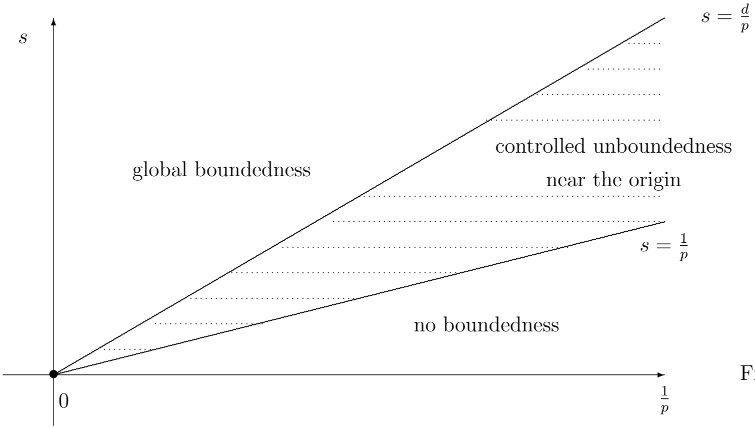
## The limiting case

Theorem 6

*Let* $d \geq 2$, $0 < p < \infty$, $1 < q \leq \infty$ *and* $s = d/p$. *Then there esists a constant* $c > 0$ *s.t.*

$$(-\log |x|)^{-1/q'} \, |f(x)| \leq c \, \| \, f \, | B^{d/p}_{p,q}(\mathbb{R}^d) \|$$

*holds for all* $0 < |x| \leq 1/2$ *and all* $f \in RB^{d/p}_{p,q}(\mathbb{R}^d)$.



$s$
$s = \frac{d}{p}$
controlled unboundedness
near the origin
global boundedness
$s = \frac{1}{p}$
no boundedness
0
$\frac{1}{p}$


Fig. 1

# 6. Compact embeddings. I

**Theorem 7**

*Let $d \geq 2$ and $1 \leq p_1 \leq \infty$. Then $RB^{s_0}_{p_0,q_0}(\mathbb{R}^d)$ is compactly embedded into $L_{p_1}(\mathbb{R}^d)$ if, and only if,*

$$p_0 < p_1 \qquad \text{and} \qquad s_0 > d\left(\frac{1}{p_0} - \frac{1}{p_1}\right).$$

- $RB^1_{2,2}(\mathbb{R}^d) = RH^1(\mathbb{R}^d)$ in the sense of equivalent norms.

- S. and Skrzypczak (2000) (many forerunners in case of Sobolev spaces).
- Cwikel and Tintarev (2011; different proof).

## 7. Homogeneous Besov spaces

Let $M \in \mathbb{N}$ and

$$\sigma_p := d \max\left(0, \frac{1}{p} - 1\right) < s < M .$$

$$\| f \,|\dot{B}^s_{p,q}(\mathbb{R}^d)\| := \left( \int_{\mathbb{R}^d} \Big[|h|^{-s} \, \| \Delta^M_h f \,| L_p(\mathbb{R}^d)\|\Big]^q \frac{dh}{|h|^d}\right)^{1/q} < \infty .$$

$$[f]_M := \{f + p : \ p \quad \text{is a polynomial of order } \le M - 1\}$$

**Definition**

*The class $[f]_M$ belongs to $\dot{B}^s_{p,q}(\mathbb{R}^d)$ if $f$ is a regular distribution s.t. $\| f \,|\dot{B}^s_{p,q}(\mathbb{R}^d)\| < \infty$.*

$\underline{\sigma_p < s < d/p}$:

$$\dot{B}^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_{t,\infty}(\mathbb{R}^d)\,, \quad t := \frac{d}{\left(\frac{d}{p} - s\right)}$$

Interpretation: there exists a representative $g \in [f]_M$ which belongs to $L_{t,\infty}(\mathbb{R}^d)$.

$$\sigma(f) := \sum_{j=-\infty}^{\infty} \mathcal{F}^{-1}[\varphi_j(\xi)\, \mathcal{F}f(\xi)]$$

$$RB^s_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)\,.$$

# 8. Decay of radial functions near infinity. II

**Theorem 8**

*Let $d \ge 2$, $0 < p < \infty$, and in addition*

$$\max\left(\sigma_p, \frac{1}{p}\right) < s < \frac{d}{p} \, . \qquad (5)$$

*Then there exists a constant $c > 0$ such that*

$$|x|^{\frac{d}{p}-s}\, |g(x)| \le c \, \| \, [g] \, |\dot{B}^s_{p,\infty}(\mathbb{R}^d)\|$$

*holds for all $g \in \dot{B}^s_{p,\infty}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d)$, $t = d/(\frac{d}{p} - s)$, and all $x \neq 0$.*

# 9. Compact embeddings. II

Strauss et all:

$$RH^1(\mathbb{R}^d) \hookrightarrow\hookrightarrow L_q(\mathbb{R}^d)\,, \qquad 2 < q < \begin{cases} \frac{2d}{d-2} & d > 2\,, \\ \infty & d = 2\,. \end{cases}$$

Can one replace $RH^1(\mathbb{R}^d)$ by the homogeneous counterpart $R\dot{H}^1(\mathbb{R}^d)$ (with an appropriate interpretation) ?
Answer: no !

$$R\dot{H}^1(\mathbb{R}^d) \not\subset L_q(\mathbb{R}^d)\,, \qquad q < \begin{cases} \frac{2d}{d-2} & d > 2\,, \\ \infty & d = 2\,. \end{cases}$$

## Theorem 9

*Let* $d \geq 2$, $0 < p < \infty$, $1 \leq p_1 \leq p_2 \leq \infty$ *and*

$$\sigma_p = d \, \max(0, \frac{1}{p} - 1) < s < \frac{d}{p} \, .$$

*Then*

$$\dot{B}^s_{p,q}(\mathbb{R}^d) \cap RL_{t,\infty}(\mathbb{R}^d) \, , \qquad t := \frac{d}{\left(\frac{d}{p} - s\right)} \, ,$$

*is compactly embedded into* $L_{p_1}(\mathbb{R}^d) + L_{p_2}(\mathbb{R}^d)$ *if*

$$p < p_1 < \frac{d}{\frac{d}{p} - s} < p_2 \leq \infty \, .$$

- S. und Skrzypczak (2010).

# 10. Compact embeddings. III - exterior domains

$$\Omega := \{x \in \mathbb{R}^d : \quad |x| \geq 1\}.$$

**Theorem 10**

*Let* $d \geq 2$, $0 < p < \infty$,

$$\max\left(\sigma_p, \frac{1}{p}\right) < s < \frac{d}{p} \qquad \text{and} \qquad \frac{d}{\frac{d}{p} - s} < p_1 < \infty\,.$$

*Then* $R\dot{B}^s_{p,\infty}(\Omega)$ *is compactly embedded into* $L_{p_1}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

- S. and Skrzypczak (2010).