Efficient railway timetabling using max-plus algebra

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Introduction

Whilst current railway timetables are in abundance and have been operational for well over a century, we will consider a different way to construct such timetables. In particular, we will use a fairly new type of mathematics - *max-plus algebra* - to develop and optimize timetables that can also handle (the inevitable occurrence of) delays. It is anticipated that such novel ways of looking at 'old' problems can provide further insight into optimization schemes; indeed, max-plus algebra has been applied in this sense to successfully describe the comings and goings of the Dutch railway system.

Assumptions

We will work at a simplified level, so that a functional railway timetable satisfies the following criteria:

1. Fixed travel times (between stations)
2. High frequency of departures at each station
3. Regular departures: one might call this "periodic" with departures occurring from a station every 30 minutes, say
4. Trains scheduled to depart must wait for all arriving trains before departing (to allow for changeover of passengers)
5. Departures occur as soon as possible, once item 4 is satisfied.

Example

Criterion 4 may be summarised as the "synchronization" criterion. It is this criterion that yields a mathematical model that is naturally described by max-plus algebra. Consider Figure 1, showing a total of four trains in a small model - one on each track. The travel times between the stations are as indicated on the arcs, where an arc pointing from station S_i to itself represents a round trip from S_i to some suburbs of that city and back again. There are two outgoing tracks (hence trains) from station S_1. Due to cri-

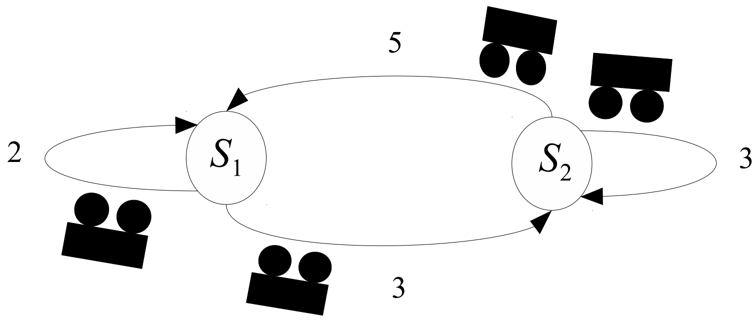

Figure 1: Railway network comprising two stations S_1 and S_2.

teria 4 and 5, these two trains depart at the same time and we denote this time by x_1. Similarly, x_2 is the common departure time of the two trains at S_2. Together, the departure times are written as the vector $\mathbf{x} \in \mathbb{R}^2$. The first departure times will be given by $\mathbf{x}(0)$. The trains thereafter leave at the time instants given by the two elements in the vectors $\mathbf{x}(1), \mathbf{x}(2), \ldots$. The kth departure times are indicated by $\mathbf{x}(k-1)$.

Because of the rules given above, particularly criterion 4, we have that departures from S_1 must wait for the train arriving from the same station (due to the previous departure), which takes 2 time units, as well as the train arriving from S_2, which takes 5 time units. Therefore, the earliest time (as implied by criterion 2) of departure at S_1 is given by the following model.

$$x_1(k+1) = \max(x_1(k) + 2, x_2(k) + 5). \tag{1}$$

Similarly, the departure times at S_2 must satisfy

$$x_2(k+1) = \max(x_1(k) + 3, x_2(k) + 3). \tag{2}$$

Then, if the initial departure times $\mathbf{x}(0)$ are given, all future departure times are uniquely determined. If, for example, $x_1(0) = x_2(0) = 0$, then we obtain the following sequence $\mathbf{x}(k)$, for $k = 0, 1, 2, \ldots$.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 11 \end{pmatrix}, \begin{pmatrix} 16 \\ 16 \end{pmatrix}, \ldots .$$

The following sequence is obtained if the initial departure times are $x_1(0) = 1$ and $x_2(0) = 0$, i.e. the first trains at S_2 (one in each direction) still leave at time 0, but the first trains at S_1 now leave at time 1.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \end{pmatrix}, \begin{pmatrix} 13 \\ 12 \end{pmatrix}, \begin{pmatrix} 17 \\ 16 \end{pmatrix}, \ldots .$$

Define the interdeparture time as the time duration between two subsequent departures along the same track. Then we see that both sequences have the same *average* interdeparture time equal to 4. The second sequence, however, has exactly this interdeparture time, whereas the first sequence has it only on average (i.e. the average of the interdeparture times 3 and 5). Thus, if these sequences were real timetable departures, with interdeparture times taken as performance measures, then most people would prefer the second timetable since it is regular.

Max-plus algebra

Equations (1) and (2) show that this problem is nonlinear. Max-plus algebra however converts this into a linear model. This is advantageous as familiar ideas such as linear systems of matrices and vectors, eigenvalues/eigenvectors come into play.

At its fundamental level, therefore, the mathematics of max-plus algebra is not a big departure from conventional linear algebra. A major difference is that the operations of multiplication and addition (as used for matrix-vector multiplication) are now replaced by addition and maximisation, respectively. Once this is embedded in the minds of the student, the rest of the work follows smoothly. Indeed, the synchronizations as described by (1) and (2) are recurrence relations, which may then be easily programmed. Some elementary graph theory is also required to abstract the railway network; in particular, circuits in the network will have important roles. The network itself will be extracted from a real railway network (most likely Spanish); thus, a comparison of the current timetable and the timetables proposed here is anticipated.

Modelling plan

Resources

- Graph theory: understanding of circuits
- Linear algebra and max-plus theory: matrices, vectors, eigenvalues, eigenvectors. Textbook and succinct worksheets will be available to help
- Straightforward programming (in MATLAB or similar) to find performance measures of the network

Tasks
The items below will be punctuated by background work to familiarise the student with max-plus algebra.

- Abstract the Spanish railway network into a simple graph theoretical interpretation
- Find total number of trains on the current Spanish railway network
- Allocate 1 train per track and find performance measure
- Delay consideration: How does the timetable respond to varying levels of delay?
- Discuss possible improvements, find new performance measure, and new response to delays
- Repeat above as far as feasible (bearing in mind the upper limit for number of trains that the network can accommodate)
- Discuss practicality of results obtained; compare with actual Spanish timetable