CHALLENGES WITH VISUALIZATION. THE CONCEPT OF INTEGRAL WITH UNDERGRADUATE STUDENTS.

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In this paper, some components of a conceptual framework for the study of visualization processes from a cognitive point of view are presented along with a discussion of examples of empirical data relating to the concept of integral: (1) the coordination of registers of representation for explaining some of the students difficulties in the understanding and learning of mathematical concepts (the integral); (2) characteristics of visualization (in Calculus): it is related to the use of the graphic register in coordination with other representations and accompanied by a global apprehension; (3) the use of the graphic register (non-visual, mixed and visual methods are identified) and the higher cognitive difficulty of visual methods.

Key- words: *Visualization, Representations, University level, Integral.*

INTRODUCTION

This paper grew out of a study conducted in 2008/2009 with a group of first year students at the Universidad Complutense de Madrid (UCM). The main aim of this study was to improve the teaching of mathematical analysis by emphasising visualization processes (Souto, 2009; Souto & Gómez- Chacón, 2009). From the beginning, a big challenge was: how to characterize visualization processes in order to be able to observe them in our particular context? The literature review related to this topic highlighted a large diversity of terms and theories around the notion of visualization; most of them referred to primary and secondary levels (Duval, 1995, 1999; Arcavi, 2003; de Guzmán, 2002; Presmeg 1985, 2006; Eisenberg & Dreyfus, 1991). Therefore, we noticed a lack of empirical studies on visualization among undergraduate students and of a corresponding adaptation of some of these theoretical elements to this level of teaching.

For the sake of brevity, in this paper we focus on cognitive aspects of visualization. We mostly use the theory of registers of semiotic representation (Duval, 1995, 1999). Firstly, some theoretical ideas from this framework are outlined and reviewed. Secondly, some examples of empirical data are analyzed. It is not our aim in this paper to provide an exhaustive analysis of the results obtained in the previous study (Souto & Gómez- Chacón, 2009), but just to analyze examples of data chosen in order to provide insight into some of the specific knowledge on individual students' reasoning in relation to visualization processes, obtained in this study.

The concept of integral has been chosen because it offers an opportunity to discuss key issues concerning visualization. Research on the concept of integral (Mundy, 1987; González- Martín & Camacho, 2004) emphasize that during the first year of university, students use the concept of integral in a very mechanical way due to the lack of coordination of the concept of area and of integral, among other reasons. Furthermore, in an attempt to improve the comprehension of the concept, other authors have recommended explicit attention to visualization (de Guzmán, 2002; González- Martín & Camacho, 2004).

ABOUT THE CONCEPTUAL FRAMEWORK

Within the cognitive approach, the theory of the registers of semiotic representation (Duval, 1995, 1999) was useful in order to describe and analyze students' difficulties in the learning of mathematical concepts. In order to explore visualization processes, it has been found fruitful to combine it with some results from research on the role of visualization in mathematical reasoning (de Guzmán, 2002; Arcavi, 2003), on individual differences in the preference to visualize (Presmeg 1985, 2006) or on reasons for a reluctance to visualize (Eisenberg & Dreyfus, 1991), since that allows us making choices of important aspects for visualization inside Duval's theory.

Understanding and learning of mathematical concepts

We agree with Duval (1995, 1999) that the only possible access to mathematical objects is through their representations in the different semiotic registers. From this perspective, the *understanding of a concept* is built through tasks that imply the use of different registers and promote the flexible coordination of representations. Therefore, *learning mathematics* implies "the construction of a cognitive structure by which the students can recognize the same object through different representations" (Duval, 1999: 12).

In this context, *improving learning* implies, in particular, to minimise difficulties, misunderstandings and mental blockings that could appear in different actions related to a register: representation, treatment and conversion (Duval, 1995). As we noted in the introduction, in our specific case - the understanding of the concept of integral – research conducted with the semiotic approach highlights as a source of difficulties the lack of coordination between both the graphic and algebraic registers, and the predominance of the latter in the students' answers. This leads us to pay special attention to the use of the graphical register and to visualization.

Visualization in mathematics education

According to Duval (Duval, 1999: 15), visualization can be produced in any register of representation as it refers to processes linked to the visual perception and then to vision. For the aim of the present study, this notion is too broad; although we take into account some other characteristics of visualization pointed out by Duval. We find more useful Arcavi's definition, which is limited to the use of figures, images and diagrams.

> "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (Arcavi, 2003: 217).

Therefore, in the frame of this research we identify visualization with the use of the graphic register. However, some remarks related to this definition are needed.

In the characterization of visualization in the context of problem solving, we find very useful the difference between *visual* and *non-visual methods* established by Presmeg in her research about preference to visualize (Presmeg, 1985). However, we have to be cautious when combining Presmeg's and Duval's approaches. For example, the following equivalence cannot be established: visual method (Presmeg) - use of graphic register (Duval). We must be cautious for two reasons. Firstly, when Presmeg (2006) talks about *visual images,* she includes mental images that belong to the world of mental representations, which are different from Duval's semiotic representations (Duval: 1995: 14). We adopt this sense of mental images in relation to visualization. Secondly, the use of the graphic register does not imply that the method is visual. Duval (1999: 14) distinguishes two types of functions for the images: the *iconic* and the *heuristic*. The latter involves a global apprehension and it is related to visualization (Duval, 1999: 14). If there is use of the graphic register but there is not global apprehension or the image performs an iconic function, we will not therefore use the term *visualization*. Thus, the relevant connection is between visual methods (Presmeg) and this heuristic function of images (Duval).

Finally, Eisenberg and Dreyfus (1991) indentify three reasons to explain the reluctance of some students to visualize: "*a cognitive one (visual is more difficult), a sociological one (visual is harder to teach) and one related to beliefs about the nature of mathematics (visual in not mathematical)"* (1991:30). With the help of Duval's approach, it will be possible to describe this particular cognitive difficulty of visualization.

PARTICIPANTS AND DATA COLLECTION

The study was conducted with a first year group of 29 mathematics students at Universidad Complutense de Madrid, 15 female and 14 male. In this first year, the students followed a course called Real Variable Analysis, in which the formal definition of the concept of integral is introduced. However, they were supposed to have learned the basic rules for integration by using primitives as well as its relation to the calculation of some areas under curves already in secondary school.

For the data collection, the instruments used were a questionnaire with problems and semi-structured interviews. The questionnaire was composed of 10 non routine problems in mathematical analysis, some taken from other studies (Mundy, 1987; works quoted in Eisenberg & Dreyfus, 1991). Most of the problems are posed in the algebraic register but they also allow a visual interpretation (Eisenberg & Dreyfus, 1991). Thus these problems allow the analysis of students' performance with regard to the coordination of registers, and particularly the use of the graphic register. The results obtained from the questionnaire required deeper investigation into affective, cognitive and sociocultural aspects of individual students. In order to do this, 6 semi-structured interviews were conducted. These were divided into several parts: individual background, tasks about beliefs and preference of visualization, questions on questionnaire's answers. In this paper, attention is paid only to the cognitive aspects of the data.

For the data analysis, we privileged the use of systemic networks for the questionnaire and transcriptions for the interviews. Systemic networks (Figure 2) allow looking simultaneously at all the students' answers to the problems. Both, students and their answers, were labelled with a number from 1 to 29 included between parentheses. In particular, systemic networks favour the observation of the following elements: strategies and kinds of representation used by each student; frequency of use and difficulties of each register; and students' conceptions.

ANALYSIS AND DISCUSSION OF RESULTS

Students' difficulties analyzed through the theory of registers of representation

Students' answers to the questionnaire were analyzed using Duval's theory of registers of representation. The results described below are based on the analysis of the systemic network (Figure 2) associated with the following problem (Figure 1), but they are representative of what happens with other problems in the questionnaire.

What's wrong in the following calculation of the integral?

$$\int_{-1}^{1} \frac{1}{x^2}dx = \int_{-1}^{1} x^{-2}dx = \frac{x^{-1}}{-1}\Big|_{-1}^{1} = \frac{-1}{x}\Big|_{-1}^{1} = \frac{-1}{1} - \frac{-1}{-1} = -2$$

Figure 1 Statement of the first problem

Firstly, the choice of representation and register is very important for solving the problem successfully. These decisions are directly related to the conception used for the integral concept. All the students who gave valid answers, placed on the top of Figure 2, were either focused on the function (global properties as continuity or asymptotes; or local at $x=0$) or they contemplated the interpretation of the integral as the area under a curve. This led two of the students to the use of the graphic register. However, most of them (22 students) interpreted the integral as a process (calculation of primitives and Barrow Rule) which leads all of them to the use of the algebraic register. In this case, the students were not able to correctly answer the question. They repeated the same calculation or made some errors (using a different primitive, interchanging the signs when applying the Barrow Rule, considering the constant of integration, miscalculations).

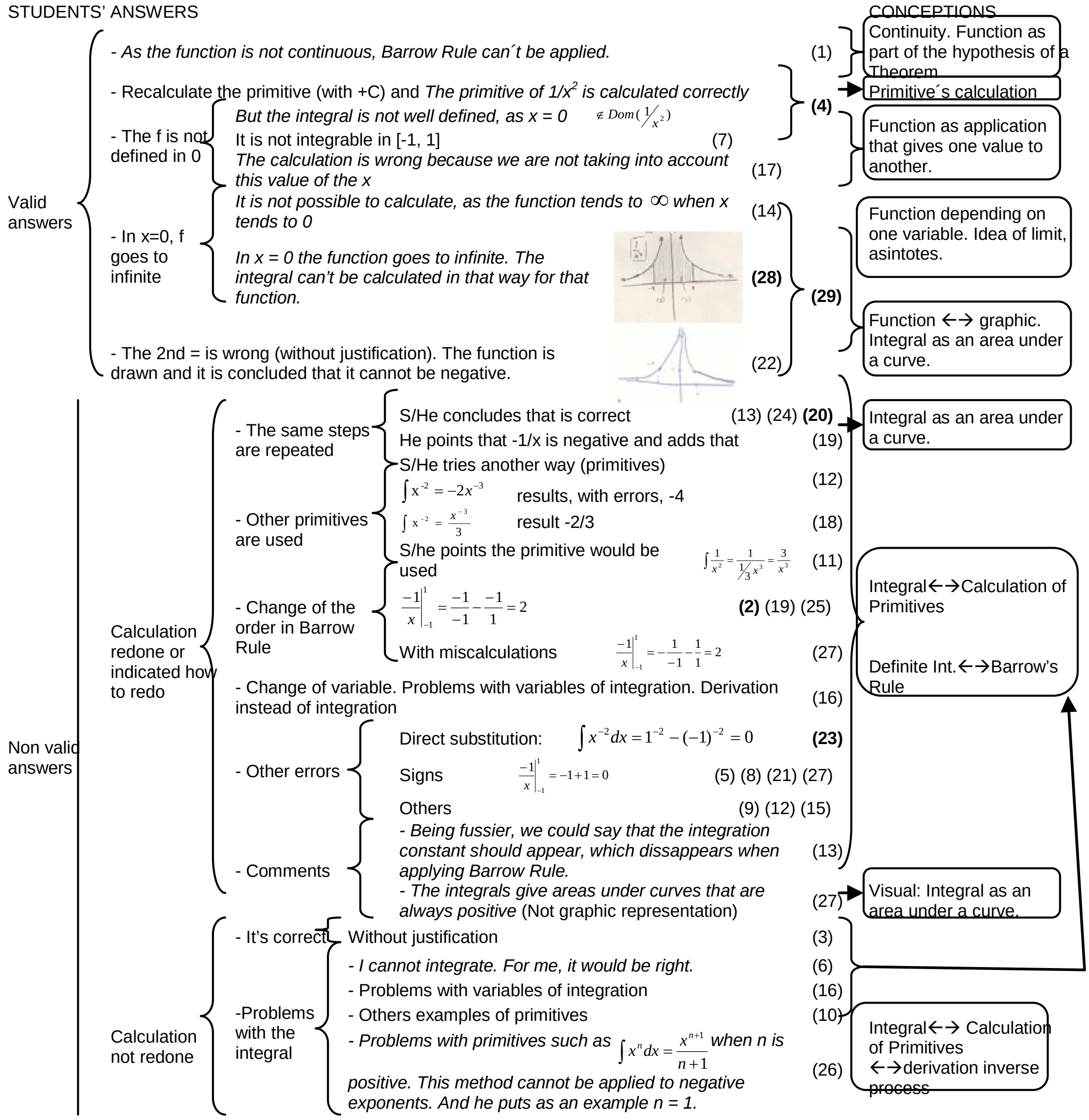

Figure 2: Systemic network associated to the problem

Secondly, we examine the way in which representations are used. The data show that the initial kind of representation chosen does not determine completely the success of the resolution. For example, student 4 was the only one who focused at the beginning on the integral as the calculation of primitives, but answered successfully. This was possible because of the flexible combination of this calculation with another argument about the domain of definition of the function. Moreover, the analysis of the answers highlights how the coordination of registers led to a better understanding of the problem (see in Figure 2, answers 28 and 29). However, there is a risk of making some errors (Figure 2, answers 19, 27) if the mobilisation of both registers is not accompanied by further reflection.

Thus, our results are coherent with previous research (Mundy, 1987; González-Martín & Camacho, 2004) described in the introduction. From a didactic point of view, the use of different kinds of representations and registers and their coordination seem to be essential. But then, how to promote their flexible coordination when teaching?

Essential for visualization: the global apprehension

Global apprehension of images is required together with the coordination of registers. However, some students do not go further than having a local apprehension and cannot see the relevant global organization (Duval, 1999: 14). Our analyses adopt these ideas as we try to show with the description of the following episode from the interviews.

1. What do you see in the image?

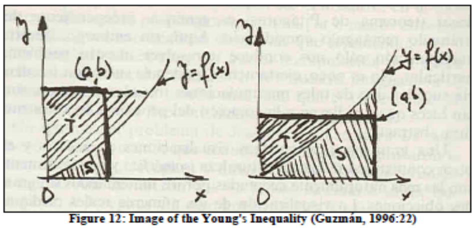

Figure 12: Image of the Young's Inequality (Guzmán, 1996:22)

2. Do you find some relation with the following heading?

Theorem 3.2 (Young's Inequality). *If f has a continuous and positive derivative on $[0, c]$ $(c > 0)$ and $f(0) = 0$, then for $a \in [0, c]$, $b \in [0, f(c)]$, we have*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx. \tag{3.3}$$

The equality holds if and only if $b = f(a)$.

Figure 13: Heading of Young's Inequality

Figure 3: Statement of the task based on Young's Inequality

The episode concerns Young's Inequality (Figure 3) and the interpretation of a graphic representation when asked for the connection with the theorem. Silvia is the name of the student chosen for the interview. She was selected because her responses to the questionnaire showed some preference for the graphic register, while she did not answer satisfactory any problem. The interview enabled us to go deeper into her difficulties with visualization.

At first, only the image was shown. Silvia detected isolated elements and even made some references to the integral as an area. Later, we showed her the statement of the inequality. She assumed the relation to the image, but it did not seem to be clear for her. She frequently requested help by asking questions. Afterwards the following conversation took place. In order to be able to continue with the interview, support is given to help her to identify correctly all the elements in the image with those in the statement. Thus, in spite of the fact that Silvia seemed to be able to coordinate the two registers, there was not any moment of clear understanding.

Interviewer: OK, what kind of explanations would you need with the drawing? Have you understood it completely, the drawing?

Silvia: Um… Well…[…]

Interviewer: OK, this [*ab*, the rectangle] is equal or less than this integral, the one which is in the drawing?

Silvia: Well, it'll be this, from 0 to *a* (she points with the finger to an interval over the *x*- axis). This one, the *S*'s.

Interviewer: OK, and the other?

Silvia: Well, *T*'s. (Silence, she seems pensive)

Interviewer: This is a little more difficult for you to see, isn't it?

Silvia: Yes.[…] Well, to understand it [the theorem], with the drawing I wouldn't understand it.

Silvia could not go beyond the mere identification of the represented units. For her, the image was only an illustration, an iconic representation that does not work as a means of visualizing the statement of Young's Inequality. Therefore, the main conjecture for Silvia's difficulties with visualization in this case is the lack of global apprehension. From a didactic point of view, the following challenges emerge: Is it possible to teach how to apprehend an image globally? If so, how can it be done?

The high cognitive requirement of visual methods

During another task in Silvia's interview, she explained why she chooses *"the way they give* [*in class*]*, the definition"* as follows: *"I don't know. It's like everything is more mechanical. In the other way* [*visual*] *you have to relate, to think.* […] *It isn't that I prefer it* [*algebraic*]*, but it's easier. So, instinctively, I do it"*. This excerpt of the interview concerns the cognitive rationale pointed out by Eisenberg and Dreyfus (1991) for the reluctance to visualize. In order to go deeper into this issue, the students' use of the graphic register in the answers to the questionnaire was analyzed. Taking into account the distinction between iconic and heuristic functions performed by images (Duval, 1999), and its relation with non-visual and visual methods (Presmeg, 1985), different kinds of techniques for solving a problem using the graphic register have been detected: *non-visual, mixed* and *visual*. The data collected from the following problem (Figure 4) of the questionnaire allow us to illustrate some characteristics of each kind of technique.

If f is an odd function in $[-a,a]$ calculate $\int_{-a}^{a}(b+(f(x)))dx$

Figure 4: Statement of the second problem

This problem was answered by 20 students, and only 8 used the graphic register. The first kind of resolution (Figure 5) appeared with higher frequency (5 out of 8 students). The images appear together with the algebraic register, in which the main reasoning takes place. The images were employed either to try to remember the definition of odd functions, or to deduce some other properties. Therefore, the image was unnecessary and it performs an iconic function. The method of resolution was considered to be *non-visual*. In fact, the example shown (Figure 5) is accompanied by an image that the student interpreted by giving an incorrect definition of odd functions. In spite of this, this misunderstanding did not affect the algebraic reasoning, which is valid.

In the other two resolutions, images were interpreted as performing a heuristic function. However, they differ according to the number of conversions made between the algebraic and the graphic registers. The second resolution (Figure 7) was given by two students. It has been called *mixed* as a first step is needed in the algebraic register, in which the additive property of integrals is applied, before converting to two graphic representations, one for each integral. As an informal conversation with the student who gives the answer in Figure 7 clarified, this conversion allowed him calculating the value of the integrals, without performing treatments in the algebraic register, and coming back afterwards to it in order to finish the evaluation of the integral. Thus, two conversions were made (algebraic- graphic- algebraic).

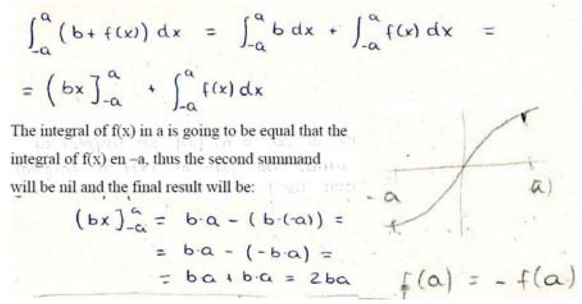

Figure 5: Non visual method

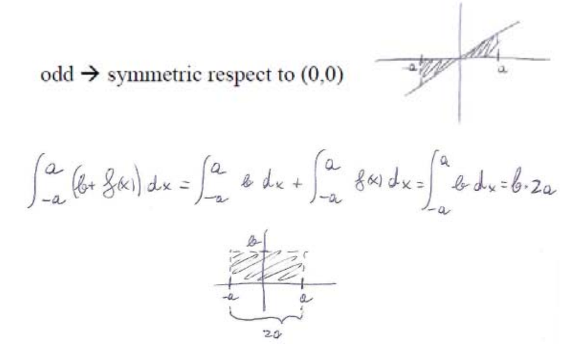

Figure 7: Mixed method

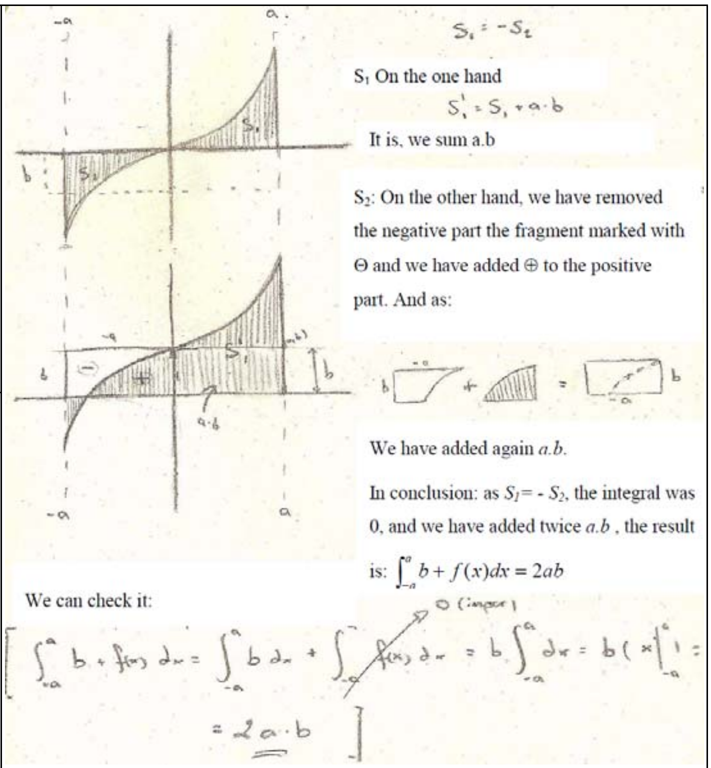

Figure 6: Visual method

The third resolution (Figure 6) is completely *visual* since it includes, at the beginning, just one conversion to the graphic register, in which the main argument is developed. In order to give this visual answer, more concepts and relations than in the non-visual ones (including the pure algebraic one) must be considered simultaneously: a visual interpretation of the integral as an area, and the odd functions as those symmetric around the origin; the recognition that adding a constant quantity to a function means a translation of its graph along the *y*-axis (this notion is not necessary in the algebraic reasoning); finally some image treatment of that sort of “cutting” and “gluing” areas, taking into account their signs. Therefore, this original answer given by only one student provides the opportunity to show how the cognitive theory of registers of semiotic representation serves to explain and to go deeper into the cognitive difficulty of visualization noted by Eisenberg and Dreyfus (1991).There is also at the end an algebraic answer used for checking. It follows a linear process consisting in the succession of several algebraic treatments, made without errors, giving as result *2ab*. Obviously, in both arguments, the result is the same. However, each kind of argument leads us to see the problem in a very different way.

Therefore, from a didactic point of view, the combination of visual and non-visual arguments when teaching seems to be advisable, since it provides complementary kinds of understanding. However, as was argued in the conceptual framework and as the empirical data have shown, this should not be misinterpreted as just using the graphic register. The following challenges for teaching emerge: how to combine visual and non-visual methods in class in order to improve the understanding of the students? How can the higher difficulty of visual arguments be handled in the class?

CONCLUSION

In this paper, we aim to: (1) present some theoretical ideas found to be relevant in a conceptual framework for a cognitive perspective of visualization; (2) show the analysis of some examples of empirical data in order to provide insight into two levels, the theoretical ideas presented and the individual students’ reasoning.

The theoretical framework of the cognitive theories of registers of semiotic representation (Duval, 1995, 1999) was useful in order to: (1) describe some difficulties of the students in the understanding and learning of mathematical concepts, in this case, the integral; (2) explore some conditions for visualization (in Calculus), related to: the explicit or implicit use of the graphic register, to the coordination with other representations (in the same or different registers) and, as Silvia’s episode showed, to the necessity of a global apprehension of the image; (3) examine the students’ use of the graphic register and the higher cognitive difficulty of visualization argued by Eisenberg & Dreyfus (1991). Moreover, visualization is related to the *heuristic function of images* (Duval, 1999) which has been identified with the *visual methods* (Presmeg, 1985). This connection led us to distinguish three different kinds of methods for solving a problem using the graphic register: *non-visual*, *mixed* and *visual*. Although this possibility has not been fully exploited in this paper, it enables us to shift our attention to individual differences in the preference to visualize.

From a didactic point of view, some challenges around a specific teaching of visualization emerge: how to use different kind of representations and registers in order to promote a flexible coordination between them? Is it possible to teach how to apprehend an image globally? How could it be done? How to combine visual, mixed and non-visual methods in class in order to improve the understanding of the students? How to manage the higher difficulty of visual methods?

NOTES

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