



Some Basic Mathematical Elements On Steel Heat Treating: Modeling, Freeware Packages And Numerical Simulation

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The basic elements of the mathematical modeling of steel heat treating, for the heating-cooling industrial procedure applied to a helical gear.

Every year since 2007, in June, a Modeling Week is held within the master program of the faculty of Mathematics of Universidad Complutense of Madrid (UCM) in cooperation with the Institute of Interdisciplinary Mathematics (IIM). During this week, students work in small groups on real industrial problems proposed

by companies under supervision of one or two qualified instructors. As it was announced in 2013, the main purpose of the VII Modeling Week was to promote the use of mathematical methods and models in research, industry, innovation, and management in the knowledge economy. That year, Prof. Ortigón Gallego

was invited to the seventh edition of this event as an instructor of one intensive course on the subject. "Some mathematical models and numerical simulation of heat treatment of steel." During this course, the basic elements of the mathematical modeling of steel heat treating were described, a very simple 3D model was



Figure 2: Induction hardening machine at work during the heating process applied to a wheel gear. Courtesy of the Stiftung Institut für Werkstofftechnik - Bremen, Germany.
(Foundation Institute of Materials Science. IWT Bremen)

considered, without taking into account mechanical effects, for the heating-cooling industrial procedure applied to a helical gear (Figure 1). Then we perform some numerical simulations of this model according to the industrial heating technique: induction or flame hardening. To do so, we provide the students with some efficient noncommercial software packages, namely Freefem++ ([1]), MEdit ([4]) and gmsh ([6]).

Freefem++ is used for preprocessing (building the tetrahedralization of the helical gear, the coil, etc.) and processing (in order to obtain the numerical approximation). Freefem++ also provides the user with some tools for postprocessing, mainly for the graphical representation of the solution of 2D problems, but it is more convenient to use gmsh for the solution of 3D problems. MEdit is a useful package for 3D visualization and can be used within Freefem++ whereas gmsh should be run in its own session.

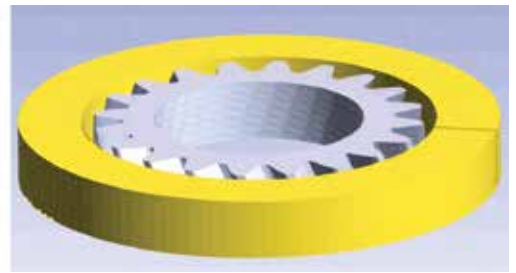
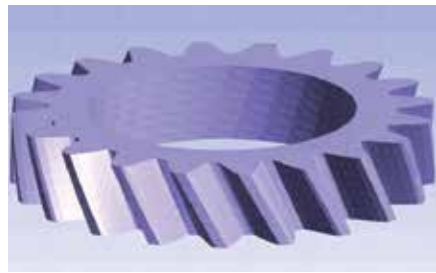


Figure 1: A specimen of a helical gear. Typical setting of the coil and the helical gear for induction hardening. These pictures have been obtained by using the postprocessing software gmsh [6].

INDUCTION HARDENING OF A HELICAL GEAR

There are many industrial hardening procedures for gears. This may depend on the size of the gear and on the final purpose of the workpiece. Induction hardening is a very general procedure widely used in the manufacture of gears. It can be applied in different manners ([2, 3, 12, 13]): (a) total induction around the workpiece, by the use of a coil surrounding the gear, or (b) local induction, by a tooth-by-tooth or a gap-by-gap techniques. In the first case, the encircling coil hardens the teeth from the tips downward. This pattern may be acceptable for certain types of gears such as splines and some gearings. Another induction technique producing a uniform contour hardening makes use of multi-frequency induction ([8, 9, 10, 12]).

Depending on the workpiece geometry and size, the heating stage can be accomplished by an induction procedure or by direct flame. Induction techniques have been successfully used in industry since the last century. During a time interval, a high frequency current passes through a coil generating a high alternating magnetic field, which induces eddy currents in the workpiece, which is placed close to the coil. The eddy currents dissipate energy in the workpiece producing the necessary heating. Figure 2 shows an induction machine during the heating process applied to a gear; we can see the coil encircling the gear.

In this paper, we first consider a helical gear encircled by a coil (made of copper) as shown in Figure 1.

In our numerical simulations we have considered very simple models for both, induction hardening and flame induction of a helical gear. We have not taken into account mechanical effects describing the deformation of the workpiece and have kept only the dynamics of the austenite and martensite phase fractions. For more complete models, the reader is referred to [5, 8, 10, 11].

The mathematical model is divided into two stages: (a) heating, and (b) cooling. During the first stage, heat is produced by electromagnetics, letting pass a high frequency current through the

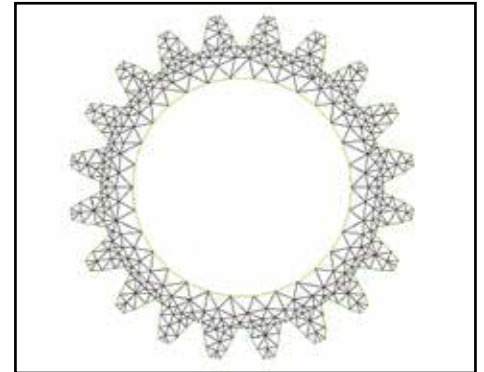


Figure 3: The gear base. The outer boundary is parametrized according to [1].

	Vertices	Triangles	Tetrahedra
Gear	2598	4696	8137
Coil	8936	7536	41496
Cube	14961	14134	89720

Table 1: Number of vertices, triangles and tetrahedra in each of the three meshes.

coil. Here, the main variables are the electric potential, the magnetic potential, the temperature and phase fraction corresponding to the austenite transformation. At the beginning of the cooling stage, the current is switched off, and the workpiece is aquaquenched; main variables in this stage are the temperature and the phase fractions corresponding to austenite and martensite.

BUILDING A 3D TRIANGULATION OF A HELICAL GEAR

This section is devoted to the tetrahedralization of a helical gear. As it was stated above, we only use noncommercial software: Freefem++, TetGen ([7]), MEdit and gmsh.

Freefem++ is an integrated software for obtaining the finite element method numerical solution to boundary value problems for PDE of the elliptic type. By means of some semidiscretization in time scheme, Freefem++ may also deal with parabolic/hyperbolic evolution problems. This includes problems arising in solid mechanics, fluid mechanics, plasma physics, etc. This

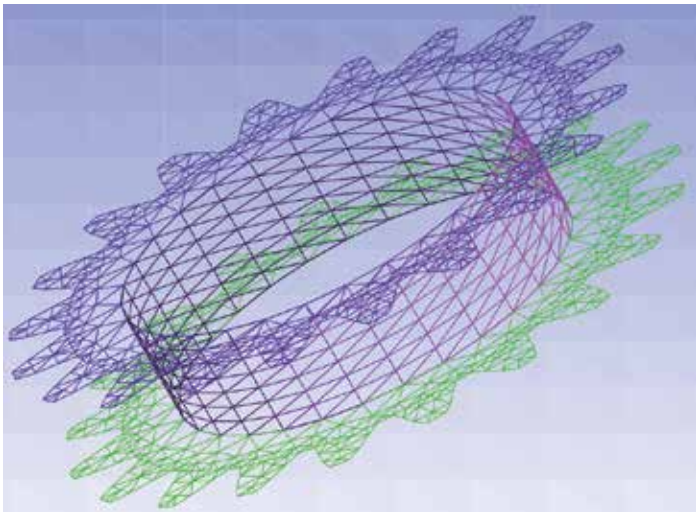


Figure 4: 3D mesh of base, inner cylindrical wall and lid of the gear.

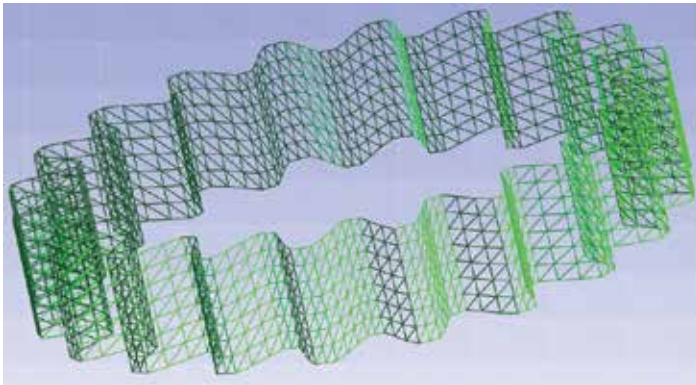


Figure 5: 3D mesh of the gear tooth surface.

software can be freely downloaded from the Internet (<http://www.freefem.org/ff++/index.htm>) and it is fully documented ([1]). Though initially developed by O. Pironneau (Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris) in the 1980s, today it is maintained by F. Hecht, (Laboratoire JLL).

The first thing to do is a triangulation of the gear base. The base has two borders, namely, a tooth boundary and an inner circle. The way to produce the tooth boundary is to use a truncation of a cosine function oscillating around an outer circle. Let R be the outer circle radius, then a parametrization of the teeth is of the form

$$\begin{cases} x = (R + d(t)) \cos t, \\ y = (R + d(t)) \sin t, \quad t \in [0, 2\pi) \end{cases} \quad (1)$$

where $d(t)$ is an oscillating function. For instance, if we want a m -tooth gear we may take

$$d(t) = \min(0.8l, \max(-0.8l, l \cos(mt + \pi))), \quad (2)$$

where l is the amplitude. Notice that, in order to produce a flat tip and bottom, we are also truncating the cosine by 20% its height and depth (Figure 3).

To start up, we load some necessary packages for 3D meshes.

```
load "msh3" // add general 3D functionality 1
load "tetgen" // 3D tetrahedralization 2
load "medit" // we may call MEdit just 3
// for 3D visualization 4
```

```
real radius = 0.2; // gear radius (cm) 5
int numteeth = 20; // number of teeth 6
real toothheight = radius/8.; 7
```

Then, we introduce some useful parameters.

```
8 //
9 //The inner cylinder has radius
10 // equals to radiusportion*radius
11 real radiusportion = 0.7;
12
13 // Half of the gear thickness (cm)
14 real thickness = 0.04;
15
16 //Number of points in the tooth boundary
17 int npteeth = 200;
18
19 // Number of points in the inner ring
20 int npinnerring = 30;
21
22 //Number of layers in the 3D mesh
23 // of the gear and coil
24 int nlayer = 7;
25 //
```

Here comes the implementation of the functions like (1) describing the boundaries.

```
26 //
27 // Truncated teeth by 20% tip and bottom
28 func real tooth(real t){return
29     min(0.8*toothheight,
30     max(-0.8*toothheight,
31     toothheight*cos(numteeth*t+pi))};
32
33 func real coronax(real t){return
34     (radius + tooth(t))*cos(t);}
35
36 func real coronay(real t){return
37     (radius + tooth(t))*sin(t);}
38
39 //Outer boundary
40 border dentadura(t=0,2*pi){
41     x=coronax(t);
42     y=coronay(t);label=123;}
```

The inner boundary is just a circle.

```
43 //Inner boundary
44 border circle1(t=0,2*pi){
45     x=radiusportion*radius*cos(t);
46     y=radiusportion*radius*sin(t);}
47
48 plot (dentadura(npteeth)
49     + circle1(npinnerring));
50 //
```

Finally, we generate the mesh. Notice that the number of the boundary points on the border circle1 is negative. This means that this arc is traversed clockwise and this results in a hole inside the domain. Also, the directive fixed border=1 means that during the automatic generation of the mesh, the original points on the boundary will be vertices of the triangulation. This is a very important issue when dealing with 3D meshes.

```
mesh Th=buildmesh(dentadura(npteeth) + 51
    circle1(-npinnerring), fixeborder=1); 52
plot (Th, ps="gearbase.eps"); 53
// 54
55
```

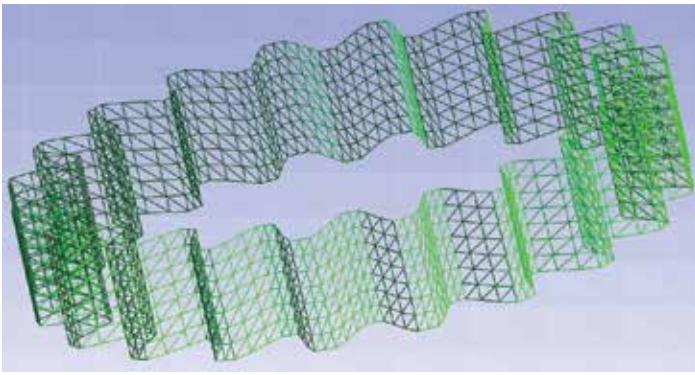



Figure 6: 3D mesh of the full gear boundary.

In order to build the gear mesh, we need to generate a 3D mesh of the whole workpiece boundary. In this case, we may distinguish four different surfaces: the base, the lid, the cylindrical inner wall and the tooth surface. Thus we build a two dimensional mesh in 3D for each of these surfaces. Then we stick out these meshes together forming a closed 3D mesh of the gear skin. This closed 3D mesh is used as a datum to generate the final gear tetrahedralization via TetGen.

Here we transform the 2D mesh Th into 3D (flat) objects, namely the base ($Thbase$) and the lid ($Th tapa$). These surfaces are assigned the reference numbers 76 and 77, respectively.

```

56 int[int] labels = [0,76];
57 mesh3 Thbase = movemesh23(Th,
58     transfo=[x,y,-thickness],
59     label=labels);
60 labels = [0,77];
61 mesh3 Th tapa = movemesh23(Th,
62     transfo=[x,y,thickness],
63     label=labels);

```

Then we generate the cylindrical inner wall. To do so, we first build a 2D rectangular mesh and transform it into the desired 3D mesh with reference number 79.

```

64 //
65 //Inner wall
66 real x0=0., x1=2.*pi,
67     y0=-thickness, y1=thickness;
68 //Auxiliary mesh of a 2D rectangle
69 mesh Thauxi =
70     square(npinnering,nlayer,
71     [x0+(x1-x0)*x,y0+(y1-y0)*y]);
72 func XX3 = radiusportion*radius*cos(x);
73 func YY3 = radiusportion*radius*sin(x);
74 func ZZ2 = y;
75 labels = [0,79];
76 mesh3 Thlaterali = movemesh23(Thauxi,
77     transfo=[XX3,YY3,ZZ2],
78     label=labels);
79

```

We stick these three meshes together, just to see that everything is OK. You can do so by saving this new mesh in a file that can be read with gmsh (or you just can call MEdit).

```

80 mesh3 Thprogress = Thbase
81     + Thlaterali + Th tapa;
82 savemesh(Thprogress,"boundaryprogress.mesh");
83 // Interactive call for MEdit
84 //medit("Boundary progress", Thprogress);
85

```

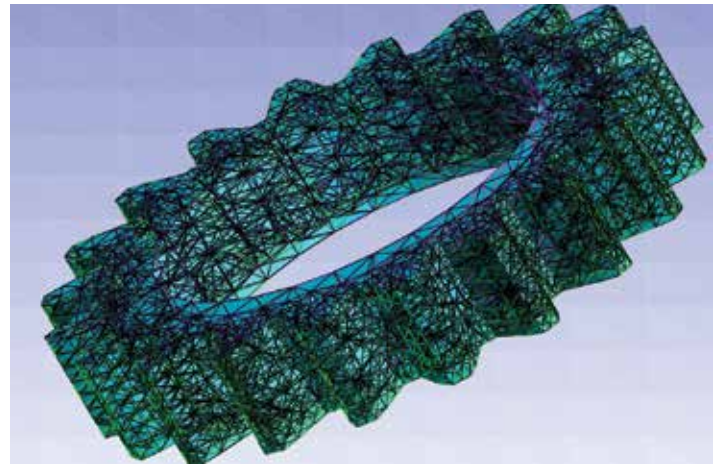


Figure 7: 3D tetrahedra mesh of a straight gear obtained with TetGen using the closed 3D boundary in Figure 6 as a datum.

Figure 4 shows the resulting mesh $Th_{progress}$. In the same way, the lateral tooth surface is obtained by a suitable transformation of a 2D rectangular mesh.

```

86 // Tooth surface
87 mesh Thauxe =
88     square(npnteeth,nlayer,
89     [x0+(x1-x0)*x,y0+(y1-y0)*y]);
90
91 func XX4 = coronax(x);
92 func YY4 = coronay(x);
93
94 labels = [0,78];
95
96 mesh3 Thlateralo = movemesh23(Thauxe,
97     transfo=[XX4,YY4,y], label=labels);

```

Now we may stick out the four surface meshes together to produce the full gear skin (Figure 6). Notice that the contact points must fit in well.

```

98 //The full gear skin
99 mesh3 GearSkin = Thbase + Th tapa
100     + Thlaterali + Thlateralo;

```

Using the 3D skin mesh $GearSkin$ as a datum to TetGen, a straight gear tetrahedralization is generated (Figure 7).

```

101 //3D gear mesh (tetrahedra)
102 real[int] domain =
103     [1.05*radiusportion*radius,0.,0.,145,0.1];
104
105 mesh3 ThGear3 =
106     tetg(GearSkin, switch="paAAQY",
107     nbofregions=1,
108     regionlist=domain);

```

The final step is to produce a twist onto the straight gear mesh resulting in the helical gear mesh shown in Figure 8.

```

109 //The helical twist
110 func XX5 = cos(2*z)*x + sin(2*z)*y;
111 func YY5 = -sin(2*z)*x + cos(2*z)*y;
112 mesh3 ThGear =
113     movemesh3(ThGear3,
114     transfo=[XX5,YY5,z]);
115 //medit("Helical Gear", ThGear);
116 // We save the mesh
117 savemesh(ThGear,"gear.mesh");

```

Remark: We have not explained every Freefem++ keyword appearing in these scripts. Most of them are self-explanatory but there are many fine points and keywords that admit their own different options. To learn more about this, the reader is referred to the various documentation available on the Internet (see, for instance, [1, 4, 6, 7]).

In the same way, we may generate a 3D tetrahedra mesh of the coil. Figure 9 shows the 3D skin mesh of the coil-gear setting. Finally, we need a large enough set containing both, the gear and the coil. This set is where the magnetic vector potential is to be computed during the heating stage. We may choose the shape of this set as simple as possible, for instance, a box or cube. Figure 10 shows the whole setting cube-coil-gear. Table 1 gives some information about the three generated meshes. It took Freefem++ 0.016275 seconds to compile the script and 2.92382 seconds to execute it on a MacBook Pro, taking into account the time spent on transferring the mesh files to the hard disk.

NUMERICAL SIMULATIONS

In this section we show some numerical simulations using the meshes and setting described in the previous section. The heating stage takes $T_a = 5.5$ seconds and the frequency is $\omega = 900$ Hz. The cooling stage takes another 9.5 seconds, that is $T_m = 15$ seconds.

Figures 11:(a)-(f) show some numerical results at two different instants of time, namely at the end of the heating stage, $t = 5.5$, and at the end of the cooling stage, $t = 15$ for the austenite, martensite and final temperature. As it is expected, induction hardening heats up the gear from the tooth tip downward, so that the whole tooth is austenized (Figure 11:(a),(b)) and transform into martensite after cooling (Figure 11:(d),(e)). Figure 11:(c) shows that almost all the austenite has been transformed, retaining the workpiece a very small amount of this phase fraction.

Another industrial technique used in the heat treatment of gears is flame hardening. This procedure can be used for both small and large gears. Usually, two techniques can be utilized: spinning and progressive heating ([12, 13]).

Figure 12:(a)-(f) show some numerical results corresponding to a flame hardening simulation. A more uniform contour can be observed in this case for both the austenite and martensite profiles, which is in good agreement with experimental results.

CONCLUSIONS

Some simplified mathematical models arising in the industrial processes of steel heat treating have

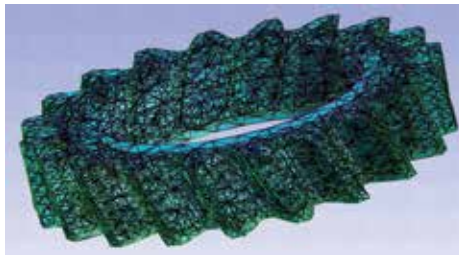


Figure 8: Final 3D tetrahedra mesh of the helical gear obtained by twisting the mesh shown in Figure 6.

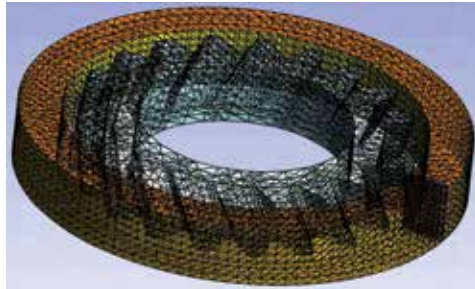


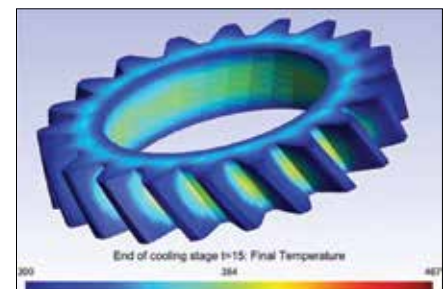
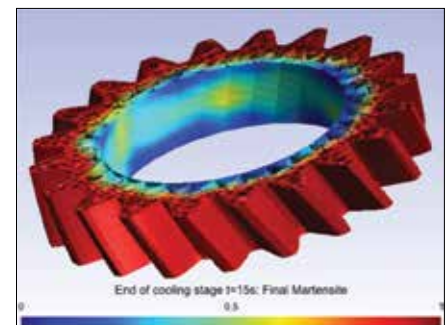
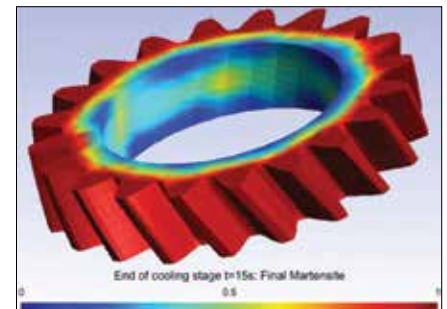
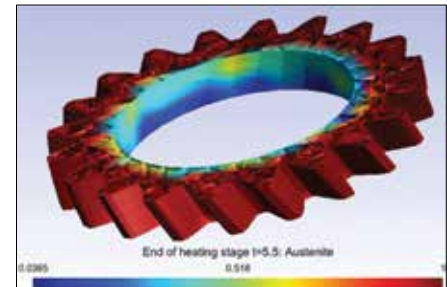
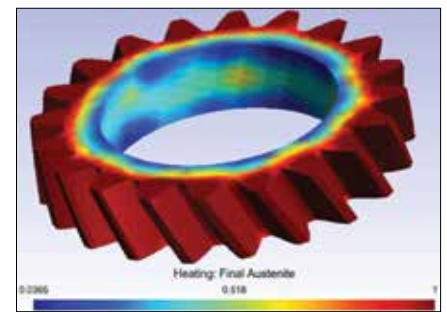
Figure 9: Coil-gear setting.

been used for the numerical simulation of the hardening (induction or flame) of a helical gear. The numerical simulations have been performed with freeware packages available in the Internet, namely Freefem++, TetGen and gmsh. We have described how to do, step by step, a 3D mesh for a helical gear. This presentation should be considered at an academic level. We have tried to put in evidence the potentiality and performance of Freefem++ as a useful FEM tool to carry out fast and efficient numerical simulations in a complex 3D geometry. Of course, it is possible to consider more sophisticated models for the description of the industrial processes of steel heat treating. For instance, by including mechanical effects, deformations, stresses, etc., other phase fractions such as bainite and pearlite. This does not represent a constraint to Freefem++ since it can handle more complicate situations. Though the 3D meshes used here are coarse, in general the numerical results are in good agreement with known experimental results. We hope that the reader gives a try to these packages and convinces himself of its easy-to-use, versatility, robustness and efficiency.

ACKNOWLEDGEMENTS

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Figure 11: Induction hardening. [a] Austenite at the end of the heating stage $t = 5.5$ s. [b] Austenite at the end of the heating stage $t = 5.5$ s. [c] Austenite at the end of the cooling stage $t = 15$ s. [d] Martensite at the end of the cooling stage $t = 15$ s. [e] Martensite at the end of the cooling stage $t = 15$ s. [f] Temperature at the end of the cooling stage $t = 15$ s.



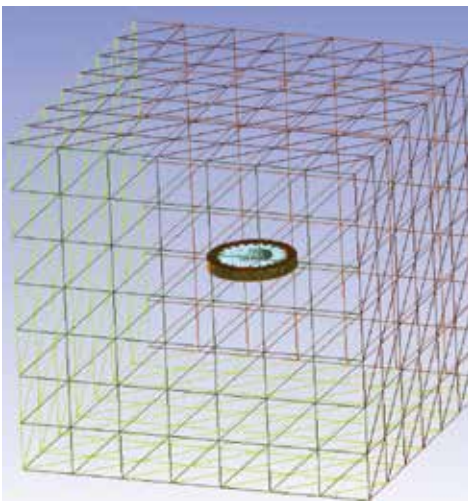



Figure 10: Cube-coil-gear setting.

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REFERENCES

- [1] S. Auliac, A. Le Hyaric, J. Morice, F. Hecht, K. Ohtsuka, O. Pironneau, FreeFem++. Third Edition, Version 3.31-2, 2014. <http://www.freefem.org/ff++/ftp/freefem++doc.pdf>
- [2] N. Barka, A. Chebak, A. El Ouafi, "Simulation of Helical Gear Heated by Induction Process Using 3D Model," *Advanced Materials Research*, 658, 266–270, 2013.

Figure 12: Flame Hardening. (a) Austenite at the end of the heating stage $t = 5.5$ s. (b) Austenite at the end of the heating stage $t = 5.5$ s. (c) Temperature at the end of the heating stage $t = 5.5$ s. (d) Martensite at the end of the cooling stage $t = 15$ s. (e) Martensite at the end of the cooling stage $t = 15$ s. (f) Temperature at the end of the cooling stage $t = 15$ s.

- [3] N. Bugliarello, B. G. Corgé, D. Giessel, D. McCurdy, R. Perkins, S. Richardson, C. Zimmerman, Heat treat processes for gears, *Gear Solutions* 38–51, July 2010.
- [4] P. J. Frey, MEdit: "An interactive mesh visualization software," Rapport technique INRIA N°. 0253, 2001. <http://www.ann.jussieu.fr/~frey/publications/RT-0253.pdf>
- [5] J. Fuhrmann, D. Hömberg and M. Uhle, "Numerical simulation of induction hardening of steel," *COMPEL*, 18, No. 3, 482–493, 1999.
- [6] C. Geuzaine J. F. Remacle, Gmsh Reference Manual, 2014. <http://geuz.org/gmsh/doc/texinfo/gmsh.pdf>
- [7] Hang Si, TetGen: "A Quality Tetrahedral Mesh Generator and a 3D Delaunay Triangulator," <http://tetgen.berlios.de/>, 2009.
- [8] D. Hömberg, "A mathematical model for induction hardening including mechanical effects," *Nonlinear Analysis: Real World Applications*, 5, 55–90, 2004.
- [9] D. Hömberg, Thomas Petzold, Elisabetta Rocca, "Simulation of multifrequency-induction-hardening including phase transitions and mechanical effects," WIAS Preprint No. 1975, 2014.
- [10] D. Hömberg, Q. Liu, J. Montalvo-Urquizo, D. Nadolski, T. Petzold, A. Schulz, "Analysis and simulation of multifrequency induction hardening", WIAS Preprint No. 1910, 2014.
- [11] J. Montalvo Urquizo, Q. Liu, A. Schmidt, "Simulation of quenching involved in induction hardening including mechanical effects," *Computational Materials Science*, 79, 639–649, 2013.
- [12] E. J. Otto, D. H. Herring, "Gear Heat Treatment. Part I," *Heat Treating Progress*, June, 2002.
- [13] V. Rudnev, D. Loveless, R. Cook, M. Black, "Induction Hardening of Gears: a Review," *Heat Treatment of Metals*, 4, 97–103, 2003.

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