Universidad Complutense de Madrid

Modelling Week - Problem 5

Option management with discontinuities in the payoff: Distribution of probability of losses

Ana Álvaro, Carl Assmann, Víctor Gallego, Emilio Pérez, Mari Carmen Prados, Alejandro Serés, Eliyas Woldegeorgis

Report

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1 Acknowledgement

This project has been accomplished in order to conclude the XI Modelling Week hosted by the Complutense University of Madrid.

In addition to this, topics such as Water Management, Neuroscience, Computer Vision, Nanotechnology, Medicine, and finance have been proposed by different companies and institutions around Europe.Thus,We have chosen "Option management with discontinuities in the pay-off: Distribution of probability of losses" proposed by Banco Popular, which was the sixth largest banking group in Spain before its purchase by the Santander group. The choice of this problem has been guided by our interest in the financial industry.
Working on this project has been very gratifying, and the problem has improved our interpersonal skills as well as professional experience in the financial market. For this reason, we are grateful to all the people who have made this project possible,and special thanks to Gerardo Oleaga, Juan Antonio Infante and Elena Castilla for helping us. More importantly, Banco Popular for presenting the topic, and the Modelling Week organising committee.

2 Introduction

Active management for derivatives including discontinuities in the pay-off (for instance, digital or binary options) is an open topic due to the value of its Greeks (specially the delta, Δ) in a neighbourhood of its discontinuities.

For a given portfolio on equity assets, a trader must be capable of replicating the position from simpler assets.The objective in this case is not the speculation but the construction of a dynamic portfolio with value approximately zero at all time.

Assuming that we are able to modify the exact composition of the portfolio almost instantly, such portfolio will be constructed and updated using sensitivities in a small neighbourhood of the studied variables: spot (S_t) , volatility (σ), and delta (Δ).

The problem in the case of the digital option is that the pay-off shows discontinuities at the strike price (K). Furthermore, Δ, that is the derivative of the present value of the option with respect to the asset price, is infinite at the strike price (K).

During the active hedging of these positions, that is when we start approaching the strike price (K) from the right and the left by strictly positive epsilon, i.e., $K - \epsilon < S < K + \epsilon$. The replication portfolio model urges us to buy or sell the asset for small movements in the value of the spot. This dynamic hedging is called "buying expensive and selling cheap". As a consequence, the agent, i.e., (trader, financial intermediary) lose the marginal profit as the financial transaction take place. This phenomena can be explained by a great Gamma, Γ (the derivative of the Delta). Hence, we propose to study the distribution of losses for different strategies of portfolio replication in these cases, and use of continuous pay-off to hedge discrete or binary ones. For instance, we will be studying binary options hedging using call spread options with different slopes.

In summary, the goal is to estimate, through simulations, the distribution of P&L (profit and losses) depending on the choice of the hedging strategy and to determine the optimal strategy, for a given risk profile.

2.1 Binary option

Consider the following financial instruments. For the sack of simplicity, we will only consider their European variant.

A digital call with a strike K and a maturity date T pays out one unit if $S(T) > K$ and nothing otherwise. The pay-off formula is thus given by:

$$B_T = \begin{cases} 0 & if \quad S_T \leq K \\ 1 & if \quad S_T > K \end{cases}$$

It other words, the general pay-off of the option can be seen as C units instead of 1, for more information see Figure 1.

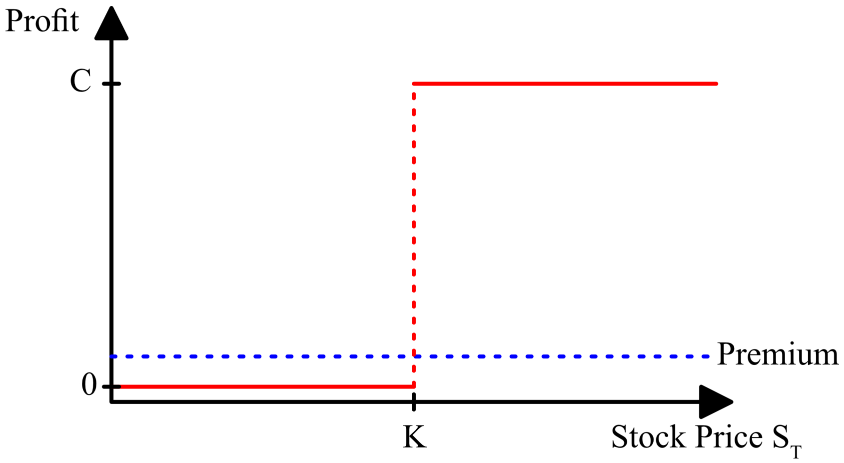


Fig. 1: Payoff for the binary option.

2.2 Call option

A call option gives the investor the right (**not the obligation**) to buy a unit of the underlying asset at an agreed price K and at a maturity date T. Consequently, the pay-off formula is given by

$$C_T = max(S_T - K, 0)$$

Note that the pay-off of the call option is a continuous function with respect to a change of S, as indicated in Figure 2.

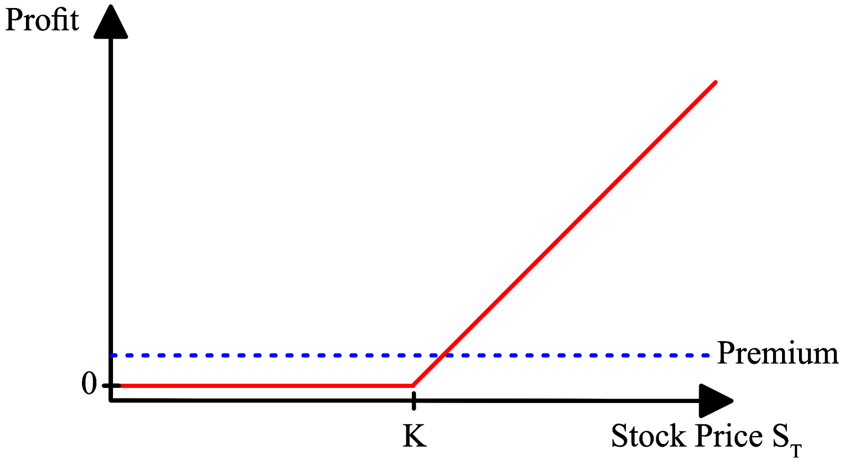


Fig. 2: Payoff for the call option.

2.3 Call spread

A call spread is an options strategy that involves purchasing call options at a specific strike price K_1 (the long position) while also selling a number of calls of the same asset and expiration date but at a higher strike K_2 (the short position). A call spread is used when a moderate rise in the price of the underlying asset is expected.

At expiration time, the call spread pay-off can be defined as the difference between the two call options with a strike price K_1 and K_2. The pay-off formula is thus given by:

$$Call\ Spread_{K_1.K_2} = Max(S_T - K_1, 0) - Max(S_T - K_2, 0).$$

This function is displayed in Figure 3. Furthermore, the main idea is to approximate the binary option using a call spread.

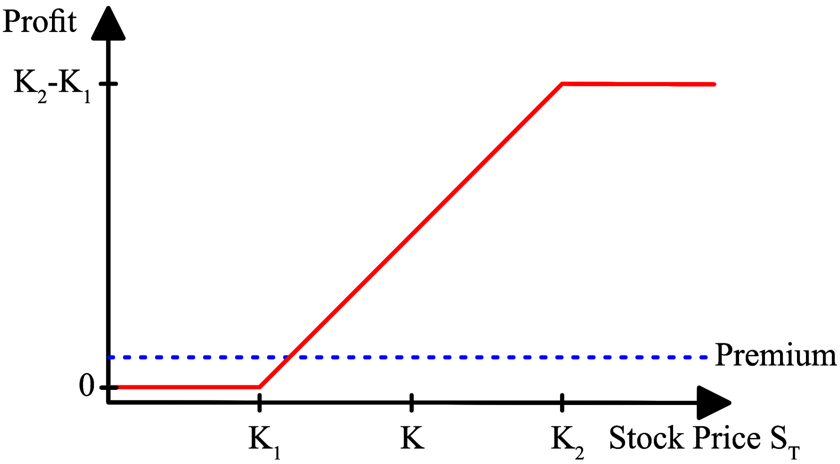


Fig. 3: Pay-off for the call spread option.

3 Black-Scholes Model

Black–Scholes model is a mathematical model of a financial market containing derivative investment instruments. From the model, one can deduce the Black–Scholes formula, which provides a theoretical estimate of the price of European-style options.

Consider the following Black-Scholes parameters:

- r is the risk-free interest rate.
- σ is an estimation of volatility.
- K is the strike price.
- T is the exercise time (that is when the contract expires and the client has the right to claim the pay-off).
- Δt is the length of the time steps.

As pointed out in the previous section, the agent has to replicate the pay-off of the digital option via assessing the underlying asset price. We also know that the digital option is path dependent, thus by simulating the possible asset prices using the Black-Scholes model and right value of epsilon one can determined the optimal strategy.

In order to obtain the underlying asset price, one has to solve geometric Brownian motion, which is given by the following:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

where $\xi \sim \mathcal{N}(0,1)$.Thus, the underlying asset is given by

$$S(t) = S_{t-\Delta_t} e^{(r-\frac{\sigma^2}{2})\Delta_t + \sigma\sqrt{\Delta_t}\xi}.$$

Notice that B_t is the standard Brownian motion. And Figure 4 shows the different values of the underlying asset obtained by the Monte Carlo simulations. Moreover, initial price for the underlying asset is 201.95.

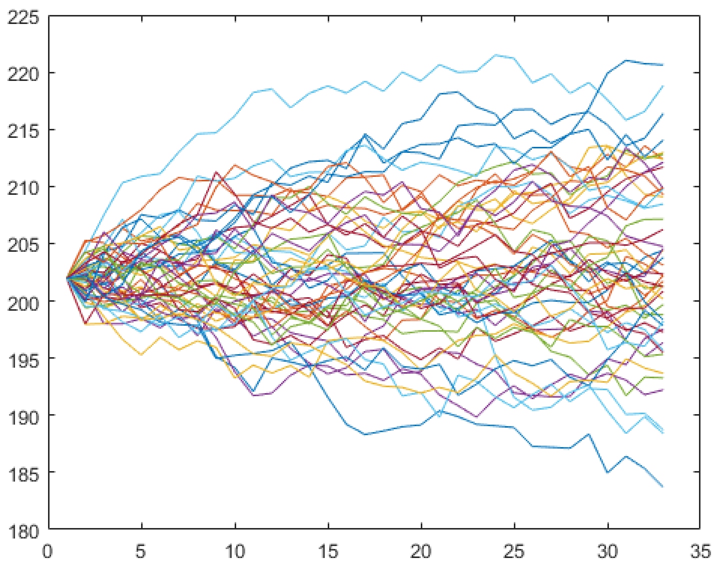


Fig. 4: Asset simulations using GBM.

After that, the price of a call option at time t can be obtained from the Black-Scholes-Merton partial differential equation. For this case, there exists an analytic formula for the solution, which is given by:

$$Call(t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_{1,2} = \frac{ln(\frac{S}{K}) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and $N(x)$ is the distribution function of the standard gaussian, i.e.,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds.$$

Figure 5 shows the graph of the valuation of a call for different values of t. Note that as $t \to T$, the graph converges to the one of the pay-off.

The price for a binary option can be obtained using the BSM model by changing contour conditions. Again, there exists an explicit formula, given by

$$Binary(t) = e^{-r(T-t)}N(d_2).$$

Figure 6 shows the graph of the valuation of a call for different values of t. Note that as $t \to T$, the graph converges to the actual payoff of the Call option. For more imofrmation see Figure 5.

In order to price the call spread, we just have to subtract two different call options at their respective strikes.

$$CallSpread(t) = Call(t, K_1) - Call(t, K_2)$$

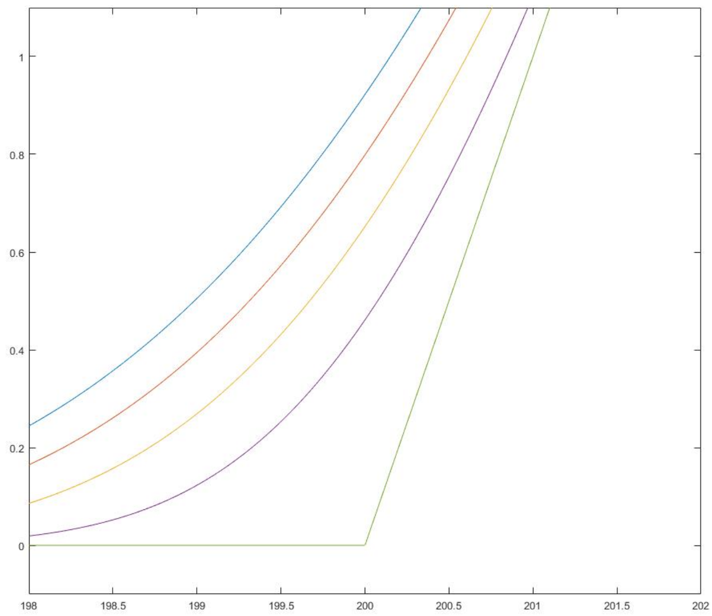


Fig. 5: Value of Call Option with $t = 0 \dots t_{end}$.

Note that K_1 is the strike of the long call position and K_2 refers to the short position. Figure 7 shows the graph of the valuation of a call for different values of t. Note that as $t \to T$, the graph converges to the one of the pay-off.

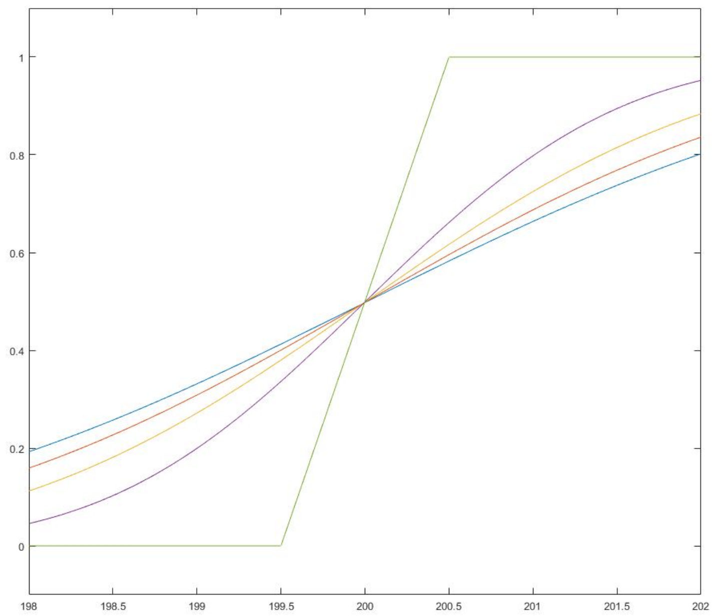


Fig. 7: Value of call spread with $t = 0 \dots t_{end}$.

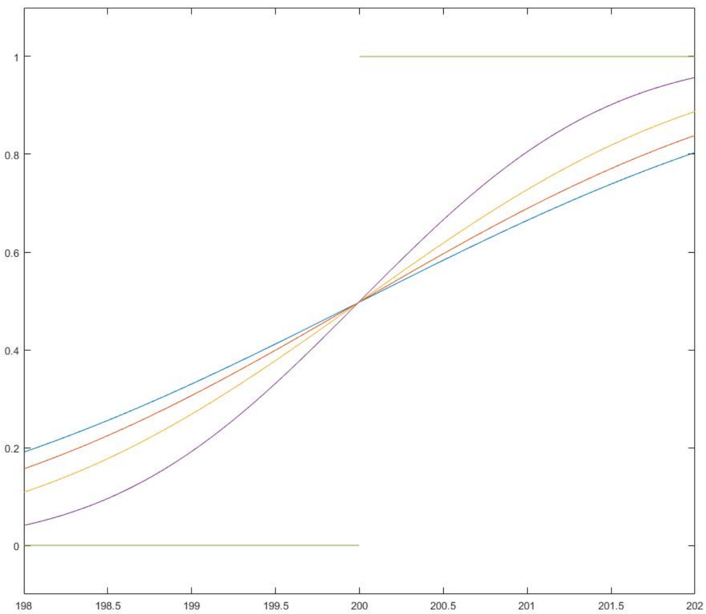


Fig. 6: Value of binary option with $t = 0 \dots t_{end}$.

It is also necessary to have formulas for the deltas Δ to hedge our positions in the market. Since the delta is just the rate of change of the price P of the option w.r.t. the price of the underlying asset, i.e, $\Delta = \dfrac{\partial P}{\partial S}$, we have:

$$\Delta_{Call}(t) = N(d_1)$$

$$\Delta_{Binary}(t) = \frac{e^{-r(T-t)}(N'(d_2))}{\sigma S\sqrt{T-t}}$$

$$\Delta_{CallSpread}(t) = \Delta_{Call}(t, K_1) - \Delta_{Call}(t, K_2).$$

Observe that Δ_{Binary} presents a ∞ discontinuity at $t = T$ (the slope is ∞ at T, due to the discontinuity of the payoff). Figures 8, 9, 10 plot the three previous deltas for different values of t.

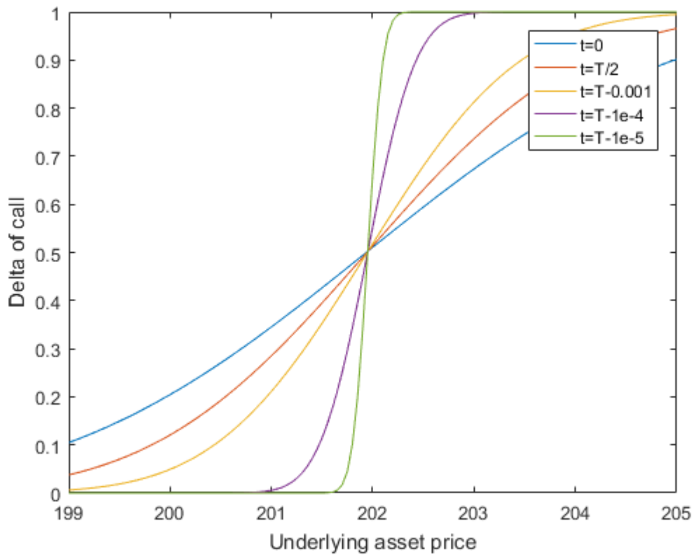


Fig. 8: Delta of call

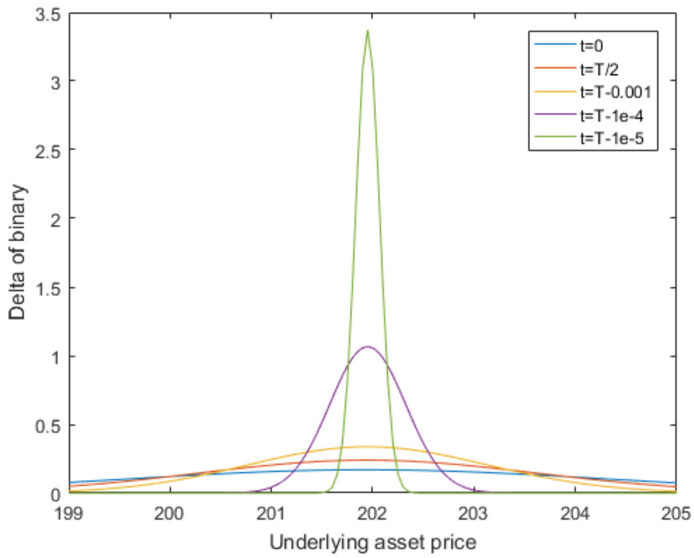


Fig. 9: Delta of the binary

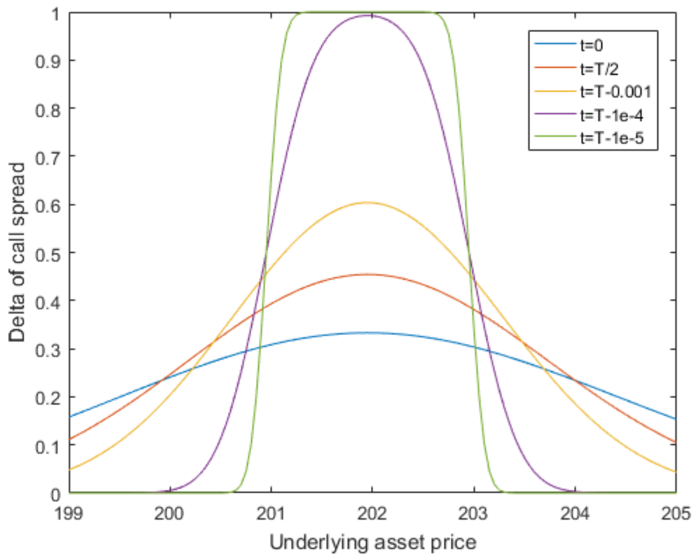


Fig. 10: Delta of call spread

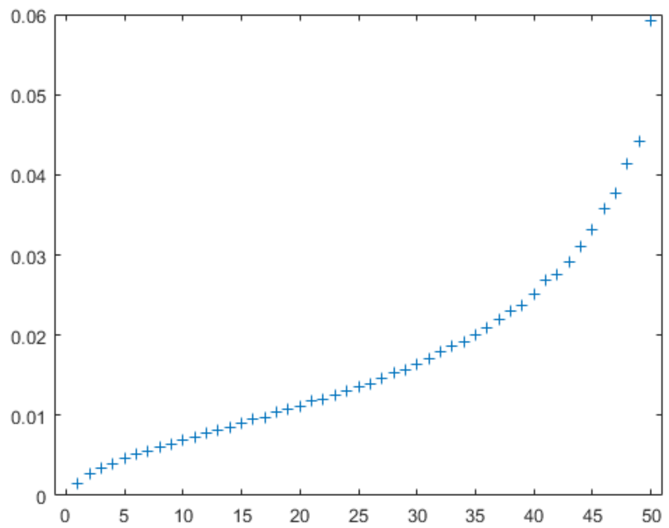


Fig. 11: Mean variation of the delta for each step over 20000 simulations of 50 steps.

In Figure 11 we can see how the closer that the time gets to the maturity date T, the bigger the changes that we have have to do to our portfolio are. That is because if the asset price value is near the strike value close to maturity, the

3.1 Hedging

To mitigate the possible losses of selling a derivative contract, we can hold a parallel investment that behaves in the opposite direction of the aforementioned contract's payment obligations. To be able to do so, we need some other asset that is also affected by the same sources of price variation as the original. In this case we will be constructing the hedge by buying or selling some units of the underlying asset, in fact, the exact quantity that we should hold at each t should be Δ_t.

We will not be able to have the perfect hedge at any time, since we are only doing transactions at discrete amounts of time, for every step that we take in our simulations, if X is the amount of units of S held, we will be changing our portfolio by:

$$(DX)_t^{t+\Delta t} = \Delta_{t+\Delta t}^{spread} - \Delta_t^{spread}$$

This dynamic adjustment will generate two types of frictions that will affect our P&L directly. The first one comes from the fact that we are holding some quantity of the asset while its price its changing. So we have to take into account:

$$\sum_{t=0}^{T-\Delta t} \Delta_t^{spread}(S_{t+\Delta t} - S_t)$$

That is the total amount of money that we have won or lost as a result of this variability in the price of the asset. The other source of profits or losses comes from the fact that the price displayed in the market is actually the mean between the price at which we can sell the asset and the price at which we buy it, that is higher.

This is known as the bid/ask spread, and it represents a real arduous that the traders need to overcome. It means that for every operation that we do with our replication portfolio, we will have to add some amount to our losses, depending on the bid/ask gap. We found a total gap of approximately 0.4% for the asset that we use to hedge, which means that we will have to count:

$$\sum_{t=0}^{T-\Delta t} (\Delta_{t+\Delta t}^{spread} - \Delta_t^{spread})^+(S_{t+\Delta t}^{bid} - S_{t+\Delta t}^{ask}).$$

As extra losses, but only when we are buying, that is when $(\Delta_{t+\Delta t}^{spread} - \Delta_t^{spread})$ is positive. We can obtain these expressions from the actual theoretical representation of the replication portfolio, if we consider that its value is tied to the ask (sell) price at each instant. We could justify that by thinking that if we had to liquidate our portfolio at any moment we would have to accept the ask price that the market is offering.

The interesting thing about adding this bid/ask constraint to our model is that it balances out with the discretization of the hedges: the more times we correct our hedge the less money it loses in comparison with the option we sold, but the more it loses by paying the bid/ask price, so we should actually see some kind of optimum value of hedges done in a day.

4 Methodology

The program has been developed in Matlab. The simulations have been realized using Monte Carlo method in order to give the new price values in a recurrent way using the previous price values for the Call and Binary options and equal to delta values. For each experiment, we generated around 2000 to 10000 simulation scenarios (the number depends in other parameters such that the number of time-steps in order to balance the total computational time).From this simulations we obtain different empirical distributions, which will be used in the next Section. In addition, all the data that are used to build the portfolio and has been updated in every step adding or quitting values, if delta is positive or negative. Finally, we have calculated the profit and losses using all data that are generated using Monte Carlo Simulations.

5 Results

5.1 Without Bid/ask

Figure 12 depicts the empirical distribution of profit & losses for a spread of $\epsilon = 0.5$. Note that the distribution is symmetrical and centred around 0.

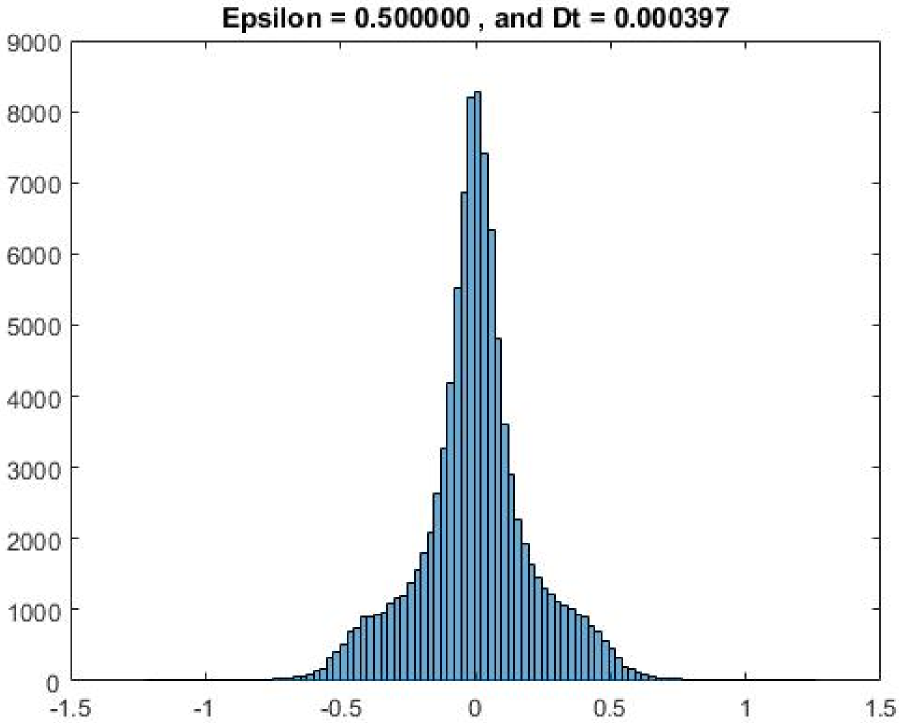


Fig. 12: Distribution of P&L with $\epsilon = 0.5$.

The reason, why this distribution has a mean equal to 0 is that in this simulation we where able to buy and sell at the exact stock price. Therefore we where able to be Δ-neutral without any costs.

To analyze the effect of the approximation using calls spread for the Δ-hedging we tested a variety of spreads (ϵ).

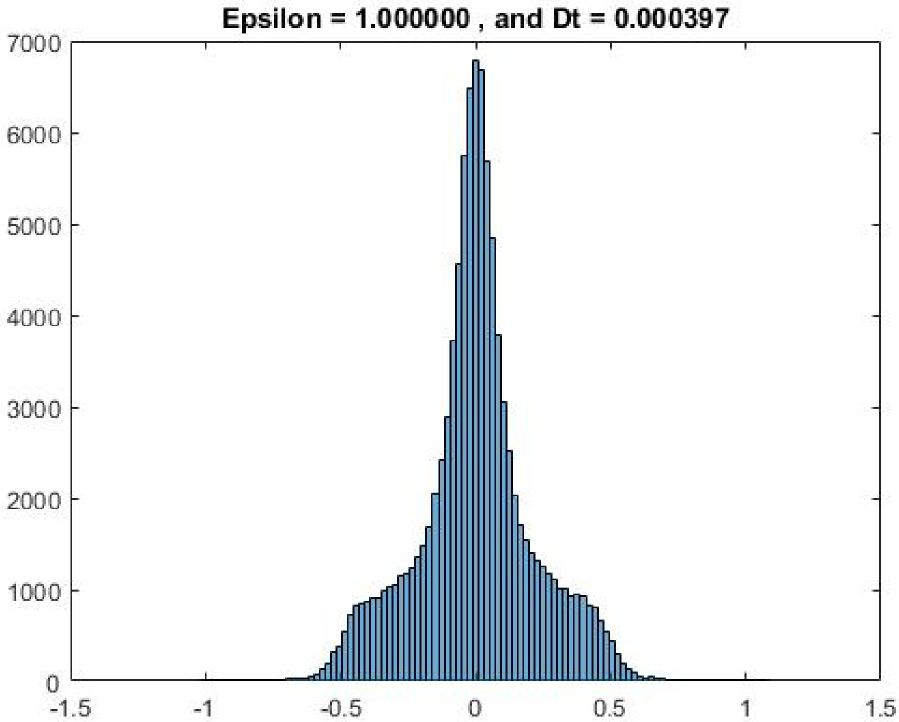


Fig. 13: Distribution of P&L with $\epsilon = 1$.

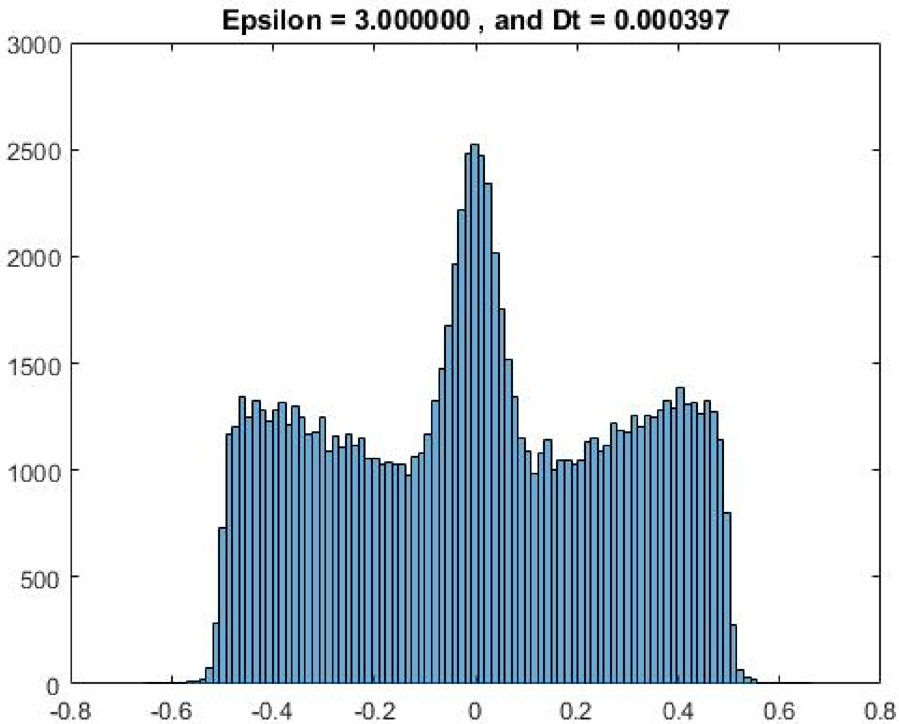


Fig. 14: Distribution of P&L with $\epsilon = 3$.

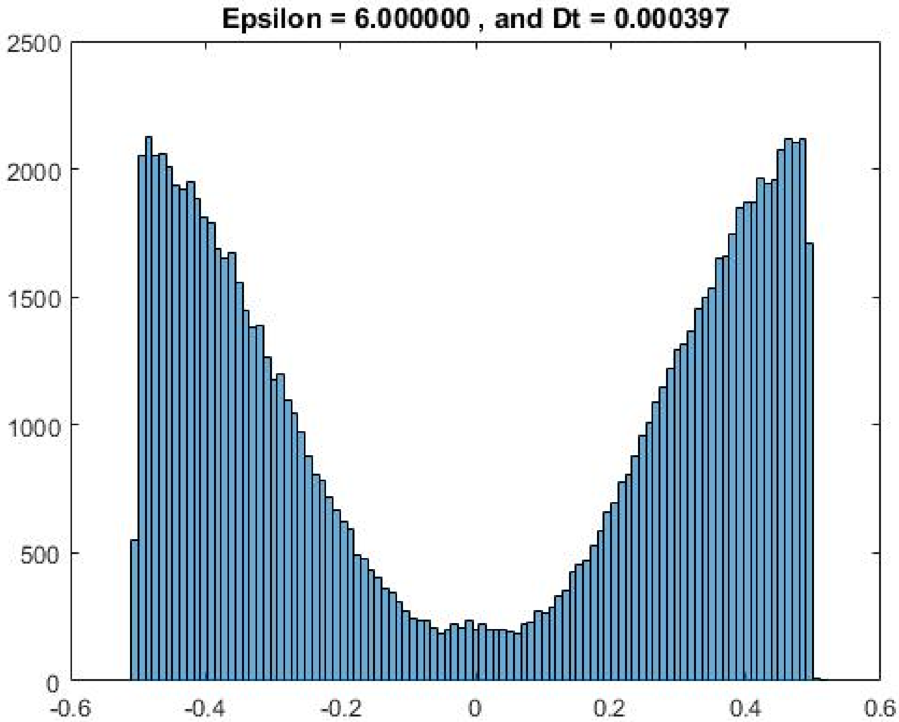
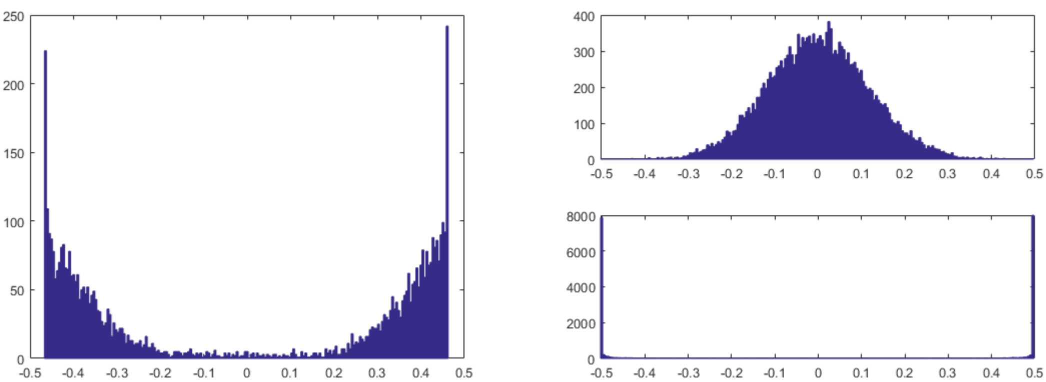


Fig. 15: Distribution of P&L with $\epsilon = 6$.

In Figure 16 we repeat the simulation with a more wide spread of $\epsilon = 10$. Now, the distribution has also mean 0, but is multimodal. This happens, because as ϵ is increased, the accuracy of the approximation has gone worse. Depending on the risk we want to take we will choose a bigger ϵ if we want to have the possibility of earning more money but with a bigger risk, or a smaller ϵ if we don't want put ourselves at risk.



(a) Distribution of P&L with $\epsilon = 10$

(b) Distribution split

Fig. 16

In this graphic 16 we have separated the variation of the prices on the asset that follows a normal distribution. The difference between the final and the initial price of the binary option is so big that it mitigates the effect of the friction of the prices.

5.2 With Bid/ask

Now we have the same graphics with a bid/ask of 0.2%.

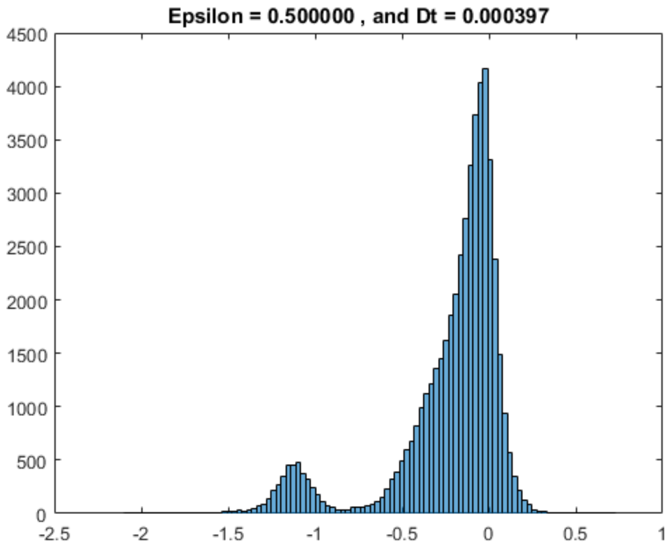


Fig. 17: Distribution of P&L with $\epsilon = 0.5$.

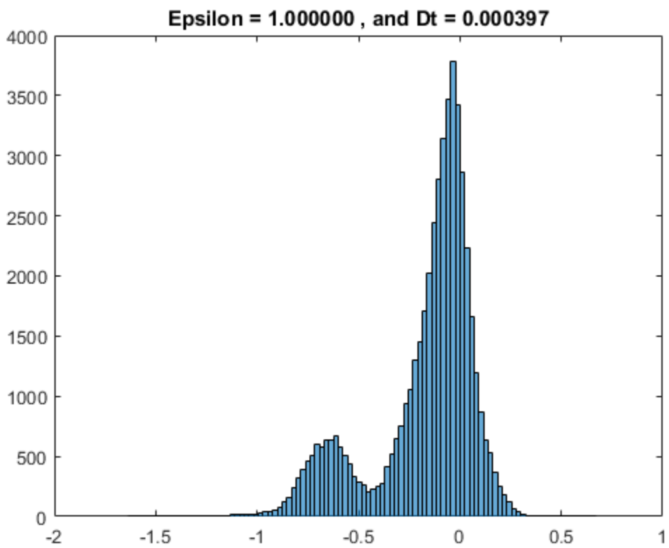


Fig. 18: Distribution of P&L with $\epsilon = 1$ and a $Bid/Ask = 0.2\%$.

With a small epsilon we can see how the distributions moved to the left, thus the mean decreased. In addition, these distributions are asymmetric.

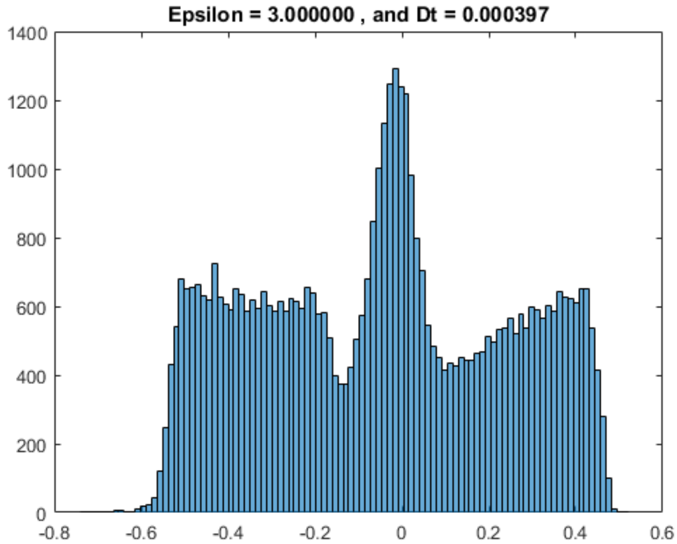


Fig. 19: Distribution of P&L with $\epsilon = 3$ and a $Bid/Ask = 0.2\%$.

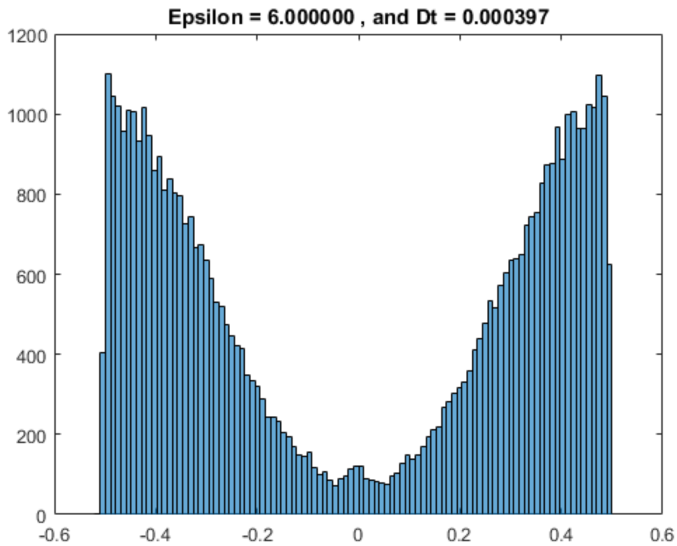


Fig. 20: Distribution of P&L with $\epsilon = 6$ and a $Bid/Ask = 0.2\%$.

However, with bigger epsilons we have similar distributions to the ones that we had without bid/ask, this is because the hedge is worst so the effect of the bid/ask is not so relevant.

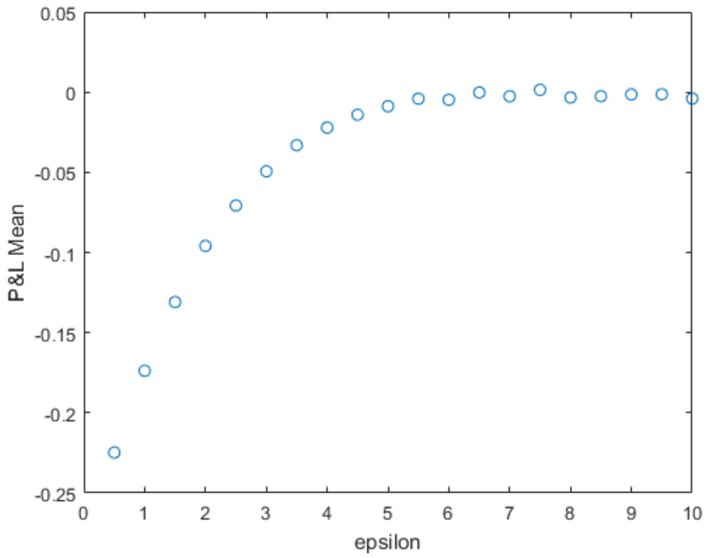
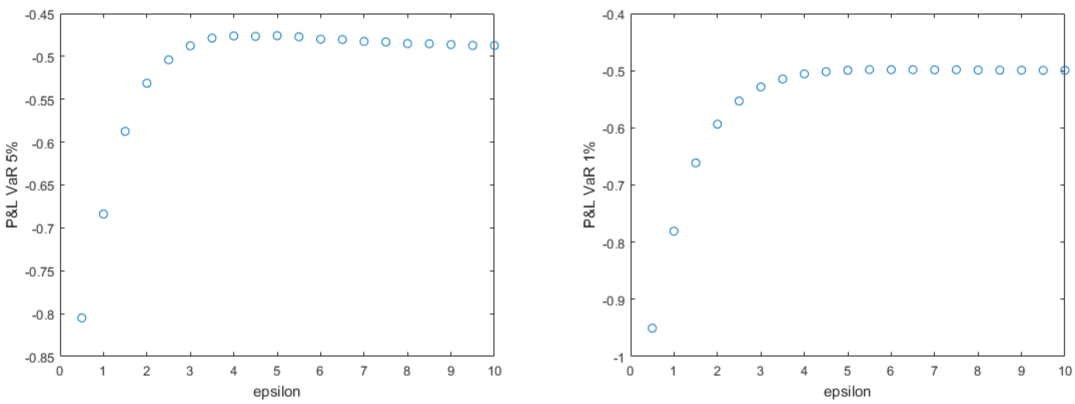


Fig. 21: P&L mean against ϵ size.

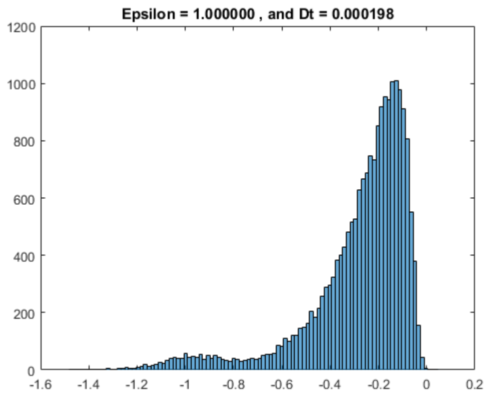
In this graphic 21 we can see that when epsilon increases the mean gets closer to zero.



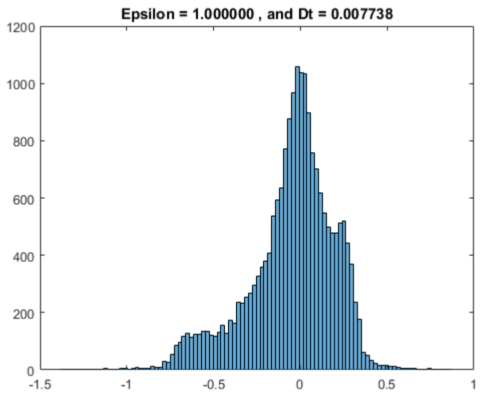
(a) P&L VaR 5% against ϵ size.

(b) P&L VaR 1% against ϵ size.

Fig. 22



(a) P&L with small dt.



(b) P&L with big dt.

Fig. 23

Figures 23 represent the value at risk of 5% and 1%. If we observe them, we will realize that there is an optimal value of epsilon where the value at risk is smaller.

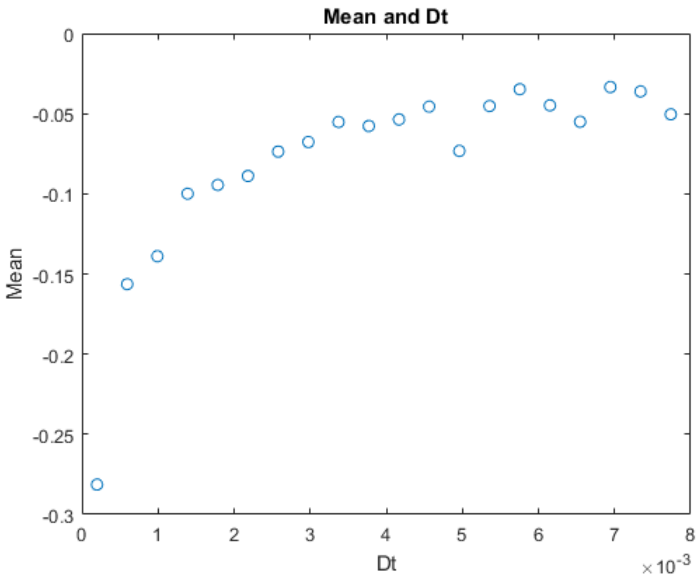


Fig. 24: Mean P&L for different dt.

Finally, we have also tested the implemented model with differents Δ_t. We can see how as Δ_t decreases, the mean also decreases because you have to buy and sell more with bid/ask.

6 Conclusions

- We successfully hedged a binary option using call spread option for delta-hedging
- Depending on the risk profile, different parameters for ϵ and Δt provide good hedging results
- we are able to manage a low and reasonable risk profile (important for banks)
- Given this hedging strategy, a fee of ~ 0.05 € would cause a mean P&L of > 0 for an 1 € binary option
- We are able to manage hedging given different parameters for volatility, Bid-Ask-spread and risk preferences

7 Outlook

Some proposals for future work are described below:

- Use an adaptive mesh so hedges become more and more frequent as we approach the time of exercise T. Thus, it may be interesting to avoid many market frictions at early timesteps, but near the end increase the hedging if necessary in order to avoid potential uncovered losses.
- Use an algorithm with a fixed threshold to decide whether to do the hedge trade at each step, depending of the amount of assets that we have to buy or sell to rebalance the replicating portfolio. That is to say, if the difference between Δ_t and Δ_{t-m}, where m is the number of steps since the last adjustment, is not big enough, we do not buy or sell assets.

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