CONTS

Pictures at a DERIVE's exhibition (interpreting DERIVE's SOLVE command)

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Abstract

General non-linear polynomial system solving defied mathematicians for many years. Approximate methods were the only alternative until the sixties, when the first general and effective method was found (Gröbner bases method). Although implementations of Gröbner bases' algorithm are incorporated to all Computer Algebra Systems, they are only known to a small number of members of the scientific community, most of them mathematicians. Although non-linear polynomial system solving is many times a huge task, there are situations where an exact solution is needed. We present an elementary introduction to the geometric interpretation of DERIVE's SOLVE command for polynomial equations and systems.

Dedicatory

This paper is dedicated to the memory of Miguel de Guzmán (1936-2004), who was one of the keynote speakers at the *Fifth International Derive & TI-89/92 Conference* (*VisiT-Me 2002*, Vienna).

Miguel studied Philosophy in Germany and Mathematics in Spain, and obtained his Ph.D. in Mathematics from Chicago University. He taught at universities in the U.S. (Chicago, St. Louis and Princeton), Sweden and Brazil, and was a professor of Mathematical Analysis at the *Universidad Complutense de Madrid* since 1982. He had published many books, translated up to five languages. But he was probably best known for been president of ICMI from 1991 to 1998.

Apart from been an excellent mathematician, I would like to underline Miguel as a human being. He was always ready to help, and he dedicated a good part of his spare time to those who most needed it.

I feel very lucky and honored, for having had him first as professor and later as colleague and friend.

1 Introduction

Research articles in Biology, Medicine, Pharmacy, Social Sciences... are usually supported by statistical analysis, directly computed by the authors themselves, thanks to the facilities provided by statistical packages like BMDP, SPSS, Minitab, Statgraphics... Very often, the users of these packages have no idea of the theory in the background. The statistical analysis supporting the hypotheses of the authors is many times restricted to check that a certain value, obtained by clicking on an icon of a windows-based piece of software, lies in an certain interval or is greater than a certain percentage. Typical examples of this situation are “hypotheses tests” and “cluster analysis”.

It is clear that new situations arise with the arrival of technology. For instance, now it is accepted to ask a student to calculate the length of the side of a depot of cubic shape, which volume has to be 10 m^3, using a calculator. The student probably doesn't know (and will never know!) the algorithm of the cubic root. That is, the process inside the calculator will be kept forever as a black-box [1]. Therefore, in the whole solving process:

real world problem
↓
rewriting the problem in the language of mathematics
↓
determining the algorithm to be used
↓
computing of the solution (applying the algorithm)
↓
adapting the solution to the real problem

learning the algorithm is neglected. This is a situation that has offended the mentality of mathematicians for a long time.

The orientation given to the subject is very important in Mathematics teaching. There are two possibilities to develop a Mathematics curriculum:

- to understand Mathematics as a beautiful subject with an interest in itself, and to present (only) topics that can be fully justified, selecting the topics and the order in which they are presented accordingly

or

- to understand Mathematics as a tool, of which most topics will be fully justified, but where some will (only) be intuitively justified.

Curiously, the first option is usually a strict rule of the School of Mathematics, meanwhile the second one is the usual approach at the School of Physics.

General non-linear polynomial system solving is not a simple task. This problem defied mathematicians for many years. Approximate methods were the only alternative until the late sixties, when the first general and effective method was found (Gröbner bases method) [2]. Although implementations of Gröbner bases' algorithm (Buchberger's algorithm) are incorporated to all Computer Algebra Systems (CASs): DERIVE, Maple, Mathematica, MuPad, Reduce, Macsyma, Axiom, CoCoA..., they are only known to a small number of members of the scientific community, most of them mathematicians.

Although non-linear polynomial system solving is many times a huge task that has to be faced using numerical methods, there are situations where an exact solution is needed (and can be easily computed!).

We present an elementary introduction to the geometric interpretation of DERIVE's SOLVE command for polynomial equations and systems (this command calculates internally a Gröbner basis of the corresponding ideal if the system isn't linear). An introduction to Ideals Theory can be found in [3] (in Spanish). An elementary introduction to Gröbner bases and a short review of their applications can be found in [4] and [5], respectively. More details about Gröbner bases can be found, for instance, in [6,7,8]. A review of applications, detailing the algebraic background, can be found in [9].

2 Solving linear equations and linear systems

2.1 Solving a linear equation (univariate case)

What does DERIVE when we ask it to SOLVE a polynomial equation in one variable, i.e., an equation of the form: univariate linear polynomial=0, like the following?

```
#1:   SOLVE(3·x - 6 = 0, x)

#2:                                  x = 2
```

We are asking DERIVE to look for the (unique) value of x that makes the equality true. From the geometrical point of view, we are looking for an isolated point on the Real Line.

As Derive can't draw 1D plots, we'll plot the point in the 2D-Plot Window introducing a fake null y-coordinate (see Figure 1).

```
#3:   [2 0]
```

Nevertheless, there is a more interesting interpretation from the geometrical point of view: we can consider this solution (i.e., this numerical value) as the x-coordinate of the point of intersection of line $y = 3x - 6$ with the x-axis (see Figure 2)

```
#4:   y = 3·x - 6
```

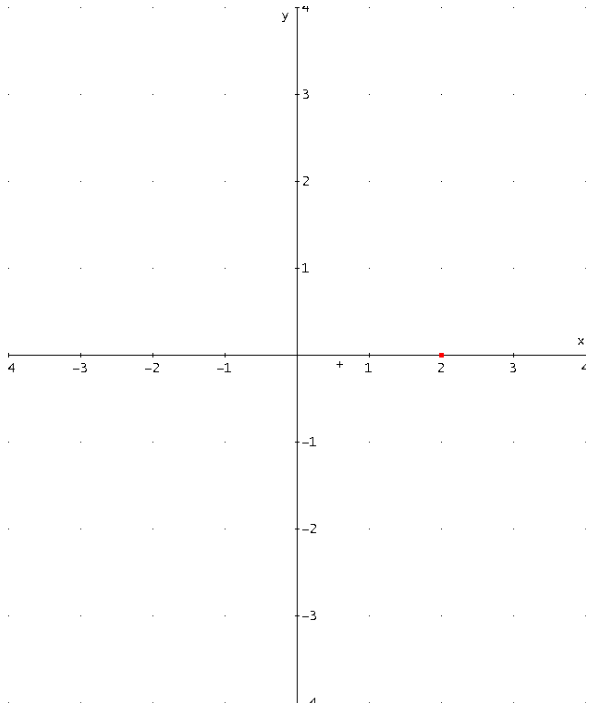

Figure 1: Solution of a linear equation in one variable.

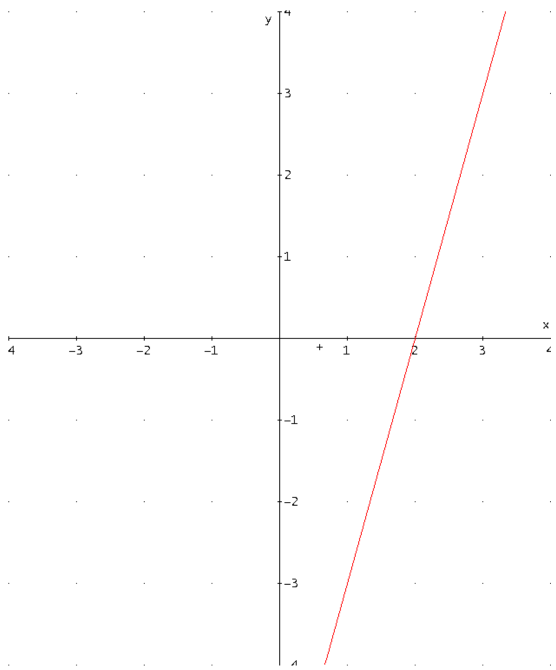

Figure 2: Another geometric interpretation of the solution of a linear equation in one variable.

2.2 Solving a linear system (univariate case)

And when we ask it to SOLVE a polynomial system (in one variable)?

```
#5:    SOLVE([3·x − 6 = 0, 2·x − 4 = 0], x)

#6:                                     [x = 2]
```

We are asking DERIVE to check if the value of *x* that satisfies the first equation makes all the equalities true (what will not be normally the case).

From the geometrical point of view, we are looking for an isolated point on the Real Line (see Figure 3). Again, as DERIVE can't draw 1D plots, we'll plot the points introducing a fake null *y*-coordinate. In this particular case a solution exists.

```
#7:    [2,0]
```

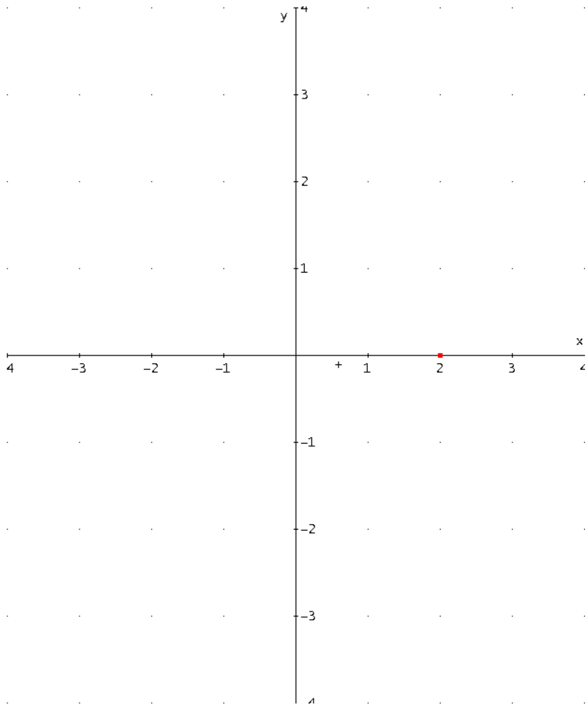

Figure 3: Solution of a polynomial system in one variable

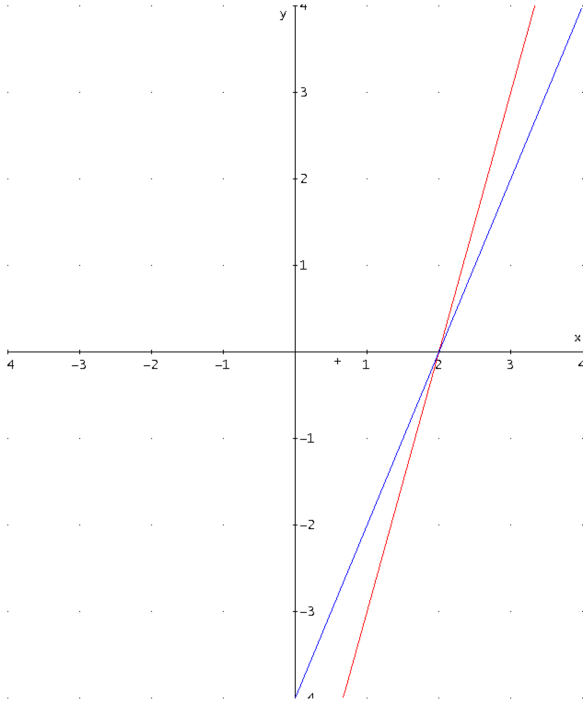

Figure 4: Another geometric interpretation of the solution of a polynomial system in one variable.

But, again, there is a more interesting interpretation from the geometrical point of view: if a solution exists, it is the x-coordinate of the point of intersection of the lines $y = 3x - 6$ and $y = 2x - 4$ and the x-axis (see Figure 4).

```
#8:   [y = 3·x - 6, y = 2·x - 4]
```

Therefore, such an intersection point exists in this example (its coordinates are (2,0)), so the only solution is $x = 2$.

2.3 Solving a linear equation (multivariate case)

What does DERIVE when we ask it to SOLVE a linear equation in two (or more) variables, i.e., an equation of the form: multivariate linear polynomial = 0, like the following?

```
#9:   SOLVE(x - y + 2·z, [x, y, z])

#10:                              x - y + 2·z = 0
```

We are asking DERIVE to look for the values of x, y and z that make the equality true. For instance ($x = 0$, $y = 0$, $z = 0$) or ($x = 2$, $y = 6$, $z = 2$) are both solutions of this equation. From the geometrical point of view (see Figure 5), the solution set is a line, plane or hyperplane (if two, three or more than three variables appear in the polynomial, respectively). That's why DERIVE returns the input with no changes.

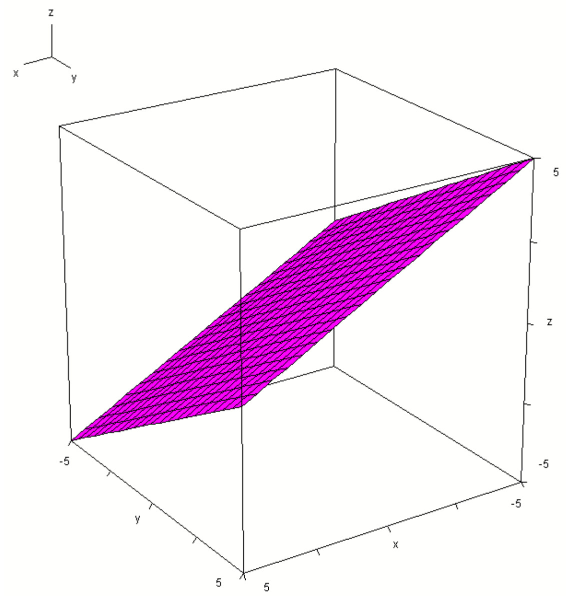

Figure 5: Geometric interpretation of the solution of a multivariate polynomial equation.

2.4 Explicit solution of a single linear equation (multivariate case)

But if we ask DERIVE to SOLVE a linear equation in two (or more) variables, i.e. an equation of the form: multivariate linear polynomial = 0, w.r.t. one single variable, like in the following example:

```
#11:    SOLVE(x − y + 2·z, z)
```

#12: $$z = \frac{y - x}{2}$$

DERIVE expresses explicitly the chosen variable as a linear function of the other variables (what is always straightforward).

2.5 Solving a linear system (multivariate case)

And when we ask DERIVE to SOLVE a linear system in two or more variables w.r.t. all variables?

```
#13:    SOLVE([y = x + 2, y = − 2·x − 1], [x, y])

#14:                        [x = −1 ∧ y = 1]
```

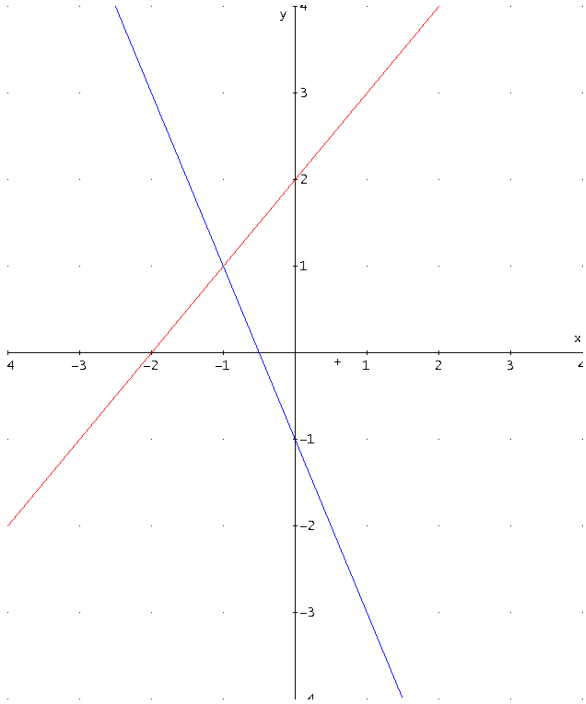

Figure 6: Geometric interpretation of the solution of a multivariate polynomial system.

We are asking DERIVE to look for the values of *x* and *y* that satisfy all the equations simultaneously (see Figure 6). From the geometrical point of view, we are asking for the intersection of the solution sets corresponding to each equation (lines in this case, where two variables appear in the polynomials; planes if three variables appear and hyperplanes if more than three variables appear). In this example such intersection is point (-1,1).

3 Solving algebraic systems: distinguishing the univariate and multivariate cases

An *algebraic equation* is an equation of the form: general polynomial = 0. *An algebraic system*, also called *polynomial system*, is a set of algebraic equations.

3.1 Solving an algebraic equation (univariate case)

What does DERIVE when we ask it to SOLVE a polynomial equation in one variable, i.e., an equation of the form: univariate polynomial = 0, like the following?

```
                3
#15:    SOLVE(x  - x = 0, x)

#16:                   x = -1 ∨ x = 1 ∨ x = 0
```

We are asking DERIVE to look for the values of *x* that make the equality true. From the geometrical point of view, we are looking for isolated points on the Real Line.

As DERIVE can't draw 1D plots, we'll plot the points in the 2D-Plot Window introducing a fake null y-coordinate and choosing Options/Display/Points/Connect=No (see Figure 7).

$$\#17: \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Nevertheless, there is a more interesting interpretation from the geometrical point of view: we can consider these solutions (i.e., these numerical values) as the x-coordinates of the points of intersection of the curve $y = x^3 - x$ with the x-axis (see Figure 8):

$$\#18: \quad y = x^3 - x$$

(in this interpretation, the intersection points (-1,0), (0,0) and (1,0) correspond to the solutions –1, 0 and 1, respectively).

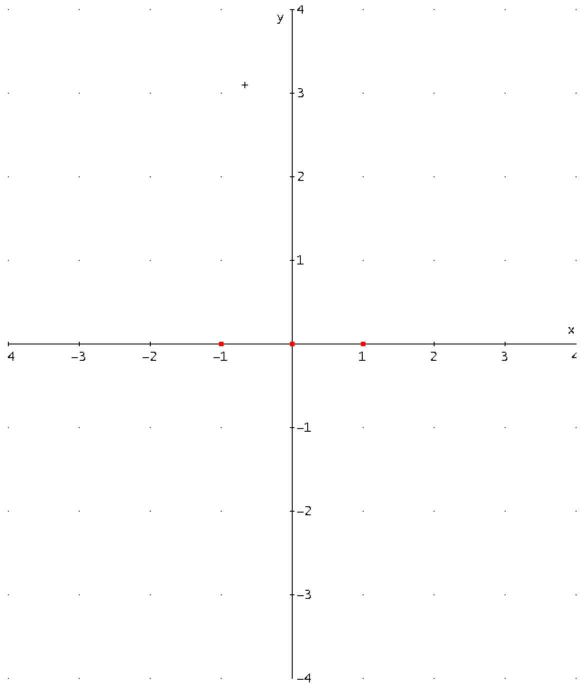

Figure 7: Solutions of a polynomial equation in one variable.

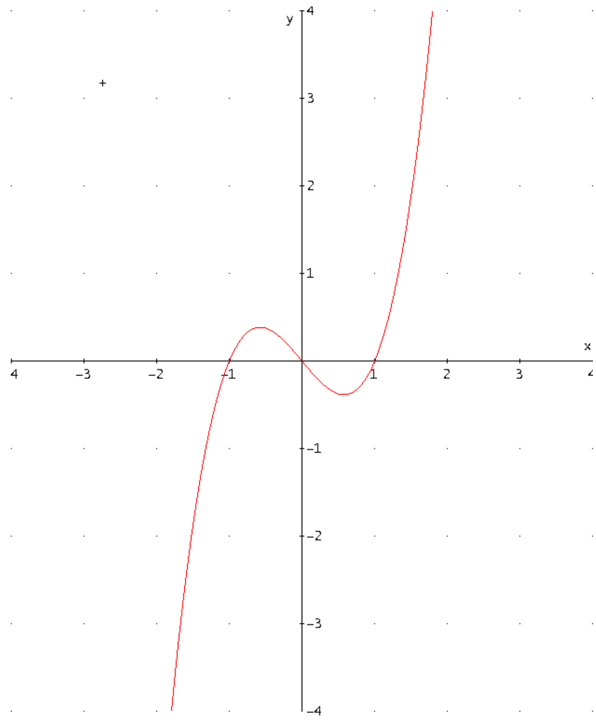

Figure 8: Another geometric interpretation of the solutions of a polynomial equation in one variable.

3.2 Solving an algebraic system (univariate case)

And when we ask DERIVE to SOLVE a polynomial system (in one variable)?

#19: $\text{SOLVE}\left(\left[x^2 - 1 = 0,\ -x^2 - x + 2 = 0\right],\ x\right)$

#20: $[x = 1]$

We are asking DERIVE to look for the values of *x* that make all equalities true. From the geometrical point of view, we are looking for isolated points on the Real Line (see Figure 9). Again, as DERIVE can't draw 1D plots, we'll plot the points introducing a fake null *y*-coordinate (and choosing Options/Display/Points/Connect=No in the 2D-Plot Window). In this particular case there is only one solution.

#21: [1,0]

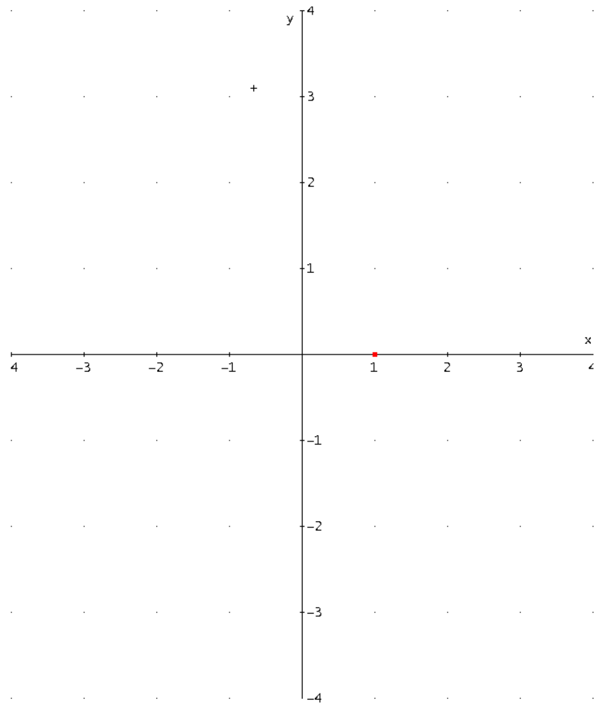

Figure 9: Solutions of an algebraic system in one variable

But, again, there is a more interesting interpretation from the geometrical point of view: we can consider the solutions as the x-coordinates of the points of intersection of the curves $y = x^2 - 1$ and $y = -x^2 - x + 2$ and the x-axis (see Figure 10).

```
#22:  [x^2 - 1 = y, - x^2 - x + 2 = y]
```

(in this interpretation, the intersection point (1,0) corresponds to the only solution $x = 1$).

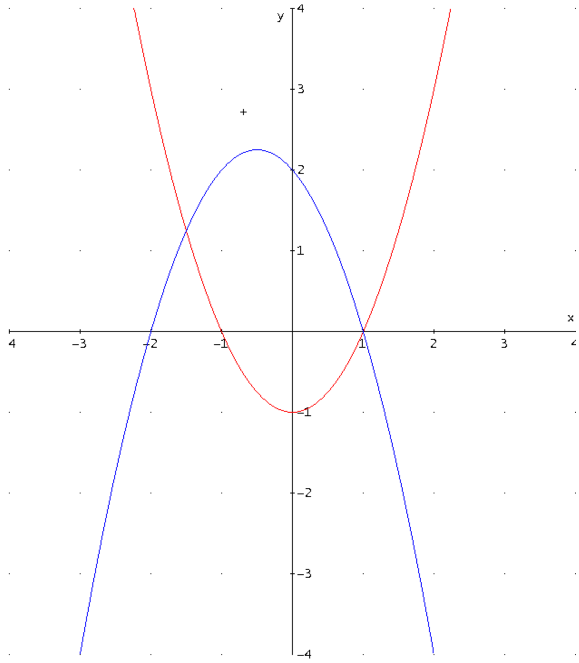

Figure 10: Another geometric interpretation of the solution of an algebraic system in one variable.

3.3 Solving an algebraic equation (multivariate case)

What does DERIVE when we ask it to SOLVE a polynomial equation in two (or more) variables, i.e. an equation of the form: multivariate polynomial = 0, like the following?

#23: $x^3 - x - y = 0$

#24: $\mathrm{SOLVE}(x^3 - x - y = 0, [x, y])$

#25: $x^3 - x - y = 0$

We are asking DERIVE to look for the values of *x* and *y* that make the equality true. For instance ($x = 0$, $y = 0$) or ($x= 2$, $y = 6$) are both solutions of the equation. From the geometrical point of view (see Figure 11), the solution set of the equation is the whole curve (or a surface, if three variables appear in the polynomial, or a hypersurface if more than three variables appear). That's why DERIVE returns the input with no changes.

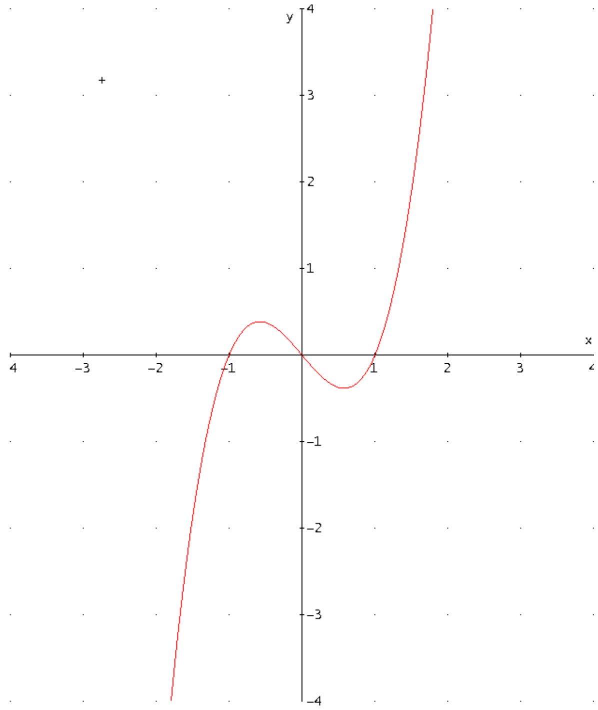

Figure 11: Geometric interpretation of the solution of a multivariate polynomial equation.

3.4 Explicit solution of a single algebraic equation (multivariate case)

What happens if we ask DERIVE to SOLVE a polynomial equation in two (or more) variables, i.e. an equation of the form: multivariate polynomial = 0, w.r.t. one single variable, like in the following example:

#26: $\text{SOLVE}\left(x^2 + \left(\frac{y}{2}\right)^2 - 4 = 0,\ y\right)$

#27: $y = -2\cdot\sqrt{(4 - x^2)} \vee y = 2\cdot\sqrt{(4 - x^2)}$

DERIVE is given an implicit expression and is asked to express the selected variable as an explicit expression in the rest of the variables. It is not always possible to achieve such an expression. And it normally happens that, although calculated, it is not a polynomial expression, like in this example (the expression involves radicals). So SOLVE is not usually used this way if we are restricting calculations to polynomial rings.

Anyway, the solution set has the same clear geometric interpretation: the solution points are the points in the corresponding implicit plot (see Figure 12):

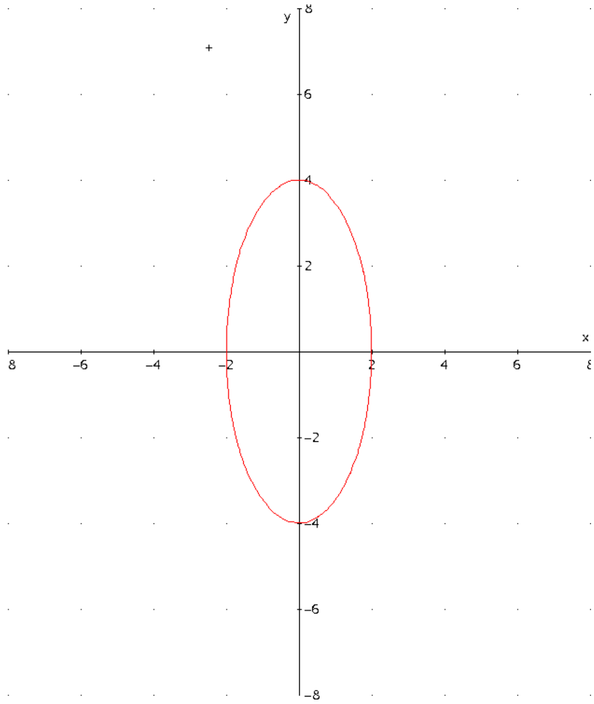

Figure 12: Geometric interpretation of the solution of a multivariate polynomial equation.

Obtaining an explicit expression can be very hard an depends on the variable chosen. For instance in the example of the previous subsection (see Figure 11), it is trivial for one variable (*y*) but it is very complicated for the other (*x*), and the expression obtained involves trigonometric functions.

#28: $\text{SOLVE}(x^3 - x - y = 0,\ y)$

#29: $y = x\cdot(x^2 - 1)$

#30: $\text{SOLVE}(x^3 - x - y = 0,\ x)$

#31: ....

In the following example (see Figure 13), one explicit expression can also be trivially obtained, meanwhile DERIVE can't do anything to express explicitly the other one:

#32: $\text{SOLVE}(x^5 - 4\cdot x^4 + x^3 + 6\cdot x^2 - y = 0,\ y)$

#33: $y = x^2\cdot(x^3 - 4\cdot x^2 + x + 6)$

$$\#34: \quad \mathrm{SOLVE}(x^5 - 4\cdot x^4 + x^3 + 6\cdot x^2 - y = 0,\ x)$$

$$\#35: \quad x^5 - 4\cdot x^4 + x^3 + 6\cdot x^2 = y$$

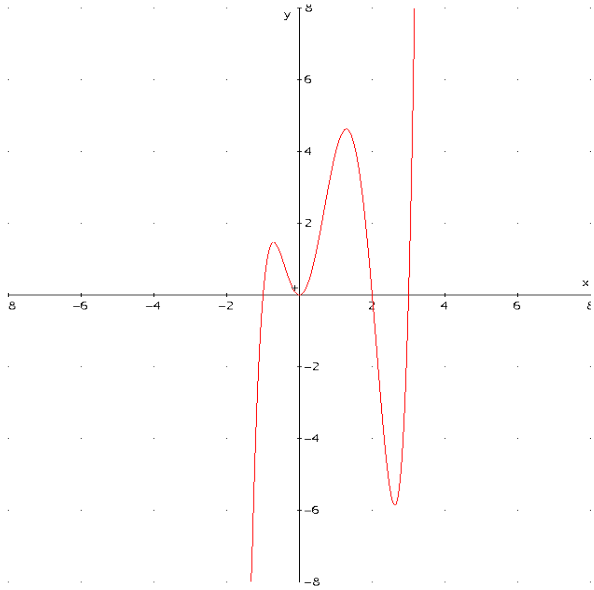

Figure 13: Geometric interpretation of the solution of a multivariate polynomial equation.

3.5 Solving an algebraic system (multivariate case)

And when we ask DERIVE to SOLVE a system (in two or more variables) w.r.t. all variables?

$$\#36: \quad \mathrm{SOLVE}\left(\left[x^2 - 1 = y,\ -x^2 - x + 2 = y\right],\ [x,\ y]\right)$$

$$\#37: \quad \left[x = 1 \wedge y = 0,\ x = -\frac{3}{2} \wedge y = \frac{5}{4}\right]$$

We are asking DERIVE to look for the values of x and y that satisfy all the equations simultaneously (see Figure 14). From the geometrical point of view, we are asking for the intersection of the solution sets corresponding to the equations (curves in this case, where two variables appear in the polynomials; surfaces if three variables appear and hypersurfaces if more than three variables appear). In this example such intersection are points (1,0) and (–3/2,5/4). The solution set of an algebraic system is also denoted *algebraic variety*.

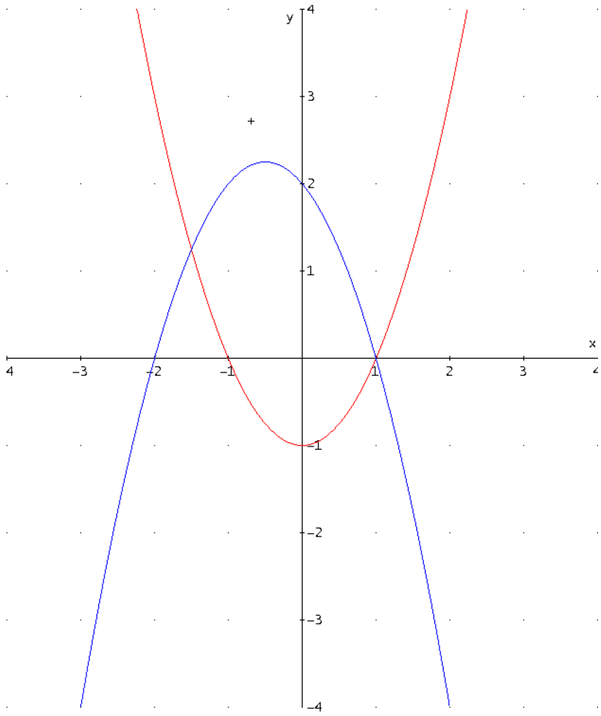

Figure 14: Geometric interpretation of the solution of a multivariate polynomial system.

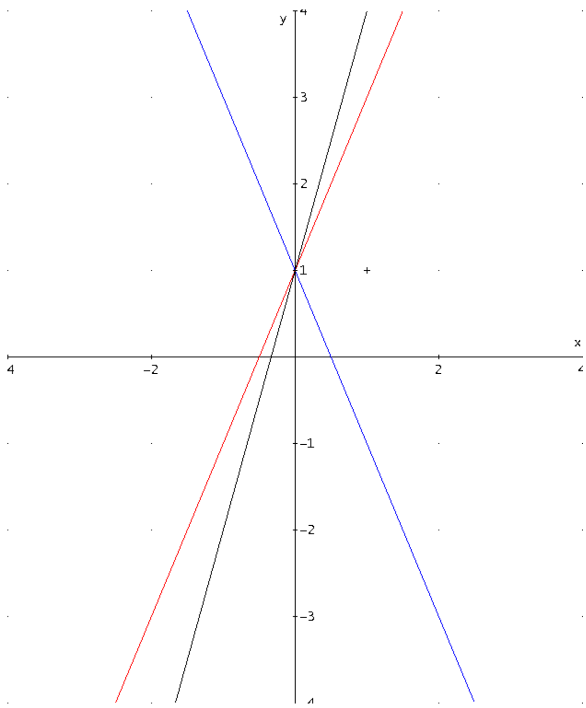

Figure 15: Geometric interpretation of the solution of a linear system in two variables.

4 Number of solutions of a linear system

The solution set of a linear system is the intersection of the solution sets (hyperplanes) corresponding to its equations (see Figure 15).

```
#38:    [2·x − y + 1 = 0, − 2·x − y + 1 = 0, 3·x − y + 1 = 0]
```

Linear systems can have: one solution, no solution or infinite solutions, as shown in the next three examples (see figures 16, 17, 18):

```
#39:    SOLVE([x − y + 1 = 0, x + y − 3 = 0], [x, y])

#40:                        [x = 1 ∧ y = 2]
```

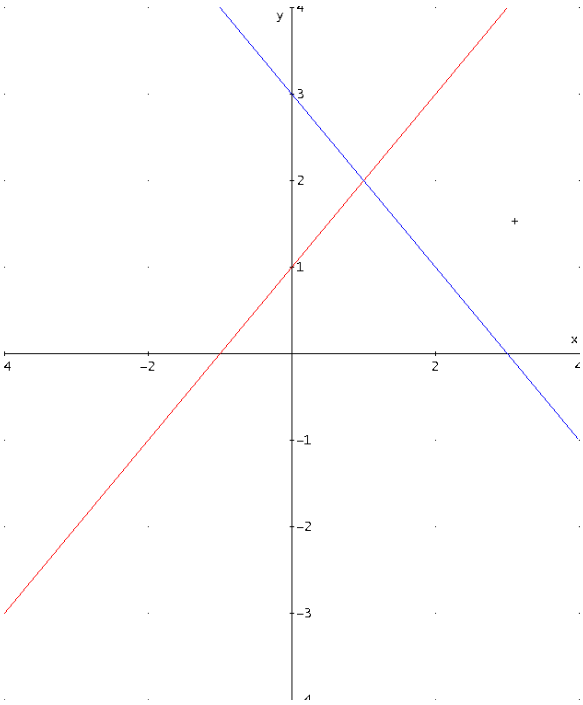

Figure 16: A linear system in two variables with a unique solution.

```
#41:    SOLVE([x − y + 1 = 0, x − y − 1 = 0], [x, y])

#42:                              []

#43:    SOLVE([x − y = 0, − 2·x + 2·y = 0], [x, y])

#44:                        [x − y = 0]
```

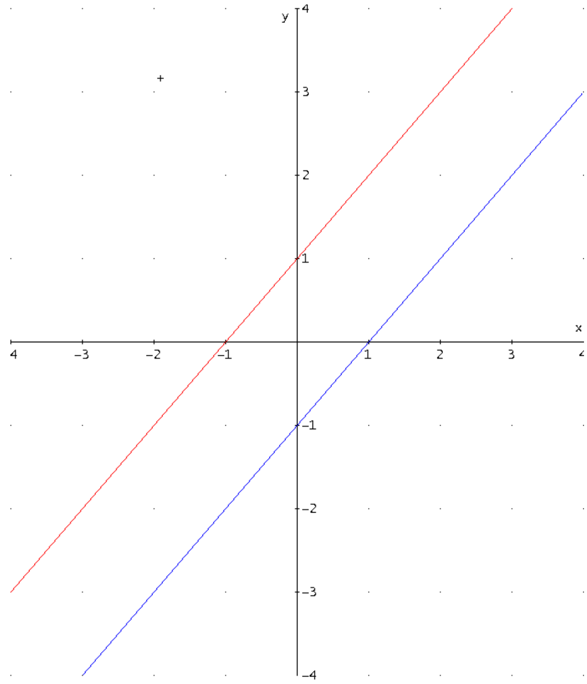

Figure 17: A linear system in two variables with no solution.

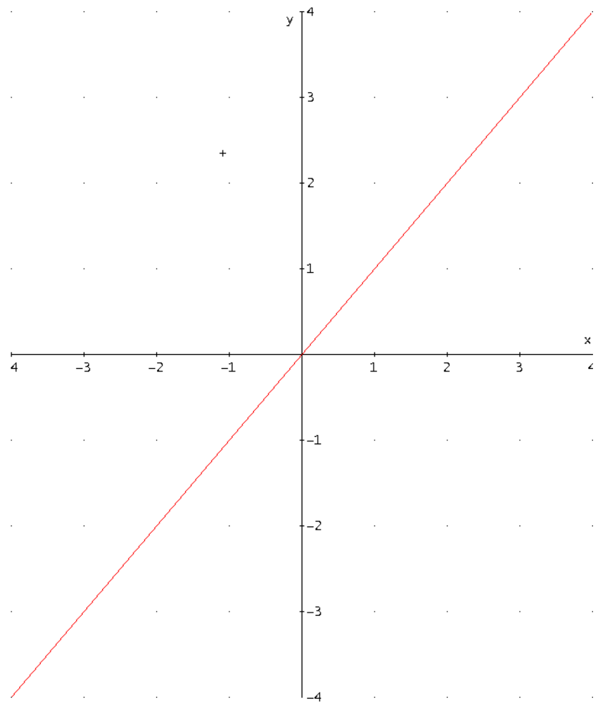

Figure 18: A linear system in two variables with an infinite number of solutions (the solution sets of both equations coincide, so the solution set of the linear system is an hyperplane).

But, if the dimension of the space is greater than 2, the solution set of the linear system could be a linear variety that is neither the empty set nor a hyperplane (i.e., a linear variety of intermediate dimension). For instance, the solution set of

#45: $\text{SOLVE}([x - y + 2\cdot z = 0,\ x - y = 0],\ [x, y, z])$

#46: $[x - y = 0 \wedge z = 0]$

is a line (see Figure 19). Clearly, all we can do is express it in a simpler way --it is not possible to express a line in the 3D space using a single equation.

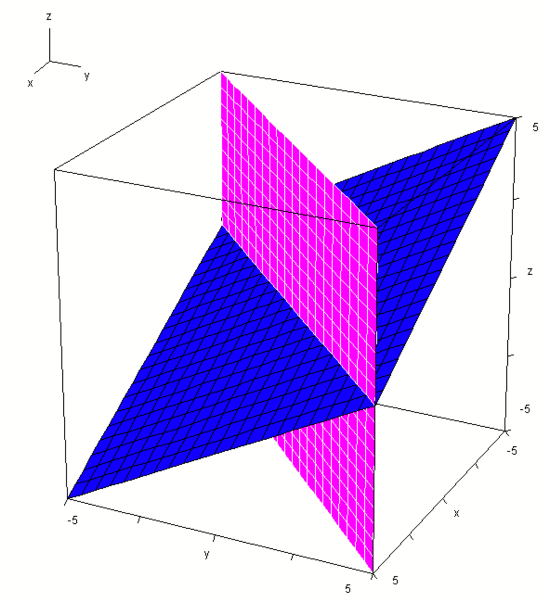

Figure 19: A linear system in three variables with an infinite number of solutions (which solution set is not an hyperplane).

5 Number of solutions of an algebraic system

As shown in the previous section, linear systems can have one solution, no solution or infinite solutions.

As said above, the solution set of an algebraic system is the intersection of the solution sets (hypersurfaces) corresponding to its equations, so algebraic systems with one solution, no solution or infinite solutions must exist (because linear systems are particular cases of algebraic systems). Let us give non-linear examples of each situation (see figures 20, 21, 22):

#47: $\text{SOLVE}\left(\left[x - y^2 = 0,\ -x - y^2 = 0\right],\ [x, y]\right)$

#48: $[x = 0 \wedge y = 0]$

#49: $\text{SOLVE}\left(\left[x - y^2 = 0,\ x - y^2 + 1 = 0\right],\ [x,\ y]\right)$

#50: $[\,]$

#51: $\text{SOLVE}\left(\left[x - y^2 = 0,\ 3 \cdot x - 3 \cdot y^2 = 0\right],\ [x,\ y]\right)$

#52: $\left[x - y^2 = 0\right]$

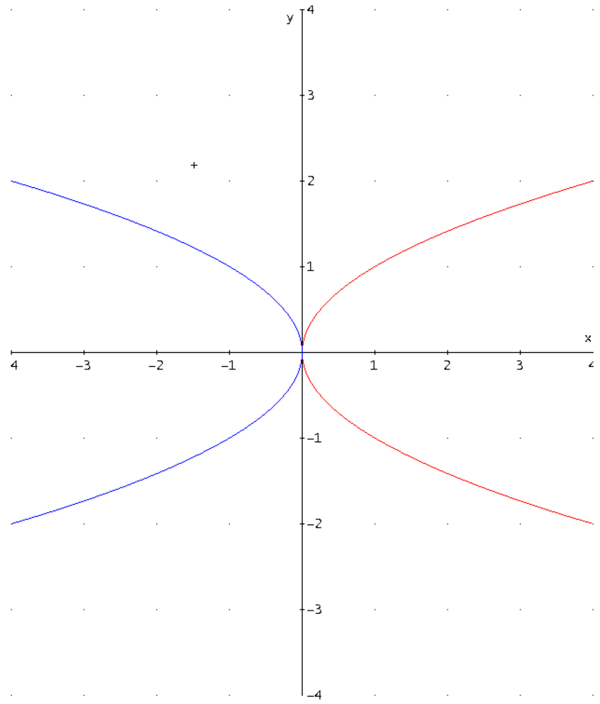

Figure 20: An algebraic system in two variables with a unique solution.

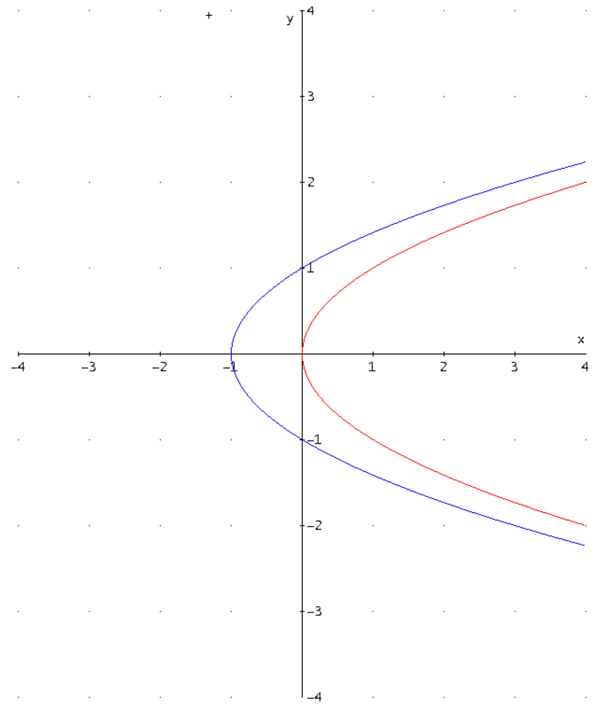

Figure 21: An algebraic system in two variables with no solution.

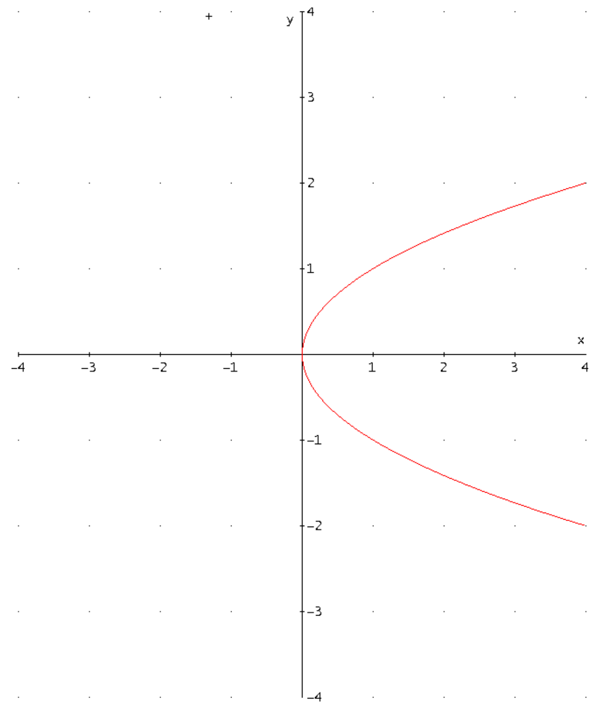

Figure 22: A linear system in two variables with an infinite number of solutions (the solution sets of both equations coincide, so the solution set of the algebraic system is a hypersurface).

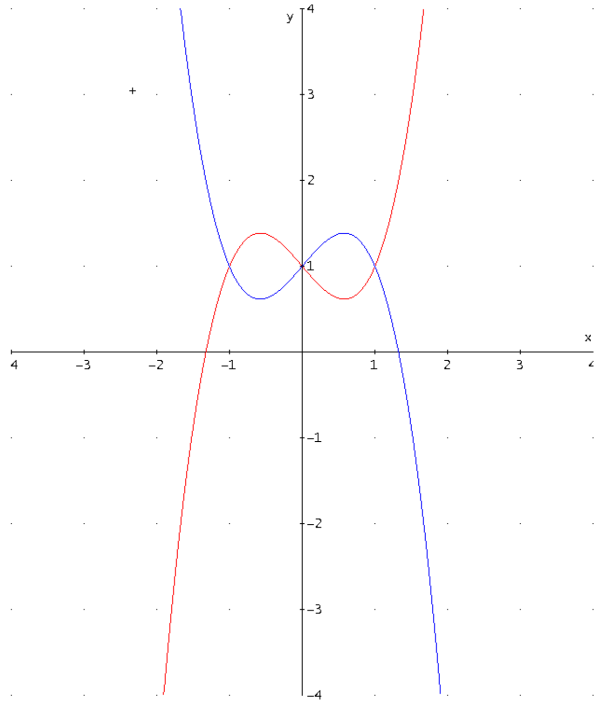

Figure 23: An algebraic system in two variables which solution set consists of three unconnected components (points).

But observe that in the case of algebraic systems a fourth case arises: the solution set of an algebraic system can consist of unconnected components. In the following example, the system has three solutions! (see Figure 23).

#53: $$\text{SOLVE}\left(\left[x^3 - x - y + 1 = 0,\ x^3 - x + y - 1 = 0\right],\ [x,\ y]\right)$$

#54: $$[x = 0 \wedge y = 1,\ x = 1 \wedge y = 1,\ x = -1 \wedge y = 1]$$

Moreover, if the dimension of the space is greater than 2, the solution set of the polynomial system could be an algebraic variety that is neither the empty set nor a hypersurface, i.e., an algebraic variety of intermediate dimension (see Figure 24).

#55: $$\text{SOLVE}\left(\left[x^2 + y^2 - z - 1 = 0,\ y - z = 0\right],\ [x,y,z]\right)$$

#56: $$\left[x^2 + y^2 - z = 1 \wedge y - z = 0\right]$$

(we shall discuss this kind of answers afterwards).

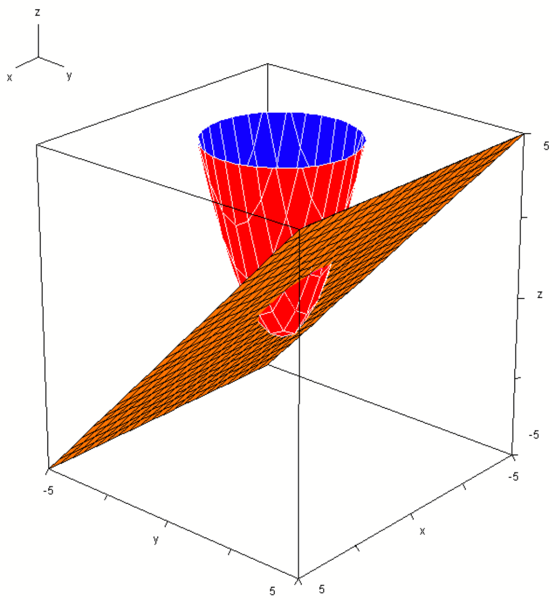

Figure 24: An algebraic system in three variables with an infinite number of solutions (which solution set is not an hypersurface).

6 Real and complex solutions of an algebraic system

Whether the system has solutions or not also depends on the set where we are looking for such solutions. The following two systems have no real solutions, as can be clearly deduced from the figures 25 and 26.

#57: $\left[x^2 + y^2 - z + 1 = 0,\ -x^2 - y^2 - z - 1 = 0\right]$

#58: $\left[x^2 + y^2 - z = 0,\ x^2 + y^2 - z + 3 = 0\right]$

But let's look for their complex solutions. The first one has complex solutions:

#59: $\text{SOLVE}\left(\left[x^2 + y^2 - z + 1 = 0,\ -x^2 - y^2 - z - 1 = 0\right],\ [x,\ y]\right)$

#60: $\left[x^2 + y^2 - z = -1 \wedge x^2 + y^2 = -1\right]$

DERIVE has expressed the solution as the intersection of one of the given elliptic paraboloids ($x^2 + y^2 - z = -1$) with the imaginary cylinder $x^2 + y^2 = -1$ and for instance ($x = i$, $y = 0$, $z = 0$) satisfies both equations. We can't illustrate it with a figure because C^n can't be drawn for $n > 1$ (we can draw R, R^2, R^3 or C).

CONTS

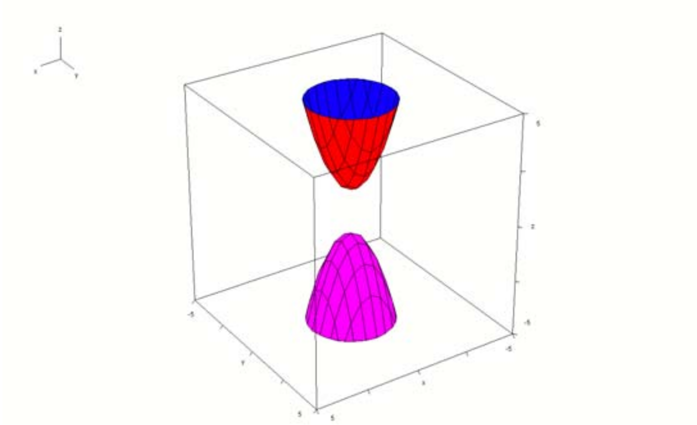

Figure 25: An algebraic system in three variables with no real solution (but with complex solutions).

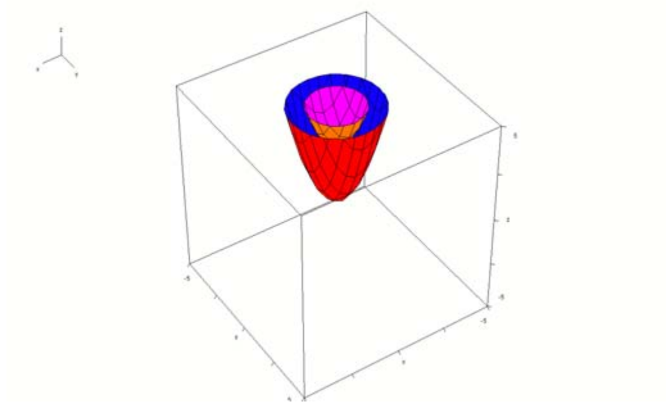

Figure 26: An algebraic system in three variables with neither real nor complex solutions.

Meanwhile the second one has no complex solutions either:

```
#61:     SOLVE([x^2 + y^2 - z = 0, x^2 + y^2 - z + 3 = 0], [x,y])

#62:                                   []
```

Let us remember that Q and R are not algebraically closed, that is, there are polynomials with coefficients in Q (respectively R) with no rational (respectively real) roots. Meanwhile C is algebraically closed. Moreover, C is the algebraic closure of Q (respectively R), that is, it is the minimum algebraically closed set that contains Q (respectively R).

7 Gröbner bases

Let us analyze the last example of Section 5. We asked DERIVE to compute

$$\#63:\quad \text{SOLVE}\left(\left[x^2 + y^2 - z - 1 = 0,\ y - z = 0\right],\ [x,y,z]\right)$$

$$\#64:\quad \left[x^2 + y^2 - z = 1 \wedge y - z = 0\right]$$

(see Figure 24). The "solution" was identical to the input. The reason is that this is the "simplest" way to describe such curve. That is not usually the case.

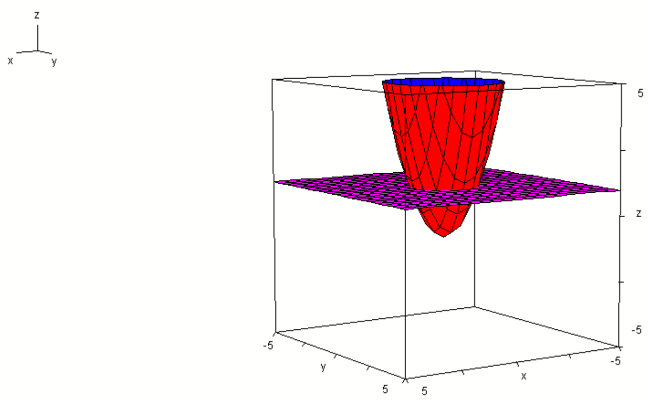

Figure 27: A circumference in the 3D space.

Let us consider the following example (see Figure 27):

$$\#65:\quad \text{SOLVE}\left(\left[x^2 + y^2 - z - 1 = 0,\ z - 1 = 0\right],\ [x,\ y,\ z]\right)$$

$$\#66:\quad \left[x^2 + y^2 = 2 \wedge z = 1\right]$$

SOLVE expresses the intersection the simplest possible way (as an intersection of a vertical cylinder and a horizontal plane, see Figure 28). In fact, this command is based in internal Gröbner bases computations.

$$\#67:\quad \text{GROEBNER\_BASIS}\left(\left[x^2 + y^2 - z - 1,\ z - 1\right],\ [x,\ y,\ z]\right)$$

$$\#68:\quad \left[z - 1,\ x^2 + y^2 - 2\right]$$

The polynomial *ideal* generated by a set of polynomials is the set of linear algebraic combinations of the given polynomials. The polynomials of the algebraic system generate an ideal that can be generated by different *bases* (sets of generators). For instance, the previous input and output are two different ways of generating the same ideal. What

Gröbner bases' algorithm provides is a canonical basis of any given ideal, that characterizes it (once the variable ordering and term ordering are chosen).

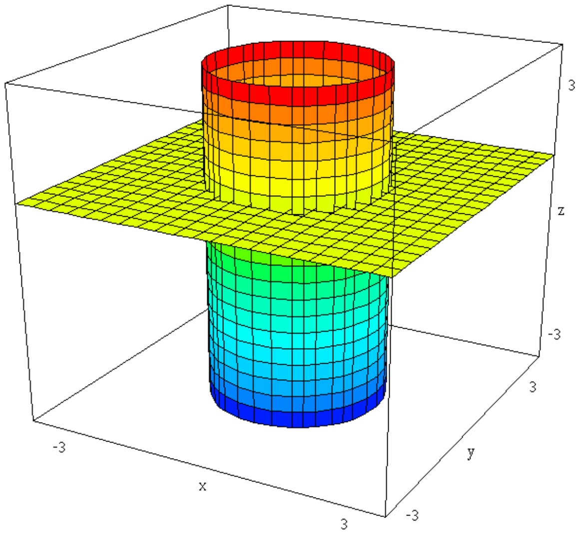

Figure 28: The same circumference in the 3D space.

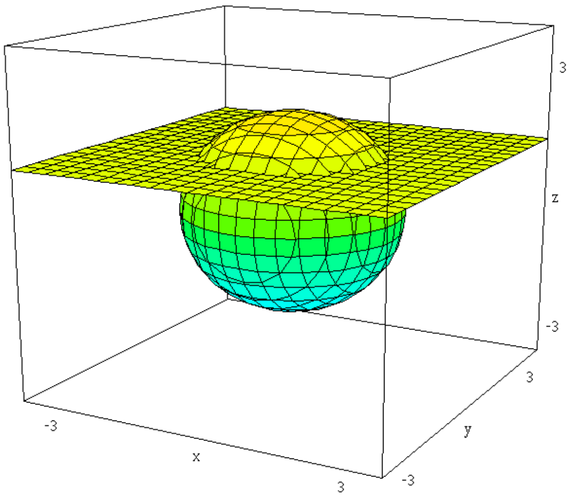

Figure 29: Another way of expressing the same circumference in the 3D space.

For example, the same circumference of Figures 27 and 28 can be described in different ways, but the corresponding Gröbner basis (once the variable order and the term order are fixed) is always the same:

- Intersection of a sphere and a horizontal plane (see Figure 29):

#69: GROEBNER_BASIS($[x^2 + y^2 + z^2 - 3, z - 1]$, [x, y, z])

#70: $[z - 1, x^2 + y^2 - 2]$

- Intersection of a sphere, a vertical cylinder and a horizontal plane (see Figure 30):

#71: GROEBNER_BASIS($[x^2 + y^2 + z^2 - 3, x^2 + y^2 - 2, z - 1]$, [x, y, z])

#72: $[z - 1, x^2 + y^2 - 2]$

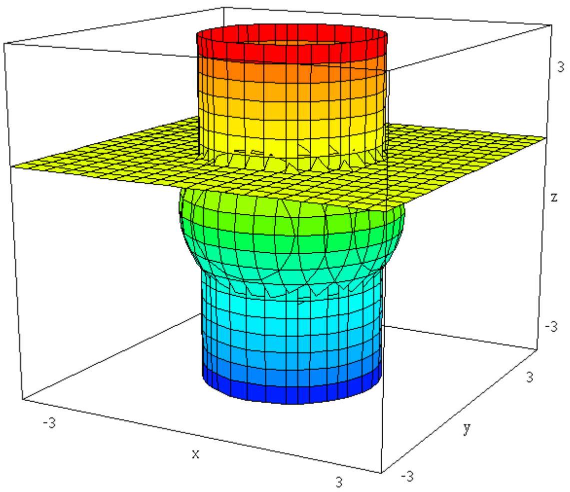

Figure 30: Yet another way of expressing the same circumference in the 3D space.

Intersection of a sphere, an elliptic paraboloid and a vertical cylinder (see Figure 31):

$$\#73: \quad \text{GROEBNER\_BASIS}\left(\left[x^2 + y^2 + z^2 - 3,\ x^2 + y^2 - z - 1,\ x^2 + y^2 - 2\right], [x, y, z]\right)$$

$$\#74: \quad \left[z - 1,\ x^2 + y^2 - 2\right]$$

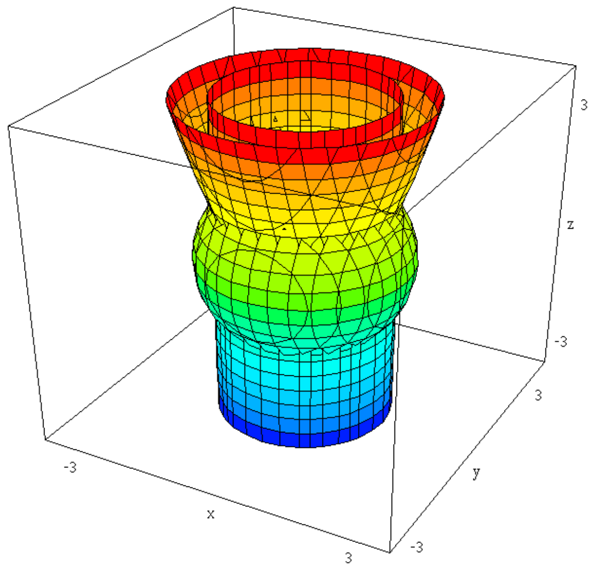

Figure 31: And another way of expressing the same circumference in the 3D space.

Despite Derive 6 can't draw implicit 3D plots, it can call DGraph 2000 (a specialized software for performing such task) [10]. Figures 29-31 plots have been created this way.

Conclusions

Understanding all the details of the Gröbner bases' calculations behind a SOLVE command requires of some algebraic background, but understanding what's going on when solving equations and systems is rather intuitive.

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[10] URL: `http://www.davidparker.com`