Biological control of rabbits

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Contents

Introduction

SIR Model

$$\begin{cases} \dot{S} = -\alpha I S \\ \dot{I} = \alpha I S - \beta I \end{cases}$$

- Based on the mass action law
- α: Rate of infections
- β: Rate of deaths
- No growth included!
- All the infected animals die

SIR model with exponential growth

$$\begin{cases} \dot{S} = -\alpha IS + \gamma S \\ \dot{I} = \alpha IS - \beta I \end{cases}$$

- Exponential growth in the absence of a disease (Malthus 1798)
- Logistic model is better
- Parameters? Nondimensionalisation

SIR model with exponential growth

$$\begin{cases} \dot{\hat{S}} = -\hat{I}\hat{S} + \kappa\hat{S} \\ \dot{\hat{I}} = \hat{I}\hat{S} - \lambda\hat{I} \end{cases}$$

Parameters that describes the dynamics of the disease:

- κ is the birth rate relative to the infection rate.
- λ is the death rate relative to the infection rate.

The higher the λ (higher mortality rate of disease), the better?

SIR model with exponential growth

$\lambda > 1$: disease with high mortality rate

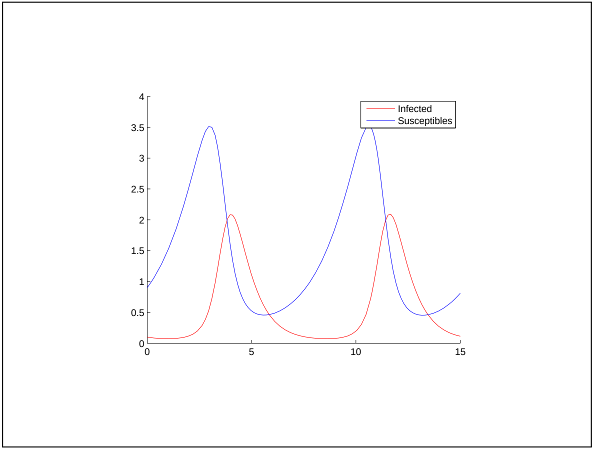


Figure: Evolution of the infected and the susceptible subjects, for $\kappa = 0.6$ and $\lambda = 1.5$

SIR model with exponential growth

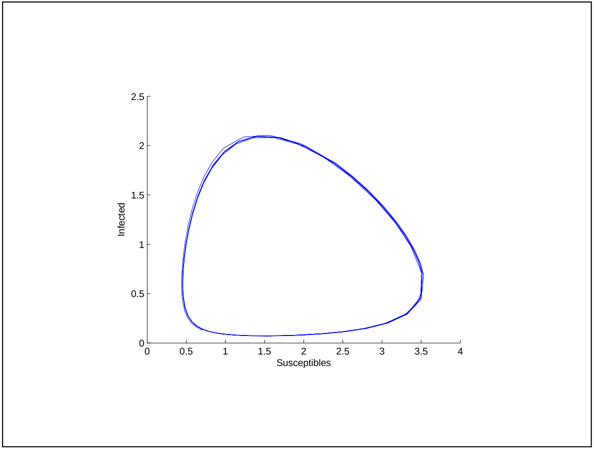


Figure: Phase portrait for $\kappa < 1$ and $\lambda > 1$

SIR model with exponential growth

$\lambda < 1$: disease with low mortality rate

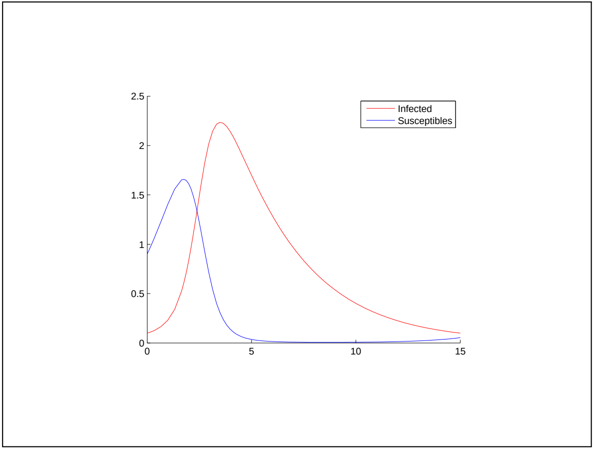


Figure: Evolution of the infected and the susceptible subjects, for $\kappa = 0.6$ and $\lambda = 0.3$

SIR model with exponential growth

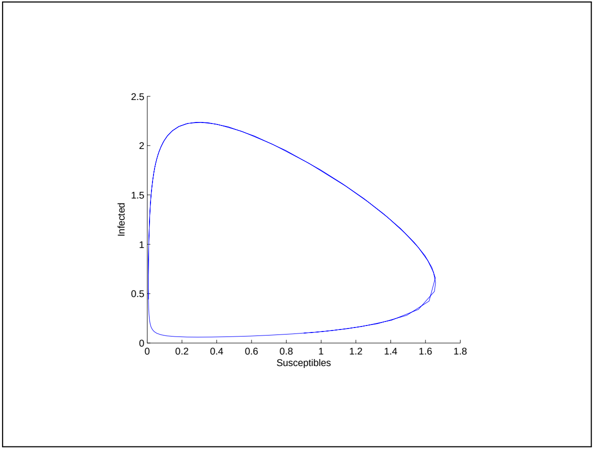


Figure: Phase portrait for $\kappa < 1$ and $\lambda < 1$

SIR model with logistic growth

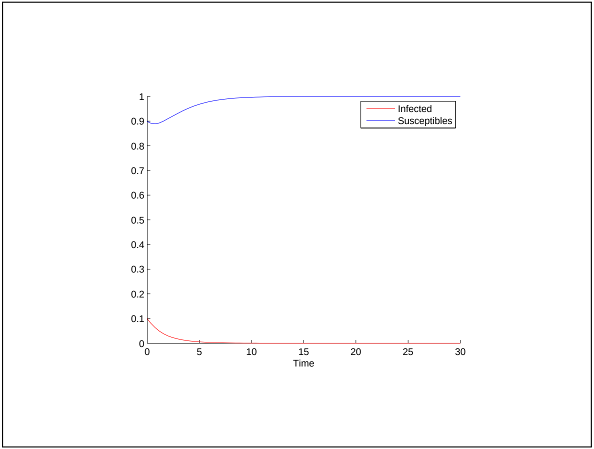
$$\begin{cases} \dot{\hat{S}} = -\hat{I}\hat{S} + \kappa S(1 - S) \\ \dot{\hat{I}} = \hat{I}\hat{S} - \lambda I \end{cases}$$

The parameters describe almost the same way the dynamics of the disease:

- κ is the birth rate relative to the infection rate.
- λ is the death rate relative to the infection rate.

SIR model with logistic growth

$\lambda > 1$: disease with high mortality rate



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SIR model with logistic growth

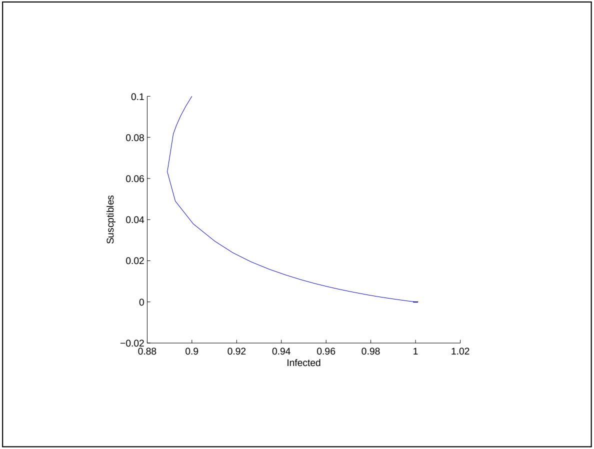

Figure: Phase portrait for $\kappa < 1$ and $\lambda < 1$

SIR model with logistic growth

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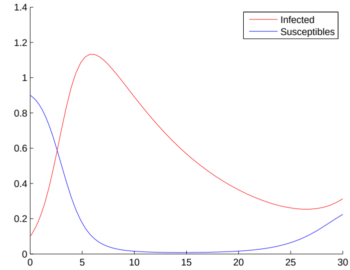

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SIR model with logistic growth

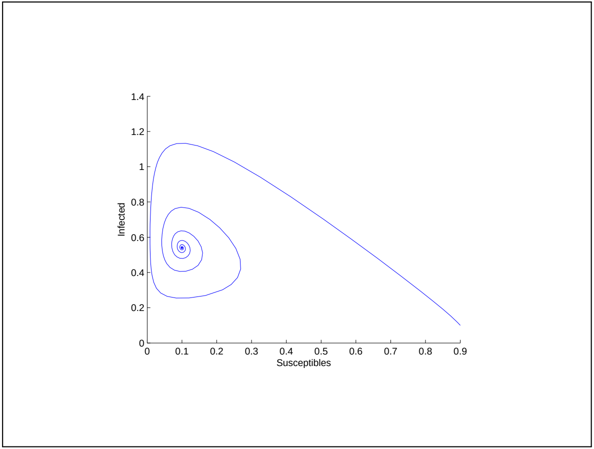
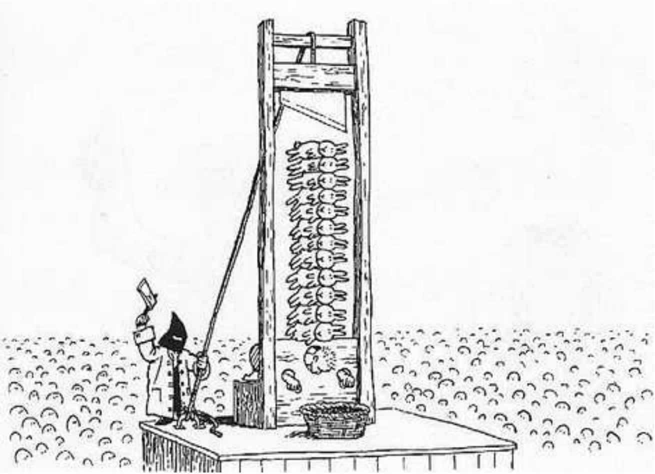


Figure: Phase portrait for $\kappa < 1$ and $\lambda < 1$

SIR model with logistic growth

Conclusion: We should seek for a contagious disease rather than a fast and agressive one, in other to spread it among the subjects

Spatio temporal dynamics



Spatio temporal dynamics

$$\dot{S}(x,t) = -S(x,t)I(x,t)$$

$$\dot{I}(x,t) = S(x,t)I(x,t) - \lambda I(x,t)$$

$x \in [0,1],\ t > 0$

Spatio temporal dynamics

$$\dot{S}(x,t) = -S(x,t)I(x,t) + D\frac{\partial^2 S}{\partial x^2}(x,t)$$

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where D is a nondimensional *diffusion parameter*

Spatio temporal dynamics

$$\dot{S}(x,t) = -S(x,t)I(x,t) + D\frac{\partial^2 S}{\partial x^2}(x,t) + \text{growth}$$
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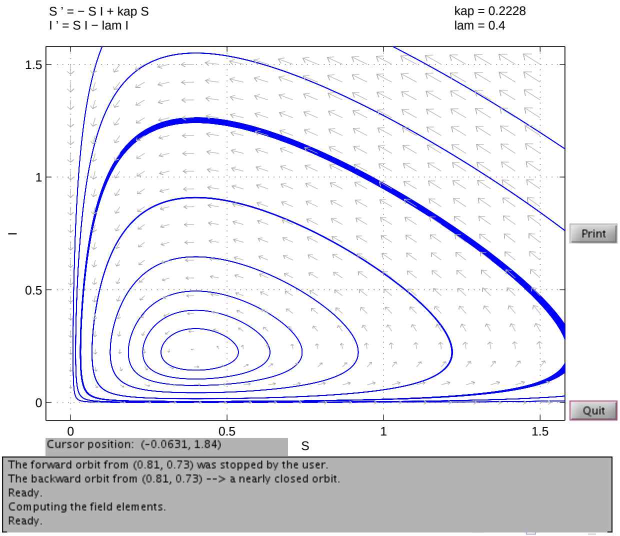
Spatio temporal dynamics



SIR with fleas population

Considering that fleas are the main vector of the disease we can neglect other transmission factors. We assume:

- $$\dot{S} = -\alpha SF + \delta S(1 - \frac{S}{K}), \tag{1}$$

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SIR with fleas population

After the nondimensionalisation the dynamic system is:

$$\begin{cases} \dot{S} = -SF + \kappa S(1 - S) \\ \dot{I} = SF - \lambda I \\ \dot{F} = \frac{1}{\varepsilon}(I - F) \end{cases} \tag{4}$$

SIR with fleas population

Two equilibrium solutions that exist for all parameter values are $A(0,0,0)$ and $B(1,0,0)$. Additional equilibrium point $C(\lambda, \kappa(1-\lambda), \kappa(1-\lambda))$ exists only if $\lambda < 1$.

A is always a saddle point while B is asymptotically stable for $\lambda > 1$.

When the point C exists it is asymptotically stable while B becomes unstable.

SIR with fleas population

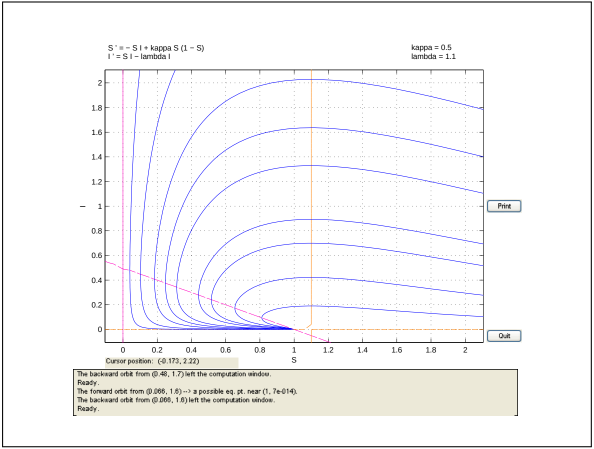

Figure: Phase portrait with $\lambda > 1$

SIR with fleas population

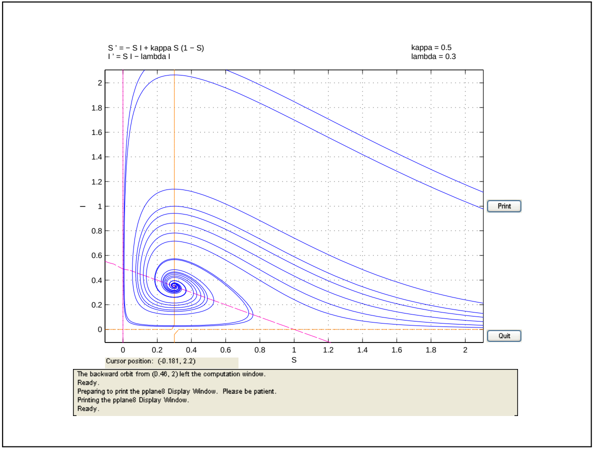

Figure: Phase portrait with $\lambda < 1$

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We can observe that:

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- When $\lambda > 1$ the situation returns to the initial condition as time goes to infinity

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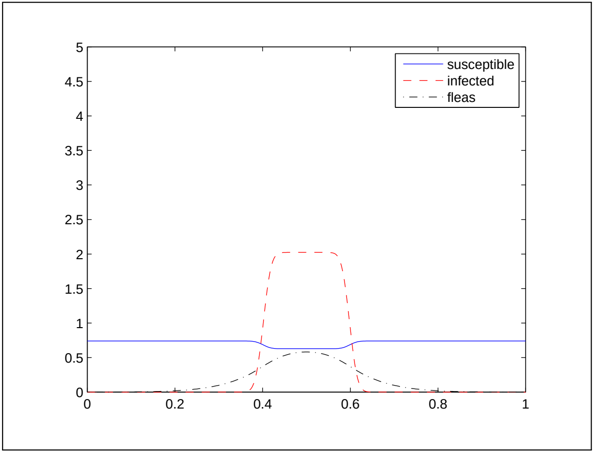
- choosing a small ε we find the dynamic system (5) is equivalent to a SIR model with logistic growth because F and I go to infinity in the same way;
- in the case $\lambda < 1$ we have an endemic disease
- When $\lambda > 1$ the situation returns to the initial condition as time goes to infinity

SIR with fleas population and space diffusion

As we have done before we introduce diffusion terms for fleas and rabbits

$$\begin{cases} \frac{\partial S}{\partial t} = -SF + \kappa S(1-S) + D_r \frac{\partial^2 S}{\partial t^2} & t > 0, x \in [0,1] \\ \frac{\partial I}{\partial t} = SF - \lambda I + D_r \frac{\partial^2 I}{\partial t^2} & t > 0, x \in [0,1] \\ \frac{\partial F}{\partial t} = \frac{1}{\varepsilon}(I - F) + D_f \frac{\partial^2 F}{\partial t^2} & t > 0, x \in [0,1] \\ S(x,0) = S_0(x) & x \in [0,1] \\ I(x,0) = I_0(x) & x \in [0,1] \\ F(x,0) = F_0(x) & x \in [0,1] \\ \frac{\partial S}{\partial x}|_{x=0,1} = 0 & t > 0 \\ \frac{\partial I}{\partial x}|_{x=0,1} = 0 & t > 0 \\ \frac{\partial F}{\partial x}|_{x=0,1} = 0 & t > 0 \end{cases} \tag{5}$$

SIR with fleas population and space diffusion



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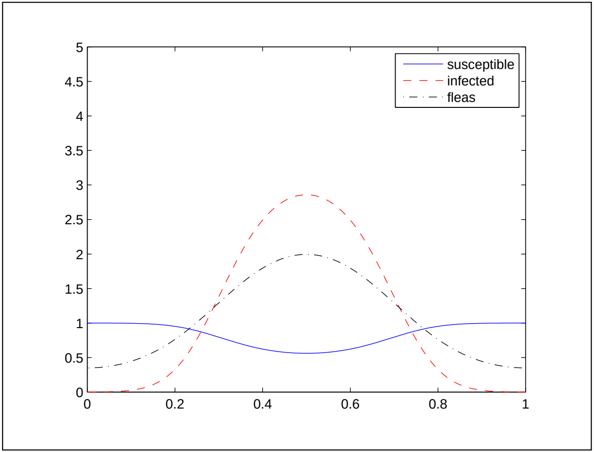

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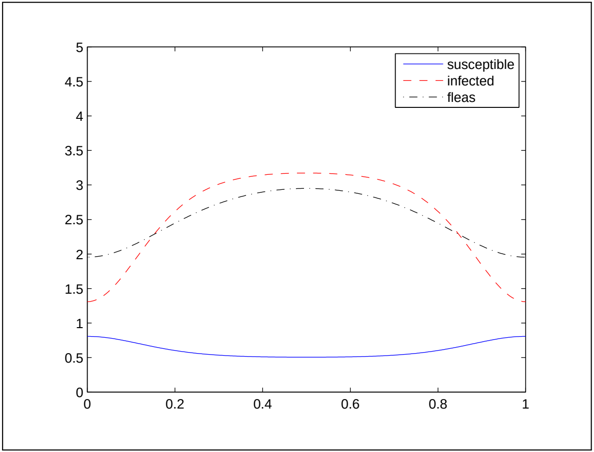
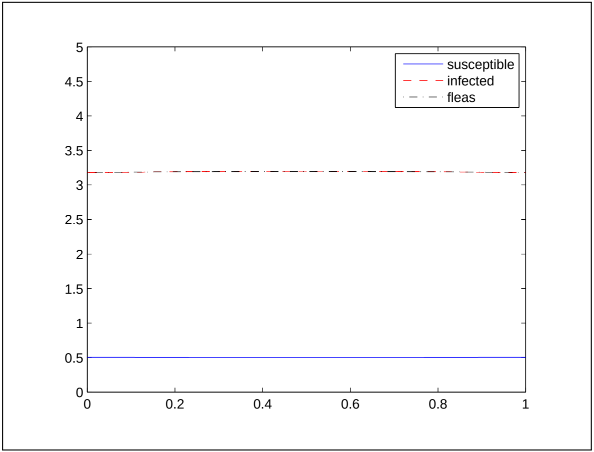


SIR with fleas population and space diffusion



Time-varying rate of infection

We now assume to have a seasonal disease:

$$\begin{cases} \dot{S} = -\alpha(t)SI \\ \dot{I} = -\alpha(t)SI - \beta I \\ \alpha(t) = Asin(\omega t) + \alpha_0 \end{cases} \tag{6}$$

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After nondimensionalisation our dynamic system becomes

$$\begin{cases} \dot{S} = -\alpha(t)\mu SI \\ \dot{I} = -\alpha(t)\mu SI - \eta I \\ \alpha(t) = 1 + \Gamma sin(t) \end{cases} \tag{7}$$

Time-varying rate of infection

We proceeded by numerical simulations and as we can see the seasonality of disease makes the populations trend periodic

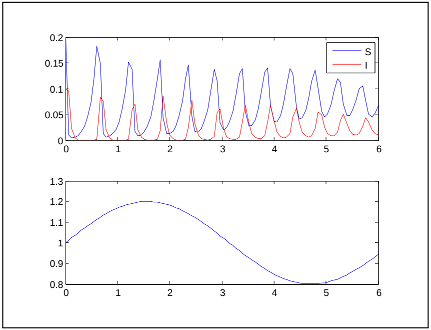

Figure: Time-varying rate of infection $\alpha(t) = 1 + \Gamma sin(t)$

Time-varying rate of infection

Disease that has an effect only in a short period of the year:

$$\alpha(t) = \begin{cases} \sigma & \textit{for} \;\; t \in [a, b] \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

Time-varying rate of infection

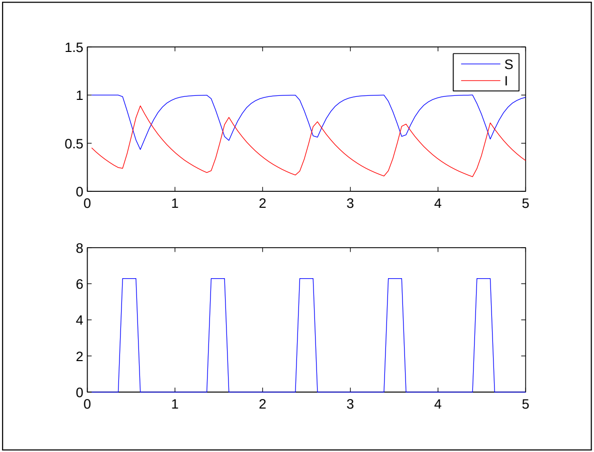

Figure: Disease that has an effect in a long period of the year

Time-varying rate of infection

It is interesting to notice that when R_0 is big enough the period of population is larger then the disease one.

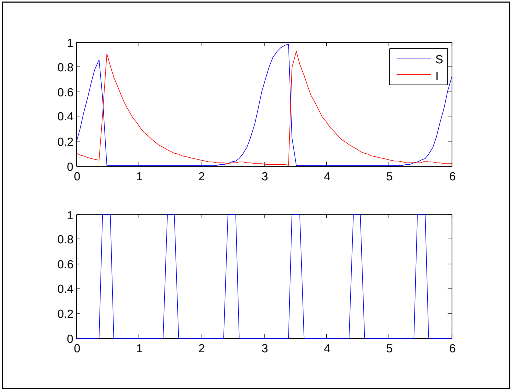

Figure: Disease with a short period of effect

Conclusions

Conclusions

- Logistic growth more realistic than exponential one
- The key parameter for virulence is R_0
- The spatiotemporal dynamics shows a spreading wave front
- Vector-driven is a more realistic transmission model
- When the disease is seasonal the variation of populations is periodic

Future work

- Two-dimensional spatial model
- Genetic immunity