Local function spaces

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Isotropic distributional (or Lebesgue-integrable) spaces on $\mathbb{R}^n$.

The Sobolev-Nikol'skij-Besov-Peetre line.
Sobolev: 1936-38, $W_p^k(\mathbb{R}^n)$ (and, mainly, in domains), $D^\alpha f$.
Nikol'skij-Besov: 1951, 1960, $B_{p,q}^s(\mathbb{R}^n) = B_q^s L_p(\mathbb{R}^n)$, $p \geq 1, s > 0$, $\Delta_h^m f$.
Peetre: 1967-75, $A_{p,q}^s(\mathbb{R}^n)$, $0 < p, q \leq \infty, s \in \mathbb{R}$, $\varphi_j(D)f$.

The Morrey-Campanato-Brudnyi line.
Morrey: 1938, $\mathcal{L}_p^r(\mathbb{R}^n)$, $p \geq 1$, $-n/p \leq r < 0$.
Campanato: 1963-65, $\mathcal{L}_p^r(\mathbb{R}^n)$, $-n/p \leq r < \infty$, $p \geq 1$,
Brudnyi: 1965-70, $\mathcal{L}_p^r(\mathbb{R}^n)$, $0 < p < \infty$.

Attempts to merge.
Kozono-Yamazaki: 1994, $B_q^s \mathcal{L}_p^r(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p > 1$, $-n/p \leq r < 0$.
Tang-Xu: 2005, $A_q^s \mathcal{L}_p^r(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p \leq \infty$, $-n/p \leq r < 0$.
Dachun Yang: 2008, $A_{p,q}^{s,\tau}(\mathbb{R}^n)$, $s, \tau \in \mathbb{R}$, $0 < p, q \leq \infty$,
T: 2012, $\mathcal{L}^r A_{p,q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $0 < p, q \leq \infty$, $-n/p \leq r < \infty$.

Comments: Sometimes domain $\Omega \subset \mathbb{R}^n$ instead of $\mathbb{R}^n$ (Sobolev, Brudnyi).
Recall $A \in \{B, F\}$. Here preference to $A = B$. Goal is to convince that merging is more than generalising.

Sobolev:

$$\|f\,|W_p^k(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \|D^\alpha f\,|L_p(\mathbb{R}^n)\|, \quad k \in \mathbb{N}, \quad 1 \le p < \infty.$$

Nikol'skji-Besov:

$$1 \le p \le \infty, \quad 0 < s < m \in \mathbb{N},$$

$$\Delta_h^1 f(x) = f(x+h) - f(x), \qquad \Delta_h^m = \Delta_h^1 \Delta_h^{m-1}, \qquad h \in \mathbb{R}^n, \quad m = 2, 3, \ldots,$$

$$\|f\,|B^s_{p,\infty}(\mathbb{R}^n)\| = \|f\,|L_p(\mathbb{R}^n)\| + \sup_{|h|\le 1} |h|^{-s}\,\|\Delta_h^m f\,|L_p(\mathbb{R}^n)\|.$$

Similarly $B^s_{p,q}(\mathbb{R}^n)$. Special case, Hölder-Zygmund spaces

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty,\infty}(\mathbb{R}^n),$$

with

$$\|f\,|\mathcal{C}^s(\mathbb{R}^n)\| = \sup_{x\in\mathbb{R}^n} |f(x)| + \sup_{x\in\mathbb{R}^n, |h|\le 1} |h|^{-s} |\Delta_h^m f(x)|.$$

Peetre:

If $\psi \in S(\mathbb{R}^n)$ then $\psi(D)f(x) = (\psi\widehat{f})^\vee(x)$ entire analytic function for any $f \in S'(\mathbb{R}^n)$.

Resolution of unity: $\varphi_0 \in S(\mathbb{R}^n)$, $\varphi = \{\varphi_j\}_{j=0}^\infty$,

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2,$$

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \qquad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.$$

$$\|f \,|B^s_{p,q}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq} \|\varphi_j(D)f \,|L_p(\mathbb{R}^n)\|^q \Big)^{1/q},$$

Similarly $F^s_{p,q}(\mathbb{R}^n)$, covering Sobolev spaces $H^s_p(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$, and classical Sobolev spaces $W^k_p(\mathbb{R}^n) = H^k_p(\mathbb{R}^n)$, $k \in \mathbb{N}_0$.

Q_{JM}, $J \in \mathbb{N}_0$, $M \in \mathbb{Z}^n$, cube in $\mathbb{R}^n$, left corner $2^{-J}M$, sides parallel to the axes, length 2^{-J+1}.

Notation:

uniform space: $\sup_{M \in \mathbb{Z}^n} \cdots Q_{0,M}$, local space: $\sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} \cdots Q_{JM}$.

$\mathcal{L}_p(\mathbb{R}^n)$, $0 < p \le \infty$, uniform L_p-space:

$$\|f \,|\mathcal{L}_p(\mathbb{R}^n)\| = \sup_{M \in \mathbb{Z}^n} \|f \,| L_p(Q_{0,M})\|.$$

$\mathcal{L}_\infty(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$.
$\mathcal{P}_k$: all polynomials of degree $\le k$ with $k \in \mathbb{N}_{-1}$, where $\mathcal{P}_{-1} = \{0\}$.

Definition 1.1. $0 < p \le \infty$, $-n/p \le r < \infty$, $k \in \mathbb{N}_{-1}$, $k + 1 > r$. Then $\mathcal{L}^r_p(\mathbb{R}^n)$: all measurable functions in $\mathbb{R}^n$, such that

$$\|f \,|\mathcal{L}^r_p(\mathbb{R}^n)\|_k = \|f \,|\mathcal{L}_p(\mathbb{R}^n)\| + \sup_{J \in \mathbb{N}, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \inf_{P \in \mathcal{P}_k} \|f - P \,| L_p(Q_{JM})\|$$

is finite.

Theorem 1.2. (i) $\mathcal{L}_p^r(\mathbb{R}^n)$ independent of k (equivalent quasi-norms).
(ii) $0 < p \leq \infty$, $r > 0$. Then $\mathcal{L}_p^r(\mathbb{R}^n) = \mathcal{C}^r(\mathbb{R}^n)$.
(iii) $0 < p < \infty$. Then $\mathcal{L}_p^{-n/p}(\mathbb{R}^n) = \mathcal{L}_p(\mathbb{R}^n)$.
(iv) $0 < p < \infty$, $-n/p < r < 0$. Let $t = n/|r|$. Then

$$L_\infty(\mathbb{R}^n) \hookrightarrow \mathcal{L}_t(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{t,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_p^r(\mathbb{R}^n) \hookrightarrow \mathcal{L}_p(\mathbb{R}^n).$$

All embeddings are *into*, but not *onto*. If $\mathcal{L}_{w,v}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_p^r(\mathbb{R}^n)$ then $\mathcal{L}_{w,v}(\mathbb{R}^n) \hookrightarrow \mathcal{L}_{t,\infty}(\mathbb{R}^n)$.
(v) $1 \leq p < \infty$, $-n/p \leq r < 0$. Then

$$\mathcal{L}_p^r(\mathbb{R}^n) \hookrightarrow \mathcal{C}^r(\mathbb{R}^n) \cap \mathcal{L}_p(\mathbb{R}^n).$$

(vi)

$$\mathcal{L}_\infty^0(\mathbb{R}^n) = \mathcal{L}_\infty(\mathbb{R}^n) = L_\infty(\mathbb{R}^n),$$
$$\mathcal{L}_p^0(\mathbb{R}^n) = bmo(\mathbb{R}^n), \qquad 0 < p < \infty.$$

Remark 1.3. $r < 0, k = -1$: Morrey 38. $r = 0$: John-Nirenberg, 61, $r > 0$: Campanato 64, Brudnyi, 65-69, 71 ($0 < p < 1$), survey 09, Peetre 69, (vi) Bennett, Sharpley 79.

$\mathcal{L}^r_p(\mathbb{R}^n)$, $0 < p < \infty$, $-n/p \le r < 0$ used for nonlinear PDEs, Navier-Stokes equation: Morrey, Giga-Miyakawa 89, M. Taylor 92. Extension to smooth spaces, including applications: Kozono-Yamazaki 1994, Mazzucato 2003.

$\varphi = \{\varphi_j\}$ above resolution of unity in $\mathbb{R}^n$.

Definition 1.4. $0 < p, q \le \infty$, $s \in \mathbb{R}$, $-n/p \le r < 0$.

$$B^s_q \mathcal{L}^r_p(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \ \|f \,|B^s_q \mathcal{L}^r_p(\mathbb{R}^n)\| < \infty\}$$

with

$$\|f \,|B^s_q \mathcal{L}^r_p(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{jsq} \|\varphi_j(D) f \,|\mathcal{L}^r_p(\mathbb{R}^n)\|^q \Big)^{1/q}.$$

Similarly $F^s_q \mathcal{L}^r_p(\mathbb{R}^n)$.

Truncation of $\varphi_j(D)f$ and localization: Dachun Yang, Wen Yuan, 08. Several papers afterwards, Sawano, ..., book Yuan, Sickel, Yang, Lecture Notes Math. 2005 (2010)

Definition 1.5. $0 < p, q \leq \infty$ ($p < \infty$ for F-spaces), $s \in \mathbb{R}$, $\tau \geq 0$.

$$\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \ \|f \,|\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)\| < \infty\}$$

with

$$\|f \,|\mathcal{B}^{s,\tau}_{p,q}(\mathbb{R}^n)\| = \sup_{J\in\mathbb{N}_0, M\in\mathbb{Z}^n} 2^{Jn\tau} \Big(\sum_{j=J}^{\infty} 2^{jsq} \|\varphi_j(D)f \,|L_p(Q_{JM})\|^q \Big)^{1/q}.$$

and similarly $\mathcal{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$.

Remark 1.6. Modification of corresponding spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ (all cubes Q_{JM} with $J \in \mathbb{Z}$, $M \in \mathbb{Z}^n$ but φ as above) in literature, also in book Yuan-Sickel-Yang.

Motivation for truncation: $bmo(\mathbb{R}^n)$, John-Nirenberg 61.

$$\|f\,|bmo(\mathbb{R}^n)\| = \sup_{J\in\mathbb{N}_0, M\in\mathbb{Z}^n} 2^{Jn} \int_{Q_{JM}} |f(x) - f_{Q_{JM}}|\,\mathrm{d}x + \sup_{M\in\mathbb{Z}^n} \int_{Q_{0,M}} |f(x)|\,\mathrm{d}x$$

with the mean value

$$f_{Q_{JM}} = |Q_{JM}|^{-1} \int_{Q_{JM}} f(y)\,\mathrm{d}y.$$

$bmo(\mathbb{R}^n) = F^0_{\infty,2}(\mathbb{R}^n)$. Extension to $F^s_{\infty,q}(\mathbb{R}^n)$ with $s\in\mathbb{R}$, $0<q<\infty$: Frazier-Jawerth 90,

$$\|f\,|F^s_{\infty,q}(\mathbb{R}^n)\| = \sup_{J\in\mathbb{N}_0, M\in\mathbb{Z}^n} \Big(2^{Jn} \int_{Q_{JM}} \sum_{j=J}^{\infty} 2^{jsq} |\varphi_j(D)f(x)|^q\,\mathrm{d}x\Big)^{1/q}$$

Motivation for brutal truncation of the entire analytic functions $\varphi_j(D)f$: points in lattices of mesh-length 2^{-j} determine

$$\varphi_j(D)f(x) = F^{-1}\varphi_j F\, f(x), \qquad x\in\mathbb{R}^n,$$

restriction to cubes Q_{JM} makes if $j\geq J$, measured in $L_\infty(\ell_q)$.

Wavelets $\psi_F \in C^u(\mathbb{R})$, $\psi_M \in C^u(\mathbb{R})$, $u \in \mathbb{N}$,

$$\int_{\mathbb{R}} \psi_M(x) x^v \, \mathrm{d}x = 0, \qquad v \in \mathbb{N}_0, \quad v < u.$$

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{w=1}^{n} \psi_{G_w}(2^j x_w - m_w), \qquad G \in G^j, \quad m \in \mathbb{Z}^n.$$

Q_{jm} cube in $\mathbb{R}^n$, $2^{-j}m$ left corner, side-length 2^{-j+1}, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$,

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{jm}.$$

Wavelet expansion:

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^{j,G}_m \, 2^{-jn/2} \Psi^j_{G,m},$$

with

$$\lambda^{j,G}_m = \lambda^{j,G}_m(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \Psi^j_{G,m}(x) \, \mathrm{d}x.$$

Wavelets as above,

$$\Psi^u = \{\Psi^j_{G,m} : \; j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n\}, \quad \text{based on } \psi_{F,u},$$

$$\operatorname{supp} \Psi^j_{G,m} \subset Q_{jm}.$$

$$\mathbb{P}_{JM} = \{j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n : \; Q_{jm} \subset Q_{JM}\}, \qquad J \in \mathbb{N}_0, \quad M \in \mathbb{Z}^n.$$

Sequence spaces $\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)$:

$$\lambda = \{\lambda^{j,G}_m \in \mathbb{C} : \; j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n\}$$

quasi-normed by

$$\|\lambda \,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(\frac{n}{p}+r)} \Big(\sum_{j=J}^{\infty} 2^{j(s-\frac{n}{p})q} \Big(\sum_{m,G:(j,G,m)\in \mathbb{P}_{JM}} |\lambda^{j,G}_m|^p \Big)^{q/p} \Big)^{1/q}.$$

Similarly $\mathcal{L}^r f^s_{p,q}(\mathbb{R}^n)$.

Definition 2.1. $0 < p, q \leq \infty$, $s \in \mathbb{R}$, $-n/p \leq r < \infty$,

$$u > \max(s + r^+, \sigma_p - s).$$

Then $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$ collects all $f \in S'(\mathbb{R}^n)$ for which

$$\|f \,|\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)\|_{\Psi^u} = \sup_{J,M} 2^{J(\frac{n}{p}+r)} \Big\| \sum_{(j,G,m)\in\mathbb{P}_{JM}} \lambda^{j,G}_m(f)\, 2^{-jn/2} \Psi^j_{G,m} \,|B^s_{p,q}(\mathbb{R}^n)\Big\|$$
$$\sim \|\lambda(f)\,|\mathcal{L}^r b^s_{p,q}(\mathbb{R}^n)\| < \infty.$$

Recall $\sigma_p = n\big(\max(1/p, 1) - 1\big)$.

Usual lift in $S'(\mathbb{R}^n)$: $I_\sigma f = F^{-1}(1+|\xi|^2)^{-\sigma/2} Ff$, $\sigma \in \mathbb{R}$.

$$I_\sigma B^s_{p,q}(\mathbb{R}^n) = B^{s+\sigma}_{p,q}(\mathbb{R}^n).$$

Theorem 2.2. (i) $\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$ is independent of Ψ^u.

$$\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{s+r}(\mathbb{R}^n).$$

$$I_\sigma \mathcal{L}^r B^s_{p,q}(\mathbb{R}^n) = \mathcal{L}^r B^{s+\sigma}_{p,q}(\mathbb{R}^n).$$

(ii) $r > 0$, then

$$\mathcal{L}^r B^s_{p,q}(\mathbb{R}^n) = \mathcal{C}^{s+r}(\mathbb{R}^n).$$

(iii) $r = 0$:

$$\mathcal{L}^0 B^s_{p,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \qquad 0 < p \le \infty, \quad s \in \mathbb{R}.$$

$$\mathcal{L}^0 L_p(\mathbb{R}^n) = bmo(\mathbb{R}^n), \qquad 2 \le p < \infty.$$

Remark 2.3. $\mathcal{L}^0 L_2(\mathbb{R}^n) = bmo(\mathbb{R}^n)$: Y. Meyer, ~ 90.

If $r < 0$ then there is an $f \in \mathcal{C}^{s+r}(\mathbb{R}^n)$ with $f \notin \mathcal{L}^r B^s_{p,q}(\mathbb{R}^n)$ for all admitted p, q, A.

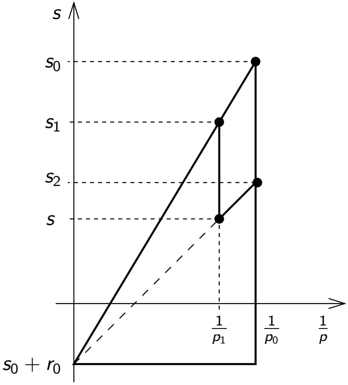

Figure: Limiting embeddings

Main problem: Necessary and sufficient conditions for

$$\mathcal{L}^{r_0} B^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{r} B^{s}_{p,q}(\mathbb{R}^n)$$

Two decisive quantities:

$$\text{differential dimension}: \quad s + r,$$
$$\text{slope}: \quad |r|p.$$

Classical case: $r = -n/p$. Then differential dimension $s - \frac{n}{p}$, slope n. Now merging parameters: From (s, p, q) to (r, s, p, q) as in Newtonian Mechanics to Special Relativity.

Proposition 3.1. Let $s_0 \in \mathbb{R}$, $0 < p_0 < \infty$, $0 < q_0 \le \infty$, $-n/p_0 \le r_0 < 0$. Let

$$p_0 \le p_1 < \infty, \qquad s_1 = s_0 + r_0\left(1 - \frac{p_0}{p_1}\right), \qquad \frac{q_1}{p_1} = \frac{q_0}{p_0}$$

$$s_1 + r_1 = s_0 + r_0 \qquad \text{(invariance of differential dimension)}.$$

Then

$$r_1 p_1 = r_0 p_0 \qquad \text{(invariance of slope)}$$

and

$$\mathcal{L}^{r_0} B^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{r_1} B^{s_1}_{p_1,q_1}(\mathbb{R}^n).$$

Furthermore

$$\mathcal{L}^{r_0} B^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{s_0+r_0}(\mathbb{R}^n).$$

Proof based on wavelet representations, above endpoint embeddings and Hölder inequalities. Disturbing point: q-conditions. Next: $q = \infty$.

Theorem 3.2. Let $s_0 \in \mathbb{R}$, $0 < p_0 < \infty$, $-n/p_0 \le r_0 < 0$. Let

$$p_0 \le p < \infty, \qquad s_0 + r_0 < s \le s_0 + r_0\left(1 - \tfrac{p_0}{p}\right)$$

(triangle in Figure). Let

$$r + s = r_0 + s_0 \qquad \text{(invariance of differential dimensions).}$$

Then $-n/p \le r < 0$ and

$$\mathcal{L}^{r_0} B^{s_0}_{p_0,\infty}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{r} B^{s}_{p,\infty}(\mathbb{R}^n).$$

Proof: Combine above Proposition 3.1 with

$$\mathcal{L}^{r_0} B^{s_0}_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^{r_1} B^{s_1}_{p,q}(\mathbb{R}^n), \quad s_0 + r_0 = s_1 + r_1, \quad s_1 < s_0,$$

Dialectical method: Contradictory thesis and anti-thesis resolve at a higher level (Hegel, Marx).
Sobolev and Morrey as a dialectical couple:

Sobolev: Offer smoothness, ask for better integrability,
Morrey: Offer (refined) integrability, aks for better smoothness.

Morrey's refinement of the (uniform) Lebesgue spaces $\mathcal{L}_p(\mathbb{R}^n)$: $0 < p < \infty$, $-n/p \le r < 0$,

$$\|f \,|\mathcal{L}^r_p(\mathbb{R}^n)\| = \sup_{J \in \mathbb{N}_0, M \in \mathbb{Z}^n} 2^{J(r+\frac{n}{p})} \|f \,| L_p(Q_{JM})\|.$$

Recall that

$$\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n) = \mathcal{L}^r F^0_{p,2}(\mathbb{R}^n), \qquad \text{if } 1 < p < \infty.$$

Main Problem repeated: Necessary and sufficient conditions for

$$\mathcal{L}^{r_0} A^{s_0}_{p_0,q_0}(\mathbb{R}^n) \hookrightarrow \mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$$

So far, classical case, $r_0 = -n/p_0$, $r = -n/p$, Sickel-T (1995). Now Morrey's refinement of L_p-spaces should be included guided by:

$$s + r \le s_0 + r_0, \quad \text{decreasing differential dimensions},$$
$$|r|p \le |r_0|p_0, \quad \text{decreasing slopes}.$$