Modelling Flow in Pipes

with Semi-Permeable Walls

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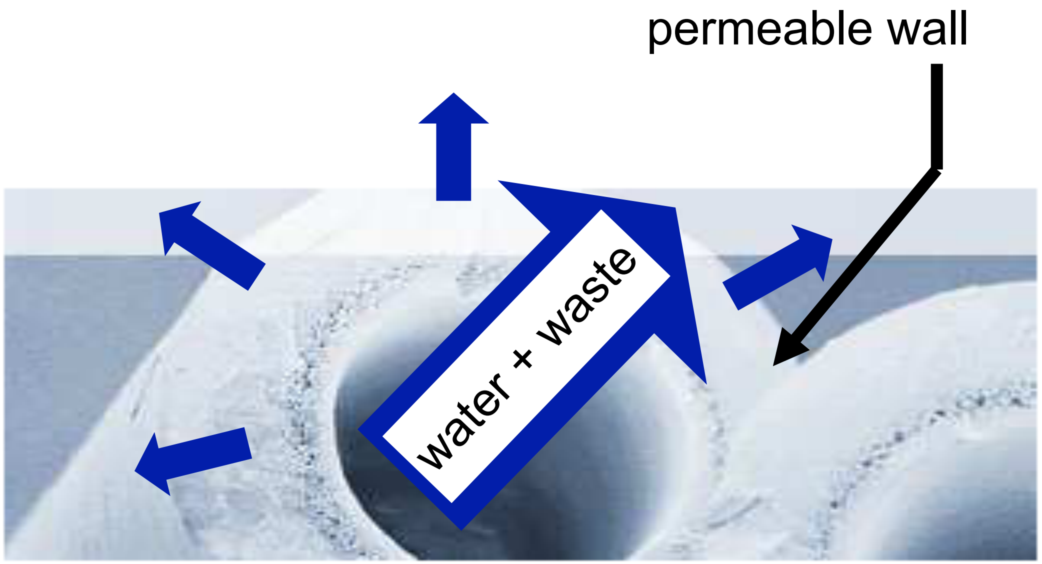
MOTIVATIONS

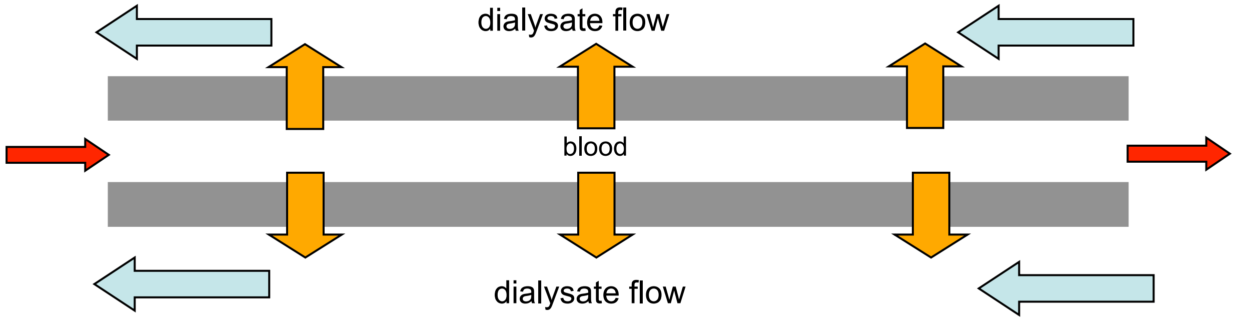
1) INDUSTRIAL APPLICATIONS – WATER FILTERS

Modern water depuration modules consist of a container housing a large number of hollow fibers whose lateral membranes is permeable to water, but not to “large” particles.

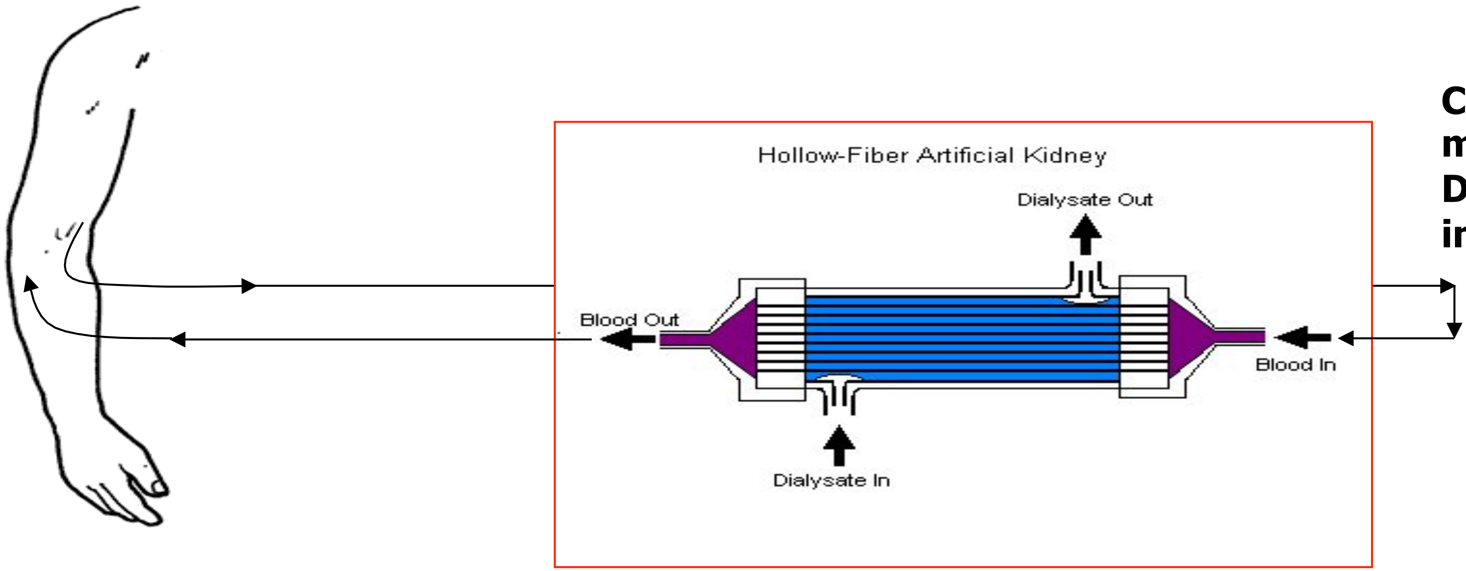


2) MEDICAL APPLICATIONS – DYALISIS

In medicine, dialysis is a method for removing waste products such as urea, as well as excess water from the blood and is used primarily to provide an artificial replacement for lost kidney function in people with renal failure.



Blood flow in semipermable hollow fibers



Compartements model Dialyzer-body interaction

3) AGRICULTURE APPLICATIONS

A widely used irrigation technique consists in delivering water by letting it filtrate through permeable pipes laid down or suspended over the ground. Several types of plants are used, according to the size of the fields to be irrigated.

Irrigation pipes

Large plants

dripping pipes (thick)

Large pressure

Small plants

"exudating pipes" (thin)

Small pressure

The above mentioned applications have a common characteristic: **the ratio between the pipe length and the pipe radius is small.**

$$\varepsilon = \frac{R^*}{L^*} << 1 \quad \text{ASPECT RATIO} <<1$$

The radial length scale and the longitudinal length scale are **well separated**. Hence, the so-called **upscaling** (or **double scale**) procedure can be used. **Upscaling** consists in the following steps:

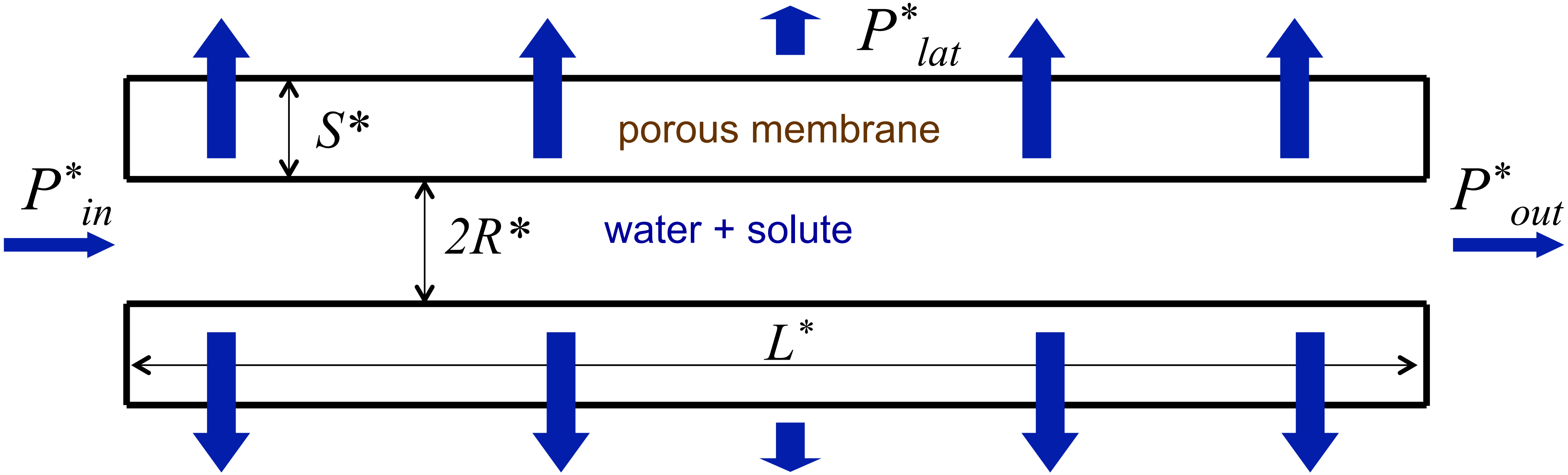
(i) Writing the governing equations

(ii) Introducing the asymptotic expansions of the main physical quantities in power of the small parameter ε

(iii) Matching the terms of equal order in ε in the governing equations as well as boundary conditions at the microscopic scale.

(iv) Averaging the relevant quantities over the fibers cross section.

PROBLEM DESCRIPTION



1) To model the dynamics of a solution whose components are a Newtonian liquid (water) and a single solute within a tube whose wall (membrane) prevents to the solute molecules to be transported across it.

2) To simulate the process considering different working conditions.

AN IMPORTANT EFFECT TO CONSIDER: OSMOSIS

The fundamental characteristic of the membrane is to be **semipermeable**, i.e. it allows water cross-flow but prevents the lateral flux of the solute. The latter therefore increases its concentration within the channel thus giving rise to an osmotic pressure that can become not negligible.

The lateral flow is driven by the pressure difference between the channel and the exterior, the so-called transmebrane pressure -TMP. The latter is then countered by osmotic pressure (which can even cause the stopping of the lateral flow).

If c_s^* is the concentration (expressed in $mol/\ell t$) of the solute, the Morse equation gives the corresponding osmotic pressure

$$P_{os}^* = \mathcal{R}^* T^* c_s^*,$$

where $\mathcal{R}^* = 0.082\ \ell t\ atm/mol\ {}^oK$, is the ideal gas constant and T^* is the absolute temperature.

MODELLING STEPS

1. DEFINITION OF THE GEOMETRICAL SETTING
   - *Cylindrical geometry*

2. DEFINITION OF THE DEPENDENT VARIABLES
   - *Solute concentration*
   - *Fluid velocity in the inner channel*
   - *Liquid discharge through the membrane*
   - *Pressure*

3. POSSIBLE SIMPLIFICATIONS
   - *The density of the solution does not depend on the solute concentration*
   - *The rheological properties of the solution do not depend on the solute concentration*

4. FUNDAMENTAL EQUATIONS DESCRIBING:
   - *Flow within the channel (Navier Stokes equation)*
   - *Flow through the membrane (Darcy's law)*
   - *Evolution equation for the solute concentration*
   - *Osmotic pressure*
   - *Boundary conditions (no-slip, flux continuity, pressure jump)*

5. SCALINGS AND CHARACTERISTIC PARAMETERS

6. ASYMPTOTIC EXPANSION - UPSCALING
   - *Definition of the approximated model*
   - *Definition of the macroscopic quantities*

7. QUALITATIVE PROPERTIES OF THE MODEL

8. SIMULATIONS

9. PHYSICAL INTERPRETATION OF THE RESULTS

TEAM TASKS

1. Define a mathematical model aimed at describing the process.

2. Introduce a double scale procedure.

3. Define the corresponding mathematical problem (BVP problem).

4. Possible qualitative properties.

5. Perform numerical simulations.