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VII UCM MODELLING WEEK

Modelling flow in pipes with semi-permeable walls (problem 1)

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1. The approach to the problem

1.1. Main applications

The modeling of the dynamics of flows through porous pipes is useful for industrial, agriculture, medical and domestic applications.

Concerning industrial, or domestic, applications polymeric membranes are very often used in filtering devices. Modern filtration modules consist of a container housing a large number of membranes which can have various shapes and can be used in different ways. Referring to modules for ultrafiltration, typical values of the internal and external fiber radius are 0.15 mm and 0.35 mm, respectively, the typical fiber length is 1 m, and the membrane pores diameter is about 0.1 μm. The membrane permeability ranges around $10^{-15} \div 10^{-16}$ m^2. Each module usually houses approximately 3000 hollow fibers and the total discharge is about 150 lt/h, when the pressure difference between inlet and outlet is $10^4 \div 10^5$ Pa.

Concerning the agriculture applications, a widely used irrigation technique consists in delivering water by letting it filtrate through pipes laid down or suspended over the ground. Two types of plants are used, according to the size of the field to be irrigated. In small plants pipes made of a permeable and flexible material (e.g. canvas) and having length of the order of 100 m and radius of the order of 1 cm are laid on the ground and connected to a reservoir (a simple barrel) whose capacity is of the order of 1 m^3 located in an elevated position so that the driving pressure is provided by gravity. In large plants pipes length is of the order of 1 km, with a diameter of the order of 10 cm (maximum). A much larger pressure is required, supplied by pumps, and therefore the pipe wall has to be thick and impermeable. Some artificial permeability is produced, for instance, by drilling dripping holes along the pipe, or anyway creating periodical permeable windows. In all cases the end of the pipe is sealed so that the flow takes place in the so called dead end configuration.

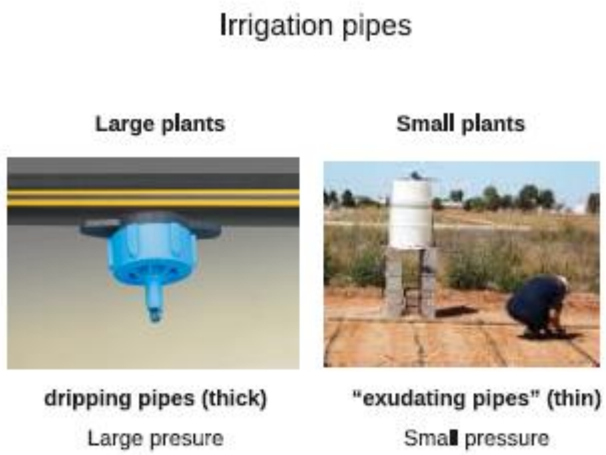

Fig. 1 Irrigation pipes

One of the most important application is dialysis, which is a process for removing waste product and excess water from the blood, and used primarily as an artificial replacement for lost kidney function in people with renal failure.

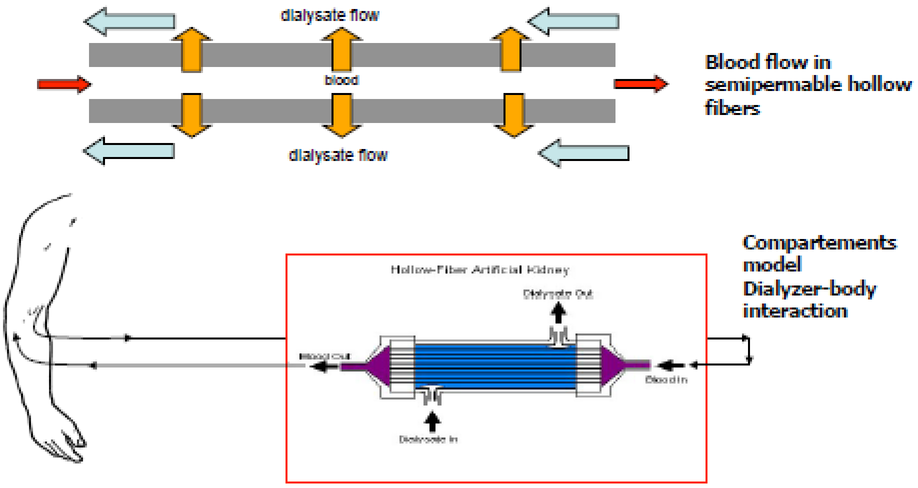

Fig. 2

Concerning artificial kidney dialysis, bundles of hollow fibers are sealed into cylindrical casing.

Fig. 3

Blood flows in the hollow fibers while dialysate flows outside see **Fig.** 2. Typical blood flow ratesare about 10 lt/h, and the membrane permeability is $10^{-16} \div 10^{-17}$ m^2. Fibers inner radius is 0.1mm, while the fiber thickness ranges around 50 μ m and the fiber length is 20 $\div$ 30 cm. In this case the filtration, or better the ultrafiltration, process is generated by osmotic and hydrostatic pressure drops over the porous membrane.

Concerning the applications above mentioned, there are essentially two flow regimes:

- Dead-end configuration, meaning that all the injected fluid is forced to cross the membrane (e.g.agriculture and some filtering devices).
- Continuous open shell mode (e.g. dialysis devices). In this configuration, the fluid enters thefiber from one side and exits from the other, but the exit flow dividedby the entrance flow is less than one.

However, the two regimes differ only in boundaryconditions.

1.2. How do hollow fibers filters work?

Typically a hollow fiber used for filtering purposes consist in a tube, i.e. a cylinder, whose walls (membrane) are porous. Usually a liquid solution is flowing in the inner channel, the flow being caused by a pressure differences between the inlet and the outlet or by gravity when the pipe is in a vertical position. In this report, we will consider open cylinders where the outlet flow divided by the entrance flow is less than one, i.e. a part of the liquid is passing through the pores of the membrane. Such a lateral flow is caused by a pressure difference between the channel and the exterior, the so-called **Trans-Membrane Pressure (TMP)**.

Fig. 4

We remark that, when dealing with semi–permeable membranes, we have to distinguish various classesof solutes possibly transported by the fluid:

- Solutes whose molecules are small enough to be transported across the membrane.
- Solutes whose molecules size and concentration are in the range to produce a non–negligibleosmotic pressure.

In this report we will consider the second hypothesis. The consequences of this hypothesis will be developed in the following sections.

1.3. Difficulties of the model

We focus on the flow of a solution (a two component mixture: solute and solvent) in a hollow fiber whose membrane is semipermeable: i.e. only the solvent can permeate through the membrane. Therefore a difference in solute concentration between inside and outside appears. This difference may give rise to a (possibly not negligible) osmotic pressure that counteracts TMP, and so opposes the lateral flow. Such a problem is quite complicated, for two reasons:

- the concentration of the solute responsible for osmosis evolves along the fiber;
- the osmotic pressure (being concentration dependent) contrasts the transmembrane flow (and the global pressure is variable).

We will illustrate the osmotic phenomenon in the next section.

1.4. The osmosis phenomenon

Osmosis it is the phenomenon of water flow through a semi-permeable membrane that may even stop the transport of salts or other solutes through it. Osmosis is a fundamental effect in all biological systems. It is applied to water purification and desalination, waste material treatment, and many other chemical and biochemical and industrial processes.

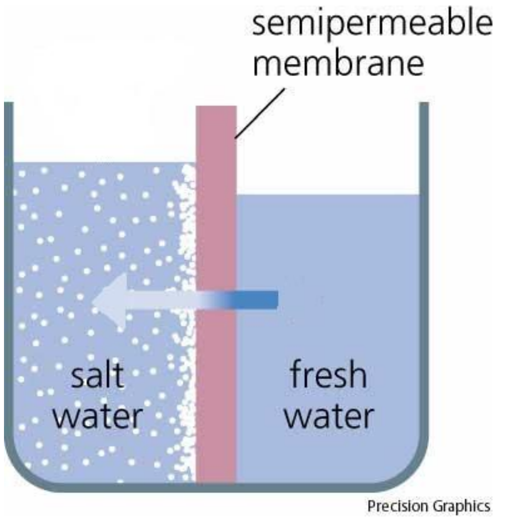

Fig. 5 Transfer water by osmosis

When two water (or other solvent) volumes are separated by a semi-permeable membrane, water will flow from the volume of low solute concentration, to the one in which the solute concentration is higher.

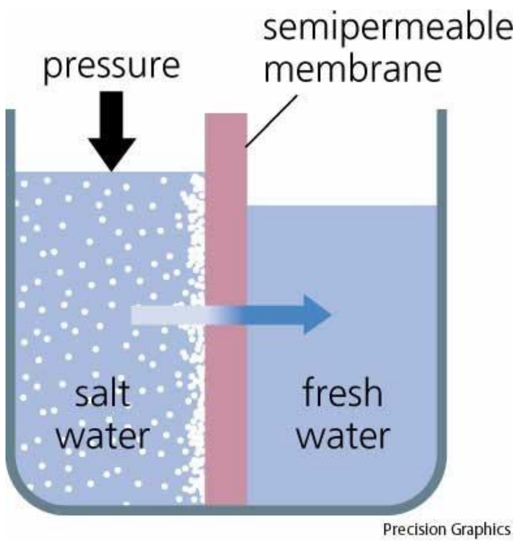

Fig. 6 Transfer water by reverse osmosis

If there are solute molecules only in one volume of the system, then the pressure on it, that stops the flow, is called the osmotic pressure.

The osmotic pressure does not depend on the solute type, or on the molecular size, but only on molar concentration, the Morse equation is:

$$P_{os}^{*} = R \cdot T \cdot c_{s}^{*}$$

where $R^{*} = 0.082\ \frac{lt \cdot atm}{mol \cdot K}$, is the ideal gas constant and T^{*} is the absolute temperature on °K.

In our framework P_{os}^{*} is to be compared with the characteristic transmembrane pressure (TMP), given rise to the dimesionless ratio:

$$Os = \frac{P_{os,c}^{*}}{\Delta p_{m,c}^{*}}$$

2. Geometry of the system

We define the parameter of the cylinder:

- L^* : length of the pipe
- R^* : radius of the section of the inner channel
- H^* : radius of the section of the pipe
- $S^*=H^*-R^*$ thickness of the porous membrane

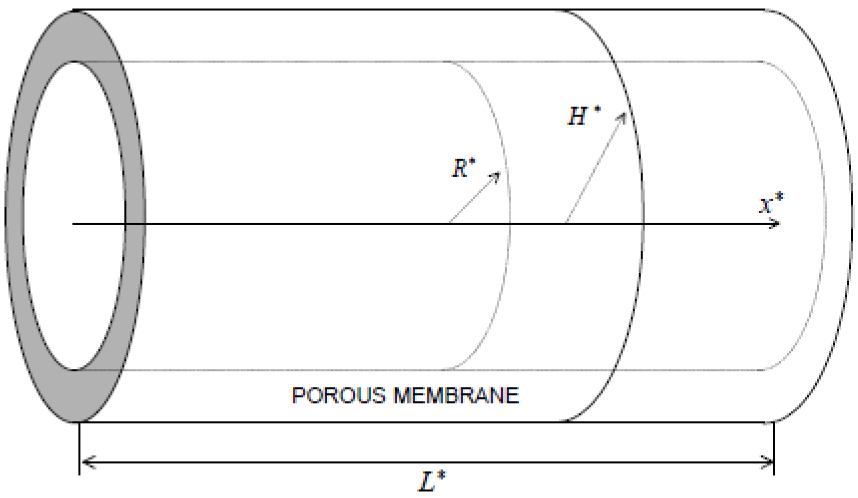

Fig. 7

We can consider that S^* and R^* have the same order of magnitude and we define $\varepsilon = \frac{R^*}{L^*} \approx 10^{-3} \div 10^{-4}$ dimensionless. The fact that the ratio between the radius and the length of the pipe is small, is one of the most important aspects of our model because it allows us to really simplify equations.

The longitudinal space variable is denoted by x^* and the radial variable is r^* where $0 \leq x^* \leq L^*$, $0 \leq r^* \leq R^*$ for the inner channel and $R^* \leq r^* \leq H^*$ for the porous membrane.

In order to simplify the problem we consider dimensionless variables by rescaling the longitudinal component using L^* and the radial one using R^*. Hence, we obtain:

$$\begin{cases} x = x^*/L^* \\ r = r^*/R^* \end{cases}$$

3. Mathematical model

3.1. Definition of the dependent variables and possible simplifications

We first describe the solution parameters:

- The concentration of the solution:

$$c_s^* = \frac{number\ of\ moles\ of\ solute\ dissolved\ into\ the\ mixture}{volume\ occupied\ by\ the\ pure\ liquid};\ [c_s^*] = mol/lt$$

- The volume fraction:

$$\nu = \frac{\Delta V_s^*}{\Delta V^*}$$

Where ΔV_s^* is the volume occupied by the solute and ΔV^* is the volume occupied by the solution as a whole. Then we have:

$$\Delta V_s^* = \nu \Delta V^*$$

Considering $\Delta V_f^* = \Delta V^* - \Delta V_s^*$ the volume of the pure fluid, we have:

$$\Delta V_f^* = (1-\nu)\Delta V^*$$

Hence, the relation between the concentration and the volume fraction is:

$$c_s^* = \theta_s^* \frac{\nu}{1-\nu} \Leftrightarrow \nu = \frac{\frac{c_s^*}{\theta_s^*}}{\frac{c_s^*}{\theta_s^*} + 1}$$

where, considering the molar mass $\mathfrak{M}_s^*$ and the density of the pure solute ρ_s^* , the molar density is:

$$\theta_s^* = \frac{\rho_s^*}{\mathfrak{M}_s^*} [\theta_s^*] = mol/lt$$

If the volume fraction is small (and this is the case we are considering) we can approximate the concentration formula in the following way $c_s^* = \theta_s^* \cdot \nu$.

We are now describing the flow by defining the velocity expressed in term of its radial and the longitudinal components.

We first consider the velocity in the inner channel, $\overrightarrow{v^*}$. Supposing that the longitudinal velocity of the solute and the fluid is almost the same and that the radial velocity of the solute is zero we obtain:

$$\begin{cases} v_x^* = v_{f,x}^* = v_{s,x}^* \\ v_{f,r}^* = \dfrac{1}{1-\nu} v_r^* \iff v_r^* = (1-\nu) v_{f,r}^* \end{cases}$$

In order to obtain dimensionless velocity we define a characteristic velocity:

$$v_c^* = \frac{\dot{V}_c^*}{\pi R^{*2}}$$

Where $\dot{V}_c^*$ is the inlet discharge. Hence, we found the following:

$$\begin{cases} v_x = \dfrac{v_x^*}{v_c^*} \\ v_r = \dfrac{v_r^*}{\varepsilon v_c^*} \end{cases}$$

In the same way we consider the velocity in the porous membrane, $\overrightarrow{q^*}$ and we obtain the dimensionless components:

$$\begin{cases} q_x = S\varepsilon \dfrac{q_x^*}{q_c^*} \\ q_r = \dfrac{q_r^*}{q_c^*} \end{cases}$$

where the characteristic velocity in the porous membrane is

$$q_c^* = \frac{\dot{Q}_c^*}{2\pi \cdot R^* \cdot L^*}$$

Being $\dot{Q}_c^*$ is the-characteristic discharge through the membrane.

The flow through the pipe is drove by the differences of pressure hence we settle $\boldsymbol{P}_{in}^*$, $\boldsymbol{P}_{out}^*$ and $\boldsymbol{P}_m^*$ as we can see in the figure:

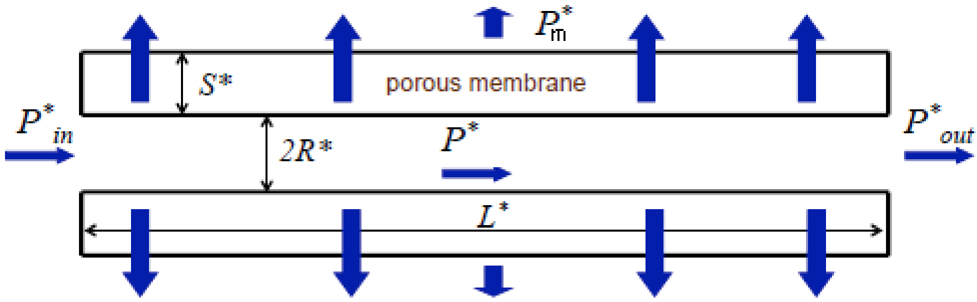

Fig. 8

We consider the following rescale

$$p^* = \boldsymbol{P}^* - \boldsymbol{P}^*_{in}$$

$$p^*_m = \boldsymbol{P}^*_m - \boldsymbol{P}^*_{in}$$

Next, we define the characteristic value for p^* and p^*_m as follow: $\Delta p^*_c = p^*_{in} - p^*_{out}$ and $\Delta p^*_{m,c} = p^*_{in} - p^*_m$. So we have these dimensionless pressures:

$$p = \frac{p^*}{\Delta p^*_c} \quad and \quad p_m = \frac{p^*_m}{\Delta p^*_{m,c}}$$

We are now interested in found relations between velocity and pressure. Concerning the inner channel we have the Poiseuille Formula:

$$v^*_c = \frac{R^{*2}}{8\mu^* L^*} \Delta p^*_c$$

whereμ^* is the solution viscosity, assumed to be independent on the solute concentration.

Concerning the velocity in the membrane we use the Darcy Law:

$$\vec{q} = -\frac{K^*}{\mu^*} \nabla p^*_m \qquad (3.1.1)$$

To obtain

$$q^*_c = -\frac{K^*}{\mu^*} \frac{\Delta p^*_{m,c}}{S^*}$$

where K* is the membrane permeability (uniform and constant).

Let us introduce the Flow Ratio F_r given by the next expression:

$$\frac{2q^*_c}{\varepsilon v^*_c} = \frac{Q^*_c}{\dot{V}^*_c} = F_r$$

Then we can consider that:

$$F_r = \frac{16}{S} \cdot Da \cdot \frac{1}{\varepsilon^2} \cdot \Gamma$$

where the Da, the Darcy's number, is defined as:

$$Da = \frac{K^*}{R^{*2}}$$

and it's a property of the pipe membrane; and we set:

$$\Gamma = \frac{\Delta p^*_{m,c}}{\Delta p^*_c}$$

that is a characteristic number expressing ~~of~~ the ratio between the lateral pressure and the inner pressure. The flow ratio is the percentage of the fluid that flows through the membrane and depends on the membrane physical characteristics as well as on TMP.

3.2. Flow equations in the inner channel

We start with continuity equation for the mixture as a whole, considering $0 \le x \le 1$ and $0 \le r \le 1$~~:~~

$$\nabla^* \cdot \vec{v}^* = 0 \Leftrightarrow \frac{\partial v_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_r) = 0 \quad (3.2.1)$$

where∇^* denotes the divergence operator with respect to x* and r* space coordinates.

The continuity equations for the solute and the fluid are:

$$\begin{cases} \dfrac{\partial \nu}{\partial t^*} + \nabla^* \cdot (\nu \vec{v}^*_s) = 0 & solute \\ \dfrac{\partial (1-\nu)}{\partial t^*} + \nabla^* \cdot \left((1-\nu)\vec{v}^*_f\right) = 0 & fluid \end{cases}$$

If we add together the two previous equations we get (3.2.1). Next, remembering that we assumed $v^*_s = v^*_f = v^*$ and using the dimensionless variables instead of x* and t*, we obtain:

$$\frac{\partial \nu}{\partial t} + \frac{\partial (\nu v_x)}{\partial x} = 0$$

We then introduce the normalized volume fraction, setting $= \frac{\nu}{\nu_{inlet}}$, so that the previous equation becomes

$$\frac{\partial \eta}{\partial t} + \frac{\partial (\eta \cdot v_x)}{\partial t} = 0$$

If we substitute the equation (3.2.1) in the previous expression we obtain

$$\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial t} = \frac{\eta}{r} \frac{\partial}{\partial r}(r \cdot v_r) \qquad (3.2.2)$$

The momentum equation for the solution flow in the inner channel is given by the Navier-Stokes equation which, ignoring the gravity, has this form:

$$\rho^* \left[\frac{\partial \vec{v}^*}{\partial t^*} + (\vec{v}^* \cdot \nabla^*) \vec{v}^* \right] = -\nabla^* P^* + \mu^* \Delta^* \vec{v}^*$$

By using the dimensionless variables and the Reynolds number:

$$Re = \frac{\rho^* v_c^* R^*}{\mu^*}$$

The Navier-Stokes equation becomes:

$$\frac{Re}{\varepsilon} \left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_r \frac{\partial v_x}{\partial r} \right] = -\frac{8}{\varepsilon^2} \frac{\partial p}{\partial x} + \frac{\partial^2 v_x}{\partial x^2} + \frac{1}{\varepsilon^2} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_x}{\partial r} \right)$$

$$\frac{Re}{\varepsilon} \left[\frac{\partial v_r}{\partial t} + v_x \frac{\partial v_r}{\partial x} + v_r \frac{\partial v_r}{\partial r} \right] = -\frac{8}{\varepsilon^4} \frac{\partial p}{\partial r} + \frac{\partial^2 v_r}{\partial x^2} + \frac{1}{\varepsilon^2} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} \right)$$

Assuming Re is small (for example $Re \leq 1$ in dialyzers) and remembering that $\varepsilon \approx 10^{-4}$ we can approximate the previous equations as follow:

$$\begin{cases} -8 \dfrac{\partial p}{\partial x} + \dfrac{1}{r} \dfrac{\partial}{\partial r} \left(r \dfrac{\partial v_x}{\partial r} \right) = 0 & (3.2.3) \\ -\dfrac{\partial p}{\partial r} = 0 & (3.2.4) \end{cases}$$

3.3. Flow equations in the membrane

We will use the continuity and the momentum equations considering $0 \leq x \leq 1$ and $1 \leq r \leq 1 + s$.

Continuity equation in the membrane is:

$$\nabla^* \cdot \vec{q}^* = 0$$

Which coupled with Darcy's Formula (3.1.1) gives rise to

$$\Delta p_m^* = 0$$

Expressing the laplacian in cylindrical coordinates and substituting the dimensionless variables we obtain

$$\frac{\partial}{\partial r} \left(r \cdot \frac{\partial p_m}{\partial r} \right) = 0 \qquad (3.3.1)$$

3.4. Boundary conditions

- **r*=H***

We know the pressure $\boldsymbol{P}_m^*$ on the lateral boundary and we assume that the pressure is continuous so we set:

$$p_m^*(x^*, H^*) = -\Delta p_{m,c}^* \quad \Longrightarrow \quad p_m(x, H) = -1$$

- **r*=0**

In the symmetry line we require the condition

$$\left.\frac{\partial v_x}{\partial r}\right|_{r=0} = 0$$

$$v_r|_{r=0} = 0$$

- **r*=R***

Continuity of the radial flux gives:

$$v_r(x, 1, t) = -\frac{SF_r}{2}\left.\frac{\partial p_m}{\partial r}\right|_{r=1}$$

For the longitudinal velocity we consider a no-slip condition:

$$v_x|_{r=1} = 0$$

We can not assume that the pressure is continuous because of the osmosis that produces the discontinuity:

$$P_m^*|_{r^*=R^*} = P^*|_{r^*=R^*} - P_{os}^*$$

Now, if we consider the dimensionless parameter:

$$P_{os} = \frac{P_{os}^*}{P_{os,c}^*} = \eta$$

We can rewrite the previous boundary condition:

$$p_m|_{r=1} = \frac{p|_{r=1}}{\Gamma} - Os \cdot \eta$$

where Os is the osmotic number.

- **x*=0**

Remembering that $p^* = \boldsymbol{P}^* - \boldsymbol{P}^*_{in}$ at x*=0 we obviously obtain:

$$p^*(0, r^*, t^*) = 0$$

In the same way we can say that:

$$c|_{x=0} = 1 \quad \Leftrightarrow \quad \eta|_{x=0} = 1$$

4. Simplification and solution of the problem.

The mathematical model, in which the ε terms have been neglected, is given by:

$$(4.1)\begin{cases} \dfrac{\partial\eta}{\partial t}+v_x\dfrac{\partial\eta}{\partial x}=\dfrac{\eta}{r}\dfrac{\partial(r\cdot v_r)}{\partial r} & (4.1.1)\\ \dfrac{\partial v_x}{\partial x}+\dfrac{1}{r}\dfrac{\partial(r\cdot v_r)}{\partial r}=0 & (4.1.2)\\ -8\dfrac{\partial p}{\partial x}+\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\cdot\dfrac{\partial v_x}{\partial r}\right)=0 & (4.1.3)\\ \dfrac{\partial p}{\partial r}=0 & (4.1.4)\\ \dfrac{\partial}{\partial r}\left(r\cdot\dfrac{\partial p_m}{\partial r}\right)=0 & (4.1.5)\\ 2\cdot v_r(x,1,t)=-SF_r\cdot\left.\dfrac{\partial p_m}{\partial r}\right|_{r=1} & (4.1.6)\\ v_x(x,1,t)=0 & (4.1.7)\\ p_m|_{r=1}=\dfrac{1}{\Gamma}\cdot p|_{r=1}-\eta Os & (4.1.8)\\ \left.\dfrac{\partial v_x}{\partial r}\right|_{r=0}=0 & (4.1.9)\\ v_r|_{r=0}=0 & (4.1.10)\\ \eta|_{r=0}=1 & (4.1.11)\\ p_m|_{r=H}=-1 & (4.1.12) \end{cases}$$

We now proceed to the derivation of the governing equations.

- **Membrane:** now we consider equations (4.1.5), (4.1.8) and (4.1.12):

$$\begin{cases} \dfrac{\partial}{\partial r}\left(r\cdot\dfrac{\partial p_m}{\partial r}\right)=0 & (4.1.5)\\ p_m|_{r=1}=\dfrac{1}{\Gamma}\cdot p|_{r=1}-\eta Os & (4.1.8)\\ p_m|_{r=H}=-1 & (4.1.12) \end{cases}$$

By the equation (4.1.5), we can deduce:

$$r\cdot\frac{\partial p_m}{\partial r}=A \quad\Rightarrow\quad \frac{\partial p_m}{\partial r}=\frac{A}{r} \quad\Rightarrow\quad \int\frac{\partial p_m}{\partial r}=\int\frac{A}{r} \quad\Rightarrow\quad p_m(r)=A\cdot ln(r)+B$$

Replacing the equation (4.1.12) and (4.1.8) in the previous expression we obtain:

$$-1=A\cdot ln(H)+B \quad\Rightarrow\quad A=\frac{-1-B}{\ln(H)}$$

$$A\cdot ln(1)+B=B=\frac{1}{\Gamma}\cdot p|_{r=1}-\eta Os$$

And replacing the values of A and B, the solution of that part of the problem is:

$$p_m(r) = \frac{-1 - \left(\frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right)}{\ln(H)} \cdot ln(r) + \frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os$$

Now we can rewrite the expression (4.1.6) as:

$$v_r(x, 1, t) = -SF_r \cdot \frac{\partial p_m}{\partial r}\bigg|_{r=1} = \frac{S \cdot F_r}{2 \cdot \ln(H)}\left(1 + \frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) \qquad (4.2)$$

- **Inner Channel:** in that chase we use (4.1.3), (4.1.4) and (4.1.9):

$$\begin{cases} -8\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_x}{\partial r}\right) = 0 & (4.1.3) \\ \frac{\partial p}{\partial r} = 0 & (4.1.4) \\ \frac{\partial v_x}{\partial r}\bigg|_{r=0} = 0 & (4.1.9) \end{cases}$$

$$8\frac{\partial p}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_x}{\partial r}\right) \quad \Rightarrow \quad \int_0^r 8\frac{\partial p}{\partial x} dr = \int_0^r \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial v_x}{\partial r}\right) dr$$

$$\frac{\partial v_x}{\partial r} = 4 \cdot r \cdot \frac{\partial p}{\partial x} \quad \Rightarrow \quad \int_0^r \frac{\partial v_x}{\partial r} dr = \int_0^r 4 \cdot r \cdot \frac{\partial p}{\partial x} dr$$

$$v_x = -2\frac{\partial p}{\partial x}(1 - r^2) \qquad (4.3)$$

which is the classical Poiseuille profile.

We define average velocity as:

$$\langle v_x^* \rangle = \frac{1}{\pi \cdot R^{*2}} \int_0^{2\pi} \int_0^{R^*} v_x^*(r^*, x^*) r^* dr^* d\phi = \frac{2\pi}{\pi R^{*2}} \int_0^{R^*} v_x^* r^* dr^*$$

$$\langle v_x \rangle = 2 \int_0^1 v(r, x) \cdot r \, dr$$

In our case, using (4.3) we have:

$$\langle v_x \rangle = -4\frac{\partial p}{\partial x} \int_0^1 (1 - r^2) \cdot r \, dr = -\frac{\partial p}{\partial x}$$

Using (4.1.10), (4.1.2) and (4.3):

$$\begin{cases} \dfrac{\partial v_x}{\partial x} + \dfrac{1}{r}\dfrac{\partial (r \cdot v_r)}{\partial r} = 0 & (4.1.2) \\ v_r|_{r=0} = 0 & (4.1.10) \\ v_r(x,1,t) = \dfrac{S \cdot F_r}{2 \cdot \ln(H)}\left(1 + \dfrac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) & (4.2) \end{cases}$$

We obtain

$$\frac{\partial}{\partial x}\left(-2\frac{\partial p}{\partial x}(1-r^2)\right) + \frac{1}{r}\frac{\partial (r \cdot v_r)}{\partial r} = 0$$

integrating the above expression, we obtain:

$$\frac{\partial^2 p}{\partial x^2} = \frac{S \cdot F_r}{\ln (H)}\left(1 + \frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right)$$

Which can also rewritten as follows:

$$\frac{\partial \langle v_x \rangle}{\partial x} = \frac{S \cdot F_r}{\ln (H)}\left(1 + \frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) \qquad (4.4)$$

Finally we use the expression (4.1.1):

$$\frac{\partial \eta}{\partial t} + v_x \frac{\partial \eta}{\partial x} = \frac{\eta}{r}\frac{\partial (r \cdot v_r)}{\partial r}$$

We multiply (4.1.1) by 2r and then integrate r between 1 and 0.

$$\frac{\partial \eta}{\partial t}\int_0^1 2r + \int_0^1 2r\, v_x \frac{\partial \eta}{\partial x} = \int_0^1 2r \frac{\eta}{r}\frac{\partial (r \cdot v_r)}{\partial r}$$

Now we use (4.2) and the general expression for average velocity and we obtain:

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\langle v_x \rangle = -\eta \ \frac{S \cdot F_r}{\ln(H)}\left(1 + \frac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) \qquad (4.5)$$

that is the **transport equations**.

In conclusion, the dynamics (within an EPSILON approximation) can be summarized by the following system:

$$\begin{cases} \dfrac{\partial \langle v_x \rangle}{\partial x} = \dfrac{S \cdot F_r}{\ln (H)}\left(1 + \dfrac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) & (4.4) \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial \eta}{\partial x}\langle v_x \rangle = -\eta \ \dfrac{S \cdot F_r}{\ln(H)}\left(1 + \dfrac{1}{\Gamma} \cdot \ p|_{r=1} \ - \eta Os\right) & (4.5) \\ \langle v_x \rangle = -\dfrac{\partial p}{\partial x} & \\ p(x=0) = 0 & \\ \eta(x=0) = 1 & \end{cases}$$

- **STATIONARY VERSION**

We now consider the stationary version for (4.4), (4.5) system, that is:

$$\begin{cases} \dfrac{\partial \langle v_x \rangle}{\partial x} = \dfrac{S \cdot F_r}{\ln (H)} \left(1 + \dfrac{1}{\Gamma} \cdot \left. p \right|_{r=1} - \eta Os \right) \\ \dfrac{\partial \eta}{\partial x} \langle v_x \rangle + \eta \dfrac{\partial \langle v_x \rangle}{\partial x} = 0 \\ \langle v_x \rangle = - \dfrac{\partial p}{\partial x} \\ p(x=0) = 0 \\ \eta(x=0) = 1 \end{cases}$$

To conclude, if we denote:

$$\left. \langle v_x \rangle \right|_{x=0} = u_{inlet}$$

The second equation gives $\frac{\partial(\eta \langle v_x \rangle)}{\partial x} = 0$, namely

$$\langle v_x \rangle = \frac{u_{inlet}}{\eta}$$

and also we define the variables:

$$\mathcal{P} = 1 + \frac{p}{\Gamma} \qquad \text{and} \qquad Z = \frac{1}{\eta}$$

Under these conditions we obtain the stationary version of the model:

$$(4.6) \quad \begin{cases} \dfrac{\partial \mathrm{Z}}{\partial \mathrm{x}} = -\left(\dfrac{\mathrm{S} \cdot \mathrm{F_r}}{\mathrm{u_{inlet}} \cdot \ln (\mathrm{H})} \right) \left[\mathcal{P} - \mathrm{Os} \cdot \dfrac{1}{\mathrm{Z}} \right] \\ \dfrac{\partial \mathcal{P}}{\partial \mathrm{x}} = - \dfrac{\mathrm{u_{inlet}}}{\Gamma} \mathrm{Z} \\ \mathcal{P}(0) = 1 \\ \mathrm{Z}(0) = 1 \end{cases}$$

It is important to remember that pressure varies along x. This is the reason way we need the value of the pressure at the outlet. The equation for it is:

$$1 - \mathcal{P}(1) = \frac{u_{inlet}}{\Gamma} \int_0^1 Z(x) dx$$

4.1. Analytic solution.

We have the system:

$$\begin{cases} \dfrac{\partial Z}{\partial x} = -\left(\dfrac{S \cdot F_r}{u_{inlet} \cdot \ln(H)}\right)\left[\mathcal{P} - Os \cdot \dfrac{1}{Z}\right] \\ \dfrac{\partial \mathcal{P}}{\partial x} = -\dfrac{u_{inlet}}{\Gamma} Z \\ \mathcal{P}(0) = 1 \\ Z(0) = 1 \end{cases}$$

Now we are going to solve it analytically, if a parameter we are going to the define is small.

First of all we consider another variable:

$$Y = \frac{Z}{\Gamma}$$

and we define:

$$B^2 = \frac{S \cdot F_r}{\ln(H) \cdot \Gamma}$$

$$\beta = \frac{Os}{\Gamma}$$

$$b = \frac{u_{inlet}}{\Gamma}$$

Now the new system of equations is ~~are:~~

$$\begin{cases} Y' = -\left(\dfrac{B^2}{u_{inlet}}\right)\left[\mathcal{P} - \beta \dfrac{1}{Y}\right] \\ \mathcal{P}' = -u_{inlet} \cdot Y \\ \mathcal{P}(0) = 1 \\ Y(0) = \dfrac{1}{\Gamma} \end{cases}$$

If β is small, we may develop a perturbative approach, looking for the solution in this form:

$$\mathcal{P} = \hat{\mathcal{P}} + \beta\chi(x) + \beta^2\chi_2(x) + \cdots$$

$$Y = \hat{Y} + \beta\xi(x) + \beta^2 \cdot \xi_2(x) + \cdots$$

- **First step $\boldsymbol{\beta = 0}$**

$$\begin{cases} \hat{Y}' = -\left(\dfrac{\mathrm{B}^2}{\mathrm{u_{inlet}}}\right)\hat{\mathcal{P}} \\ \hat{\mathcal{P}}' = -\mathrm{u_{inlet}} \cdot \hat{Y} \\ \hat{\mathcal{P}}(0) = 1 \\ \hat{Y}(0) = \dfrac{1}{\Gamma} \end{cases}$$

To solve this system we derive two first equations and we obtain the following system:

$$\hat{P}'' = -u_{inlet} \cdot \hat{Y}' = u_{inlet} \cdot \frac{B^2}{u_{inlet}} \hat{P} = B^2 \cdot \hat{P}$$

$$\begin{cases} \hat{P}'' = B^2 \cdot \hat{P} \\ \hat{Y}'' = B^2 \cdot \hat{Y} \\ \hat{P}(0) = 1 \\ \hat{Y}(0) = 1 \\ \hat{P}'(0) = -b \\ \hat{Y}'(0) = -\dfrac{B^2}{u_{inlet}} \end{cases}$$

The solution of the homogeneous system is:

$$\hat{P}(x) = \cosh(Bx) - \frac{u_{inlet}}{\mathrm{B} \cdot \Gamma} \sinh(Bx)$$

$$\hat{Y}(x) = \frac{1}{\Gamma}\cosh(Bx) - \frac{B}{u_{inlet}} \sinh(Bx)$$

- **Second step $\boldsymbol{\beta \neq 0}$**

Now we consider:

$$\mathcal{P} = \hat{\mathcal{P}} + \beta\chi(x)$$

$$Y = \hat{Y} + \beta\xi(x)$$

Our new system is:

$$\begin{cases} \hat{Y}' + \beta \cdot \xi' = -\left(\dfrac{\mathrm{B}^2}{\mathrm{u_{inlet}}}\right)\left[\hat{\mathcal{P}} + \beta \cdot \chi - \dfrac{\beta}{\hat{Y} + \beta \cdot \xi}\right] \\ \hat{\mathcal{P}} + \beta \cdot \chi' = -\mathrm{u_{inlet}} \cdot (\hat{Y} + \beta \cdot \xi) \\ \hat{\mathcal{P}}(0) + \beta \cdot \chi(0) = 1 \\ \hat{Y}(0) + \beta \cdot \xi(0) = \dfrac{1}{\Gamma} \end{cases}$$

that reduces to:

$$\begin{cases} \xi' = -\left(\frac{B^2}{u_{inlet}}\right)\left[\chi - \frac{1}{\hat{Y}}\right] \\ \chi' = -u_{inlet} \cdot \xi \\ \chi(0) = 0 \\ \xi(0) = 0 \end{cases}$$

Namely :

$$\begin{cases} \chi''(x) = B^2 \cdot \left(\chi - \frac{1}{\hat{Y}(x)}\right) \\ \chi(0) = 0 \\ \chi'(0) = 0 \end{cases}$$

We solve this system using the method of variation of constants.

The homogenous solution is:

$$\chi(x) = C_1 \cdot \chi_1(x) + C_2 \cdot \chi_2(x)$$

$$\chi_1(x) = \cosh(Bx)$$

$$\chi_2(x) = \sinh(Bx)$$

The wronskian is:

$$\mathcal{W} = \begin{vmatrix} \chi_1 & \chi_2 \\ \chi_1' & \chi_2' \end{vmatrix} = \begin{vmatrix} \cosh(Bx) & \sinh(Bx) \\ B \cdot sinh(Bx) & B \cdot \cosh(Bx) \end{vmatrix} = B$$

We apply the method of constants variations:

$$\chi(x) = C_1(x) \cdot \chi_1(x) + C_1(x) \cdot \chi_1(x)$$

$$C_1'(x) = \frac{1}{B}\begin{vmatrix} 0 & \sinh(Bx) \\ -\frac{B^2}{\hat{Y}(x)} & B \cdot \cosh(Bx) \end{vmatrix} = \frac{B \cdot \sinh(Bx)}{\hat{Y}(x)}$$

$$C_1'(x) = \frac{1}{B}\begin{vmatrix} \cosh(Bx) & 0 \\ Bsinh(Bx) & -\frac{B^2}{\hat{Y}(x)} \end{vmatrix} = -\frac{B \cdot \cosh(Bx)}{\hat{Y}(x)}$$

Integrating:

$$C_1(x) = B \cdot \int_0^x \frac{\sinh(Bx')}{\hat{Y}(x')} dx' + K1$$

$$C_2(x) = -B \cdot \int_0^x \frac{\cosh(Bx')}{\hat{Y}(x')} dx' + K_2$$

So,

$$\chi(x) = \left[B \cdot \int_0^x \frac{\sinh(Bx')}{\hat{Y}(x')} dx' + K1 \quad \right] \cdot cosh(Bx) + \left[-B \cdot \int_0^x \frac{\cosh(Bx')}{\hat{Y}(x')} dx' + K_2\right] \cdot sinh(Bx)$$

Now, we have to determine the constants $K_1$and K_2, using the boundary conditions:

$$\chi(0) = 0$$

$$\chi'(0) = 0$$

$$\chi'(x) = \left[B \cdot \int_0^x \frac{\sinh(Bx')}{\hat{Y}(x')} dx' + K1 \quad \right] \cdot B \cdot sinh(Bx) + \cosh(Bx) \cdot \frac{B \cdot \sinh(Bx)}{\hat{Y}(x)} +$$

$$+ \left[-B \cdot \int_0^x \frac{\cosh(Bx')}{\hat{Y}(x')} dx' + K_2\right] \cdot B \cdot cosh(Bx) - \sin(Bx) \frac{B \cdot \cosh(Bx)}{\hat{Y}(x)}$$

We obtain the system:

$$\begin{cases} K_1 = 0 \\ B \cdot K_2 = 0 \end{cases} \Rightarrow K_1 = K_2 = 0$$

$$\chi(x) = \left[B \cdot \int_0^x \frac{\sinh(Bx')}{\hat{Y}(x')} dx'\right] \cdot cosh(Bx) + \left[-B \cdot \int_0^x \frac{\cosh(Bx')}{\hat{Y}(x')} dx'\right] \cdot sinh(Bx) =$$

$$= B \cdot \int_0^x \frac{\sinh(Bx') \cdot \cosh(Bx) - \sinh(Bx) \cdot \cosh(B \cdot x')}{\hat{Y}(x')} dx' =$$

$$= B \cdot \int_0^x \frac{\sinh(B(x' - x))}{\hat{Y}(x')} dx'$$

So the solution is the following:

$$\boldsymbol{\mathcal{P} = \hat{\mathcal{P}} + \beta\chi(x) = \cosh(Bx) - \frac{u_{inlet}}{B \cdot \Gamma} \sinh(Bx) + \beta \cdot \left[B \cdot \int_0^x \frac{\sinh(B(x' - x))}{\hat{Y}(x')} dx'\right]}$$

4.2. Numerical simulation of system (4.6)

In case the parameter β is not small the previously illustrated perturbative approach is not reliable. We therefore solve numerically system (4.6).

If the flow is driven prescribing the inlet and outlet pressures (namely the pressure difference) $\mathcal{P}_{out}$, is given, then the inlet velocity appearing in system is actually an unknown parameter,. We therefore should use a procedure of successive approximations, mapping a guess of u_{inlet} into a new value.

The equations are discretized in space using a finite difference schema, and solved using Matlab operator ode45.

We solve numerically system (4.6) considering the following parameters:

$\Gamma = 2$ (Pressure ratio)

$S = 0.5$ (Membrane thickness)

$H = 1 + S = 1.5$ (Diameter of the pipe)

$LnH = 0.4$

$F_r = 0.1$ (Flow ratio)

We first consider the simple case in which the inlet velocity u_{inlet} is given. This case corresponds to the so-called discharge driven dynamics, i.e. the inlet flux is prescribed. In particular, here we are considering two cases:

- **Case 1:** inlet velocity fixed $u_{inlet} = 1$, and we vary the osmosis parameter O_s, between 0 and 0.7.

Running the Matlab program we obtained the following results:

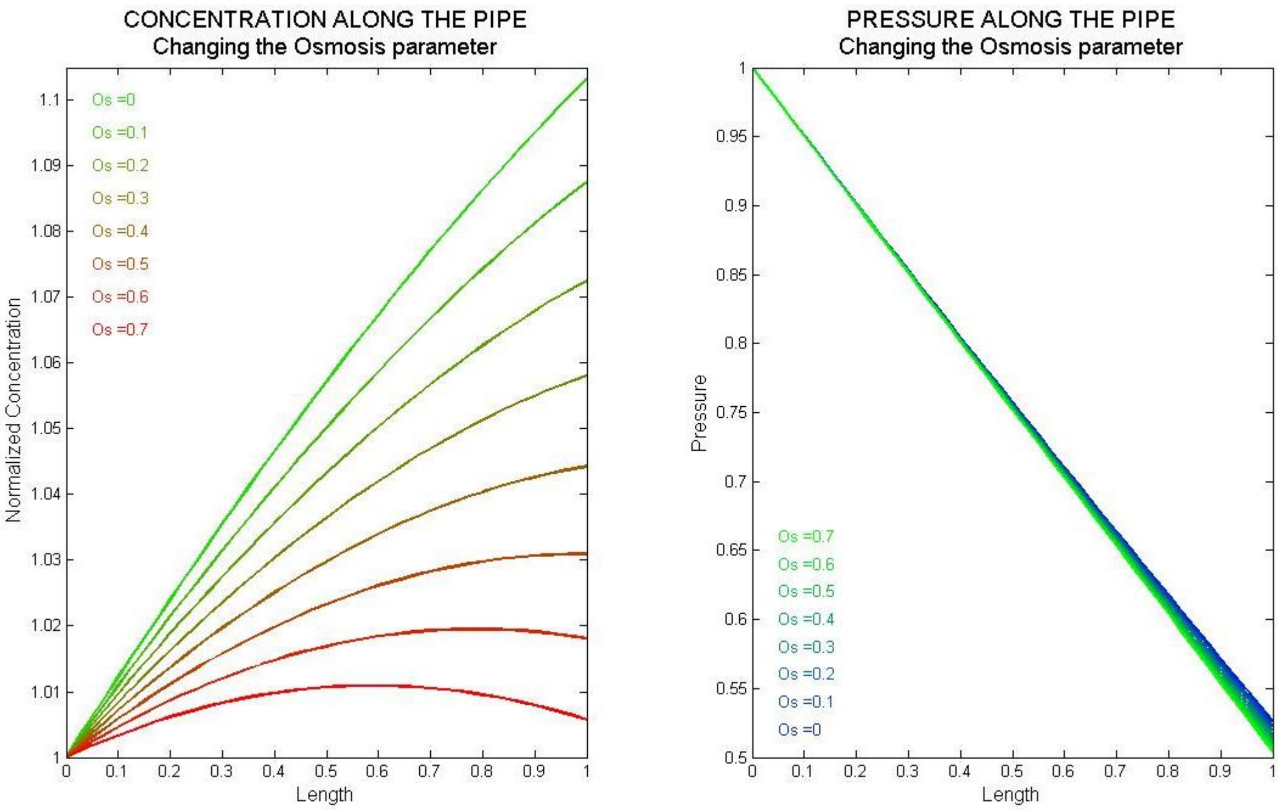

Fig. 9 Evolution of the normalized concentration and pressure along the pipe for different osmosis parameters.

In the left graphic we can see how the solute concentration strongly depends on osmosis parameter. When the osmosis parameter increase, the concentration reaches a maximum in the pipe and then decreases, meanning that liquid is sukced from the outside. In this cases, which occur when Os>0.4, the TMP is compensed by the osmotic pressure close to the pipe outlet. In particular, the osmotic pressure exceeds TMP, cousing an entering flux. That makes at the end of the pipe we have less solute concentration .

In the right graphic it shown the evolution of the pressure for the same osmosis parameter values, and we can see that the effect of osmosis parameter in the pressure along the pipe is inappreciable.

- **Case 2:** Numerical simulation considering the osmosis parameter fixed and varying the inlet velocity u_{inlet}

For this simulation we takes the osmosis parameter$Os = 0.3$, which is considered a normal value and the velocity u_{inlet}changing between 1 and 2.

Running the Matlabprogram to solve it, weobtain the following results:

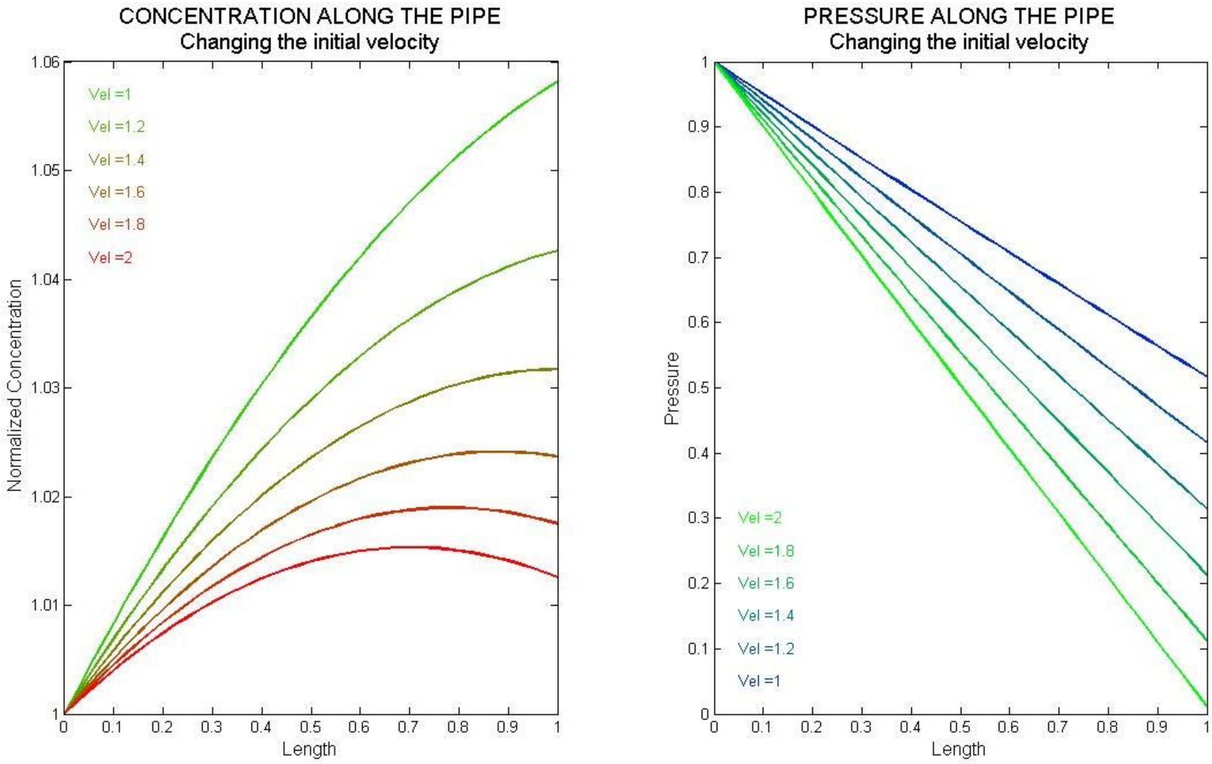

Fig. 10 Evolution of the normalized concentration and pressure along the pipe for different inlet velocities

In this case, the velocity influences in both, concentration and pressure along the pipe.

For low velocity, 1, the concentration of solute inside always increasing along the pipe, but as the velocity increases, reaches a maximum inside the pipe. That is, by increasing the velocity will reach a point where the pressure compensates osmosis.

This agree with the pressure graph which shows that at higher velocity the pressure along the pipe decreases faster, which makes it compensated by osmosis.

We considered then the case in which the flow is driven giving the inlet and outlet pressure (pressure driven dynamics). In particular, the pressure at the end of the pipe is such that $\mathcal{P}_{out} = 0$ Concerning Os, we have set.

$Os = 0.3$ (Osmosis parameter)

Now u_{inlet} is not a known parameter. We have implemented the following algorithm:

Step 0: *(Initiation)*

Initialize the inlet velocity: $u_{inlet}^{(0)} = 1$

Solve the system: $\begin{cases} \frac{\partial Z}{\partial x} = -\left(\frac{SF_r}{u_{inlet} lnH}\right)\left[\mathcal{P} - Os\frac{1}{Z}\right] \\ \frac{\partial \mathcal{P}}{\partial x} = -\frac{u_{inlet}}{\Gamma} Z, \\ \mathcal{P}|_{x=0} = 1, \\ Z|_{x=0} = 1. \end{cases}$

Get$Z^{(0)}$ y $\mathcal{P}^{(0)}$

Step n:

Calculate $u_{inlet}^{(n)} = \frac{\Gamma}{\int_0^1 Z^{(n-1)} dx}$

Solve the system: $\begin{cases} \frac{\partial Z}{\partial x} = -\left(\frac{SF_r}{u_{inlet}^{(n)} lnH}\right)\left[\mathcal{P} - Os\frac{1}{Z}\right] \\ \frac{\partial \mathcal{P}}{\partial x} = -\frac{u_{inlet}^{(n)}}{\Gamma} Z, \\ \mathcal{P}|_{x=0} = 1, \\ Z|_{x=0} = 1. \end{cases}$

Get$Z^{(n)}$y$\mathcal{P}^{(n)}$.

Calculate $\int_0^1 Z^{(n)}$ integrating by Simpson

Calculate $u_{inlet}^{(n+1)} = \frac{\Gamma}{\int_0^1 Z^{(n)} dx}$

Tolerance Step:

If $\left|u_{inlet}^{(n+1)} - u_{inlet}^{(n)}\right| \geq 10^{-4}$ or $iter < 100$ then *Step 1*else *Stop*

Note: A priori we do not know the convergence of the problem, so we add a constraint for the "while", a maximum of 100 iterations.

We obtained this results:

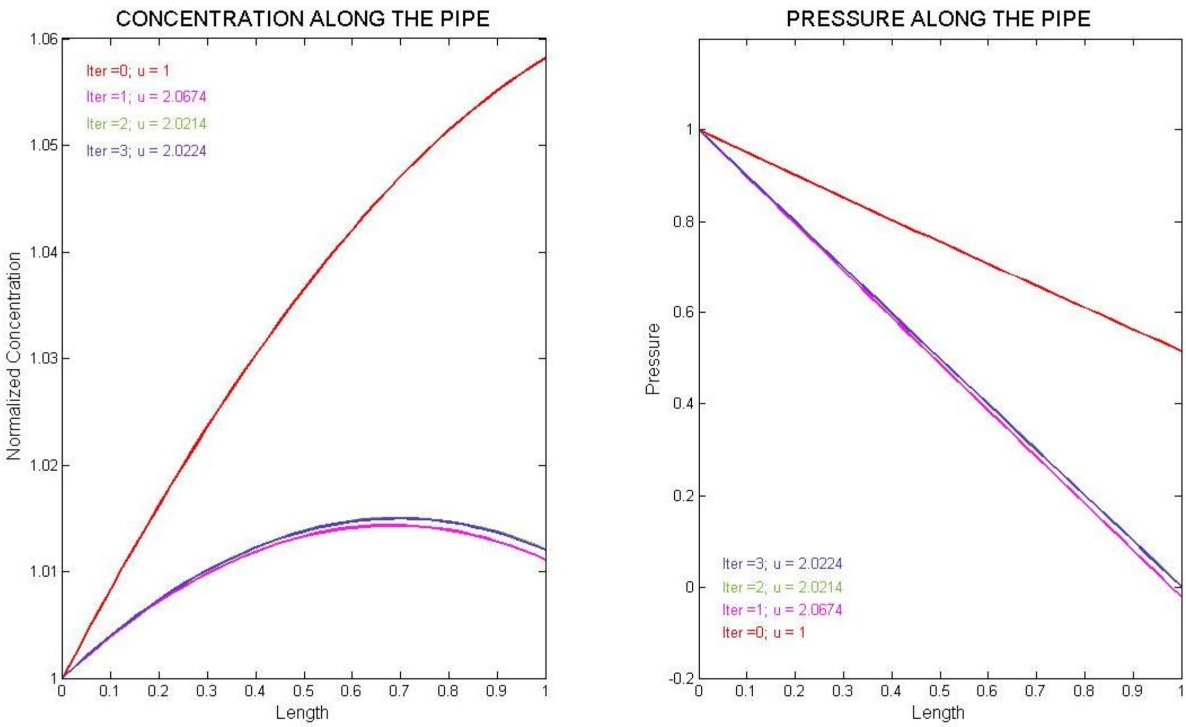

Fig. 11 Initial velocity calculation

We note that the iterative procedure converges very quickly. In the first iteration to the second, the solution is much improved in four iterations and obtain a value of $u_{inlet} = 2.0224$. In the graph to the right we can see that the value of the pressure at the end of the pipe to the initial velocity value is 0, which is what we were trying to achieve.

5. Conclusions.

Here is a peculiar aspect of the problem, that has emerged during our study.

An important constraint that must be always fulfilled is :

$$\mathcal{P}'(x) < 0 \qquad\qquad \forall\ x \in [0, 1]$$

Indeed, if$\mathcal{P}'(x)$ is negative, the flow goes form inlet to outlet. We want to avoid flow inversion phenomena, that occurs when ever $\mathcal{P}'(x)$ becomes positive.

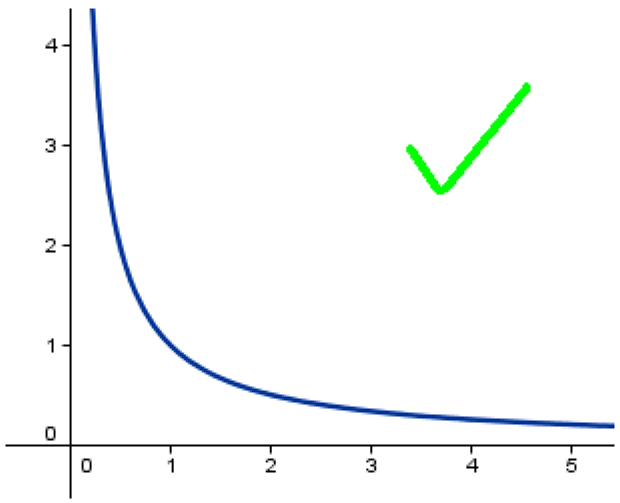

Fig. 12 Negative slope.

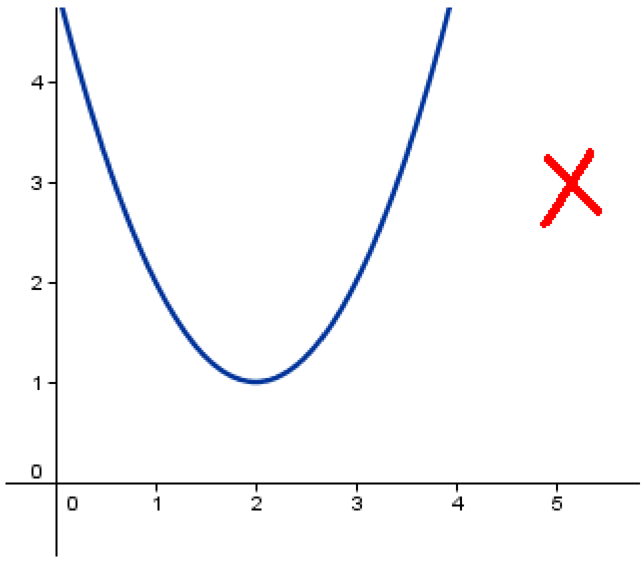

Fig. 13 Pressure with slope positive and negative.

So, working with the case in which the parameter β is small,

$$\mathcal{P}'(\mathrm{x}) = -\frac{u_{inlet}}{\Gamma} \cdot \cosh(Bx) + B \cdot \sinh(Bx) =$$

$$= B \cdot \cosh(Bx) \cdot \left[\tanh(Bx) - \frac{u_{inlet}}{B \cdot \Gamma}\right]$$

We know that the graph of the function tanh is:

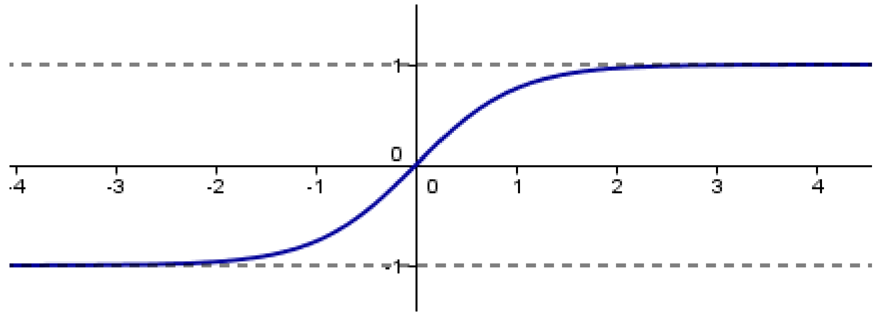

Fig. 14 tanh function

So, if $\left[\tanh(Bx) - \frac{u_{inlet}}{B \cdot \Gamma}\right] < 0$. i.e. $\frac{\mathrm{u_{inlet}}}{\mathrm{B \cdot \Gamma}} > 1$, $\mathcal{P}'(\mathrm{x})$ is always negative.

In our case, considering the following values:

S=0.5

Fr=0.1

ln(H)=0.4

$\Gamma = 2$

$\mathrm{u_{inlet}} = 2.0224$

$$B = \sqrt{\frac{S \cdot F_r}{\ln(H) \cdot \Gamma}} = 0.2483$$

$$\frac{u_{inlet}}{B \cdot \Gamma} = 4.0723 > 1$$

We can see that the values considered in the simulation fulfill the above condition.