Four-dimensional homogeneous Lorentzian manifolds

Giovanni Calvaruso[1]

[1]Università del Salento, Lecce, Italy.
Joint work(s) with A. Fino (Univ. of Torino) and A. Zaeim (Payame noor Univ., Iran)

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Outline

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A pseudo-Riemannian manifold (M, g) is *(locally) homogeneous* if for any two points $p, q \in M$, there exists a (local) isometry ϕ, mapping p to q.

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"Algebraic": to classify all pairs $(\mathfrak{g}, \mathfrak{h})$, formed by a Lie algebra $\mathfrak{g} \subset \mathfrak{so}(p, q)$ and an isotropy subalgebra $\mathfrak{h}$, such that $\dim(\mathfrak{g}/\mathfrak{h}) = p + q = \dim M$.

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Three and four-dimensional cases provide some nice examples of these two different approaches.

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The Lorentzian analogue of the latter result also holds [Calvaruso, 2007], leading to a classification of 3D homogeneous Lorentzian manifolds, which has been used by several authors to study the geometry of these spaces.

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The downside of Komrakov's classification is that one finds **186** different pairs $(\mathfrak{g}, \mathfrak{h})$, with $\mathfrak{g} \subset \mathfrak{so}(p, q)$ and $\dim(\mathfrak{g}/\mathfrak{h}) = p + q = 4$,

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The downside of Komrakov's classification is that one finds **186** different pairs $(\mathfrak{g}, \mathfrak{h})$, with $\mathfrak{g} \subset \mathfrak{so}(p, q)$ and $\dim(\mathfrak{g}/\mathfrak{h}) = p + q = 4$, and each of these pairs admits a family of invariant pseudo-Riemannian metrics, depending of a number of real parameters varying from 1 to 4.

On the other hand, a locally homogeneous Riemannian 4-manifold is either locally symmetric, or locally isometric to a Lie group equipped with a left-invariant Riemannian metric [Bérard-Bérgery, 1985].

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This leads naturally to the following

QUESTION:

To what extent a similar result holds for locally homogeneous Lorentzian four-manifolds?

Self-adjoint operators

Let $\langle .,. \rangle$ denote an inner product on a real vector space V and $Q : V \to V$ a self-adjoint operator.

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In the Lorentzian case, self-adjoint operators are classified accordingly with their eigenvalues and the associated eigenspaces *(Segre types)*.

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4. *Segre type* $[1, 3]$: Q has two real eigenvalues (which coincide in the degenerate case), one of which has multiplicity three and each associated to a one-dimensional eigenspace.

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- Segre type $[11, z\bar{z}]$: $Q = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix}$, $d \neq 0$.

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- Segre type $[1,3]$: $$Q = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & a & 1 & -1 \\ 0 & 1 & a & 0 \\ 0 & 1 & 0 & a \end{pmatrix}.$$

Nondeg. Segre types	$[111,1]$	$[11,z\bar{z}]$	$[11,2]$	$[1,3]$
Degenerate S. types	$[11(1,1)]$	$[(11),z\bar{z}]$	$[1(1,2)]$	$[(1,3)]$
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QUESTION:

For which Segre types of the Ricci operator, is a locally homogeneous Lorentzian four-manifold necessarily either Ricci-parallel, or locally isometric to some Lorentzian Lie group?

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Starting from the description of the Lie algebra of the transitive groups of isometries, such spaces have been classified into 8 classes: A_1, A_2, A_3 (admitting both Lorentzian and neutral signature invariant metrics), A_4, A_5 (admitting invariant Lorentzian metrics) and B_1, B_2, B_3 (admitting invariant metrics of neutral signature).

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Recently, we obtained an explicit description of invariant metrics on these spaces, which allowed us to make a thorough investigation of their geometry.

Let $M = G/H$ a homogeneus space, $\mathfrak{g}$ the Lie algebra of G and $\mathfrak{h}$ the isotropy subalgebra.
The quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ identifies with a subspace of $\mathfrak{g}$, complementar to $\mathfrak{h}$ (not necessarily invariant).

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The quotient $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ identifies with a subspace of $\mathfrak{g}$, complementar to $\mathfrak{h}$ (not necessarily invariant).
The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely determines the isotropy representation

$$\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{m}), \qquad \rho(x)(y) = [x, y]_{\mathfrak{m}} \quad \forall x \in \mathfrak{h}, y \in \mathfrak{m}.$$

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Invariant pseudo-Riemannian metrics on M correspond to nondegenerate bilinear symmetric forms g on $\mathfrak{m}$, such that

$$\rho(x)^t \circ g + g \circ \rho(x) = 0 \quad \forall x \in \mathfrak{h}.$$

Such a form g on $\mathfrak{m}$ uniquely determines the corresponding Levi-Civita connection, described by $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$, such that

$$\Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y), \qquad \forall\, x, y \in \mathfrak{g},$$

where $v : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{m}$ is determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_m, [z, y]_m) + g(y_m, [z, x]_m), \quad \forall\, x, y, z \in \mathfrak{g}.$$

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The curvature tensor corresponds to $R : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$, such that

$$R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y]),$$

for all $x, y \in \mathfrak{m}$.

Such a form g on $\mathfrak{m}$ uniquely determines the corresponding Levi-Civita connection, described by $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$, such that

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for all $x,y \in \mathfrak{m}$.
The Ricci tensor ϱ of g, with respect to a basis $\{u_i\}$ of $\mathfrak{m}$, is given by

$$\varrho(u_i,u_j) = \sum_{r=1}^{4} R_{ri}(u_r,u_j), \qquad i,j = 1,..,4.$$

A1) $\mathfrak{g} = \mathfrak{a}_1$ is the decomposable 5-dimensional Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{s}(2)$.There exists a basis $\{e_1, ..., e_5\}$ of $\mathfrak{a}_1$, such that the non-zero products are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_4, e_5] = e_4$$

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and $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 + e_4\}$. So, we can take

$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_5, u_4 = e_3 - e_4\}$$

and have the following isotropy representation for h_1:

$$H_1(u_1) = u_4,\ H_1(u_2) = -u_1,\ H_1(u_3) = -\tfrac{1}{2}u_4,\ H_1(u_4) = 0.$$

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Consequently, with respect to $\{u_i\}$, invariant metrics g are

$$g = \begin{pmatrix} a & 0 & -\frac{a}{2} & 0 \\ 0 & b & c & a \\ -\frac{a}{2} & c & d & 0 \\ 0 & a & 0 & 0 \end{pmatrix}, \quad a(a-4d) \neq 0.$$

A2) $\mathfrak{g} = \mathfrak{a}_2$ is the one-parameter family of 5-dimensional Lie algebras:

$$\begin{array}{lll} [e_1, e_5] = (\alpha + 1)e_1, & [e_2, e_4] = e_1, & [e_2, e_5] = \alpha e_2, \\ [e_3, e_4] = e_2, & [e_3, e_5] = (\alpha - 1)e_3, & [e_4, e_5] = e_4, \end{array}$$

where $\alpha \in \mathbb{R}$, and $\mathfrak{h} = \mathrm{Span}\{h_1 = e_4\}$. Hence, we can take

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With respect to $\{u_i\}$, invariant metrics have the form

$$g = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & 0 & 0 \\ -a & 0 & b & c \\ 0 & 0 & c & d \end{pmatrix}, \quad ad \neq 0.$$

A3) $\mathfrak{g} = \mathfrak{a}_3$ is described by

$$[e_1, e_4] = 2e_1, \quad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2,$$
$$[e_2, e_5] = -\varepsilon e_3, \quad [e_3, e_4] = e_3, \quad [e_3, e_5] = e_2,$$

with $\varepsilon = \pm 1$ and $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3\}$. Thus,

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and with respect to $\{u_i\}$, invariant metrics are given by

$$g = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ a & 0 & c & d \end{pmatrix}, \quad ab \neq 0.$$

A4) $\mathfrak{g} = \mathfrak{a}_4$ is the 6-dimensional Schroedinger Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{n}(3)$, where $\mathfrak{n}(3)$ is the 3*D* Heisenberg algebra.

$$\begin{array}{llll} [e_1, e_2] = 2e_2, & [e_1, e_3] = -2e_3, & [e_2, e_3] = e_1, & [e_1, e_4] = e_4, \\ [e_1, e_5] = -e_5, & [e_2, e_5] = e_4, & [e_3, e_4] = e_5, & [e_4, e_5] = e_6 \end{array}$$

and $\mathfrak{h} = \mathrm{Span}\{h_1 = e_3 + e_6, h_2 = e_5\}$. Therefore, we take

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$$\mathfrak{m} = \mathrm{Span}\{u_1 = e_1, u_2 = e_2, u_3 = e_3 - e_6, u_4 = e_4\}$$

and from the isotropy representation for h_1, h_2, we conclude that with respect to $\{u_i\}$, invariant metrics are of the form

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & a & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{2} \end{pmatrix}, \quad a \neq 0.$$

A5) $\mathfrak{g} = \mathfrak{a}_5$ is the 7-dimensional Lie algebra described by

$$\begin{array}{llll} [e_1, e_2] = 2e_2, & [e_1, e_3] = -2e_3, & [e_1, e_5] = -e_5, & [e_1, e_6] = e_6, \\ [e_2, e_3] = e_1, & [e_2, e_5] = e_6, & [e_3, e_6] = e_5, & [e_4, e_7] = 2e_4 \\ [e_5, e_6] = e_4, & [e_5, e_7] = e_5, & [e_6, e_7] = e_6. & \end{array}$$

The isotropy is $\mathfrak{h} = \mathrm{Span}\{h_1 = e_1 + e_7, h_2 = e_3 - e_4, h_3 = e_5\}$.
So,

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$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{4} & 0 \\ 0 & \frac{a}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{a}{8} \end{pmatrix}, \quad a \neq 0.$$

Segre types of the Ricci operator

$$\textbf{A1:}\ \begin{pmatrix} -2a^{-1} & 0 & a^{-1} & 0 \\ 0 & -2a^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{16d(a+4d)}{a^2(a-4d)} & -\frac{2c}{a^2} & -2\,a^{-1} \end{pmatrix} \quad [1(1,2)],$$

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$$\textbf{A2:} \begin{pmatrix} -\frac{3\alpha^2}{d} & 0 & -\frac{b(3\alpha-2)}{ad} & 0 \\ 0 & -\frac{3\alpha^2}{d} & 0 & 0 \\ 0 & 0 & -\frac{3\alpha^2}{d} & 0 \\ 0 & 0 & 0 & -\frac{3\alpha^2}{d} \end{pmatrix} \quad [(11,2)], \; [(111,1)]\,,$$

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A3: $$\begin{pmatrix} -3\,b^{-1} & 0 & 0 & \frac{d+\varepsilon b}{ab} \\ 0 & -3b^{-1} & 0 & 0 \\ 0 & 0 & -3b^{-1} & 0 \\ 0 & 0 & 0 & -3b^{-1} \end{pmatrix}$$ $[(11,2)]$, $[(111,1)]$,

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A4: $$\begin{pmatrix} -3a^{-1} & 0 & 0 & 0 \\ 0 & -3a^{-1} & 0 & 0 \\ 0 & -\frac{5b}{a^2} & -3a^{-1} & 0 \\ 0 & 0 & 0 & -3a^{-1} \end{pmatrix} \quad [(11,2)], \; [(111,1)]\,,$$

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On the other hand, there exist non-reductive homogeneous Lorentzian 4-manifolds with Ricci operator of Segre type either [1(1,2)] or [(11,2)] (which are not Ricci-parallel).

Thus, for such Segre types of the Ricci operator, a result similar to the one of Bérard-Bérgery cannot hold!!!

Geometry of non-reductive examples

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However, there exist non-reductive (pseudo-Riemannian) g.o. 4-spaces.

Non-reductive Ricci solitons

A Ricci soliton is a pseudo-Riemannian manifold (M, g), together with a vector field X, such that

$$\mathcal{L}_X g + \varrho = \lambda g,$$

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Ricci solitons are the self-similar solutions of the Ricci flow

$$\frac{\partial}{\partial t} g(t) = -2\varrho(t).$$

Einstein manifolds are trivial Ricci solitons.

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There exist plenty of examples of *4D* homogeneous Ricci solitons, both Lorentzian and of neutral signature $(2, 2)$.
In particular, for non-reductive Lorentzian four-manifolds, we have the following.

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(c) M is of type $A4$ and g satisfies $b \neq 0$. In this case, $\lambda = -\frac{3}{a}$.

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In particular, all the known examples of Ricci solitons on non-compact homogeneous Riemannian manifolds are isometric to some solvsolitons, that is, to left-invariant Ricci solitons on a solvable Lie group.

Obviously, non-reductive Ricci solitons are not isometric to solvsolitons.

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As there exist four-dimensional non-reductive homogeneous Lorentzian four-manifolds, with Ricci operator of Segre type either $[1(1, 2)]$ or $[(11, 2)]$, which are neither Ricci-parallel nor locally isometric to a Lie group,

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As there exist four-dimensional non-reductive homogeneous Lorentzian four-manifolds, with Ricci operator of Segre type either $[1(1, 2)]$ or $[(11, 2)]$, which are neither Ricci-parallel nor locally isometric to a Lie group, the above result is optimal.

Given a $4D$ (locally) homogeneous Lorentzian manifold (M, g), Given a $4D$ (locally) homogeneous Lorentzian manifold (M, g), the Ricci operator Q of (M, g) has the same Segre type at any point and constant eigenvalues.

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We can build a (local) pseudo-orthonormal basis $\{e_i\}$, such that

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We can build a (local) pseudo-orthonormal basis $\{e_i\}$, such that

- the Ricci operator takes one of the canonical forms for the corresponding Segre type,
- and the components of R, ∇R, $\nabla^2 R \dots \nabla^k R$... remain constant.

Let Γ_{ij}^k denote the coefficients of the Levi-Civita connection of (M,g) with respect to $\{e_i\}$, that is,

$$\nabla_{e_i} e_j = \sum_k \Gamma_{ij}^k e_k,$$

Since $\nabla g = 0$, we have:

$$\Gamma_{ij}^k = -\varepsilon_j \varepsilon_k \Gamma_{ik}^j, \quad \forall i,j,k = 1,\dots,4,$$

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The curvature of (M,g) can be then completely described in terms of Γ_{ij}^k.

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Proof: There are four distinct possible forms for the nondegenerate Ricci operator Q of (M, g).
Using a case-by-case argument, we showed that for any of them, (M, g) is isometric to a Lie group.

Example: suppose that Q is of type $[11, 2]$.

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Denoted by $\{\omega^i\}$ the coframe dual to e_i with respect to g, by the definition of the Ricci tensor we get

$$\begin{aligned} Ric = \; & q_1\omega^1 \otimes \omega^1 + q_2\omega^2 \otimes \omega^2 + (1+q_3)\omega^3 \otimes \omega^3 \\ & -2\omega^3 \circ \omega^4 + (1-q_3)\omega^4 \otimes \omega^4, \end{aligned}$$

where $q_i \neq q_j$ when $i \neq j$.

So, $$\nabla Ric = 2\sum_{k=1}^{4}\Big(\Gamma^2_{k1}(q_1-q_2)\omega^1\circ\omega^2\otimes\omega^k$$
$$+(\Gamma^4_{k1}-\Gamma^3_{k1}(1+q_3-q_1))\omega^1\circ\omega^3\otimes\omega^k$$
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The constancy of the components of ∇Ric then implies that Γ_{ij}^{k} is constant for all indices i, j, k. So, (M, g) has a Lie group structure with a left-invariant Lorentzian metric.

A similar argument works for a “good” degenerate Segre type:

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Theorem

Let (M, g) be a simply connected, complete four-dimensional homogeneous Lorentzian manifold. If the Ricci operator of (M, g) is of degenerate type $[(11)(1, 1)]$, then either (M, g) is Ricci-parallel, or it is a Lie group equipped with a left-invariant Lorentzian metric.

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If the Ricci operator is of degenerate Segre type $[(111, 1)]$, then (M, g) is Einstein and so, Ricci-parallel.

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The starting point is, once again, Komrakov's classification and description of homogeneous pseudo-Riemannian 4-manifolds with nontrivial isotropy.

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Remark

Also the result of Bérard-Bérgery was not obtained by direct proof, but using some classification results.

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The description of a homogeneous space M as a coset space G/H does not exclude the fact that M is also isometric to a Lie group (EXAMPLE: the standard three-sphere).

We report now the details for one example in Komrakov's list. The homogeneous space is $M = G/H$, where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is the 5D (reductive) Lie algebra

$$(*) \quad [e_1, u_1] = u_3, \quad [e_1, u_3] = -u_1, \quad [u_1, u_3] = e_1 + u_2,$$

with $\mathfrak{h}$ =span(e_1) and $\mathfrak{m}$ =span($u_1, .., u_4$).

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To show that M is isometric to a Lie group, it suffices to prove the existence of a 4D subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$, such that the restriction of the map $\mathfrak{g} \to T_o M$ to $\mathfrak{g}'$ is still surjective.

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This last condition implies that $\mathfrak{g}' = \mathfrak{m} + \varphi(\mathfrak{m})$, where $\varphi : \mathfrak{m} \to \mathfrak{h}$ is a $\mathfrak{h}$-equivariant linear map.

As the isotropy subalgebra is spanned by e_1, (*) yields that a linear map $\varphi : \mathfrak{m} \to \mathfrak{h}$ is $\mathfrak{h}$-equivariant when $\varphi(u_1) = \varphi(u_3) = 0$.

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Then, again from (*), we find that $\mathfrak{g}'$ is a subalgebra when $c_2 = 1$ (for any value of c_4).

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we conclude that M is locally isometric to the 4D simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}'$ =span(v_1, v_2, v_3, v_4), explicitly described by

$$[v_1, v_2] = -v_3, \quad [v_1, v_3] = v_2, \quad [v_2, v_3] = -v_1.$$

Ricci-parallel homogeneous spaces

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Theorem

If (M, g) is a *4D* Ricci-parallel homogeneous Lorentzian manifold, then its Ricci operator is of one of the following degenerate Segre types: $[(111, 1)]$, $[(11)(1, 1)]$, $[1(11, 1)]$, $[(111), 1]$, $[(11, 2)]$.

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- If Q is of Segre type $[1(11, 1)]$, then (M, g) is locally reducible and isometric to a direct product $\mathbb{R} \times \mathbb{M}^3_1(k)$.
- If Q is of Segre type $[(11, 2)]$, then (M, g) is a Lorentzian Walker manifold and Q is two-step nilpotent.

A pseudo-Riemannian manifold (M, g) is (locally) conformally flat if g is locally conformal to a flat metric.

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Moreover, it admits as univeral covering either a space form $\mathbb{R}^n, \mathbb{S}^n(k), \mathbb{H}^n(-k)$, or one of Riemannian products $\mathbb{R} \times \mathbb{S}^{n-1}(k), \mathbb{R} \times \mathbb{H}^{n-1}(-k), \mathbb{S}^p(k) \times \mathbb{H}^{n-p}(-k)$.

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In pseudo-Riemannian settings, the problem of classifying conformally flat homogeneous manifolds is more complicated and interesting, as conformally flat homogeneous pseudo-Riemannian manifolds need not to be symmetric.

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Theorem [Honda-Tsukada]

An $n(\geq 3)$-dimensional conformally flat homogeneous pseudo-Riemannian manifold M^n_q with diagonalizable Ricci operator is locally isometric to either:

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(iii) A product manifold of an $(n-1)$-dimensional pseudo-Riemannian manifold of constant curvature $k \neq 0$ and a one-dimensional manifold.

Theorem

Let (M, g) denote a $4D$ conformally flat homogeneous Lorentzian manifold. Then, there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3, e_4\}$, with e_4 time-like, such that Q takes one of the following forms:

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(I) If the minimal polynomial of Q does not have repeated roots:

(Ia) $diag(r, \dots, -r)$;

(Ib) $$\begin{pmatrix} t & 0 & 0 & 0 \\ 0 & \pm t & 0 & 0 \\ 0 & 0 & r & s \\ 0 & 0 & -s & r \end{pmatrix}, \quad \begin{array}{l} s \neq 0, \\ r^2 + s^2 = t^2. \end{array}$$

(II) If the minimal polynomial of Q has a double root:

$$\begin{pmatrix} \pm r & 0 & 0 & 0 \\ 0 & \pm r & 0 & 0 \\ 0 & 0 & r+\frac{\varepsilon}{2} & -\frac{\varepsilon}{2} \\ 0 & 0 & \frac{\varepsilon}{2} & r-\frac{\varepsilon}{2} \end{pmatrix},$$

(II) If the minimal polynomial of Q has a double root:

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(III) If the minimal polynomial of Q has a triple root:

$$\begin{pmatrix} \pm r & 0 & 0 & 0 \\ 0 & r & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & a & \frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

Consequently, the possible Segre types of the Ricci operator Q for a $4D$ conformally flat homogeneous Lorentzian manifold, are the following:

Case	Ia	Ib	II	III
Non degenerate type	—	$[11, z\bar{z}]$	—	[1,3]
Degenerate types	$[(11)(1,1)]$ $[1(11,1)]$ $[(111),1]$ $[(111,1)]$	$[(11),1\bar{1}]$	$[(11),2]$ $[1(1,2)]$ $[(11,2)]$	$[(1,3)]$

At any point $p \in M$ and for any index k, consider the Lie algebra

$$\mathfrak{g}(k,p) = \{Y \in \mathfrak{so}(q,n-q) : Y.R(p) = Y.\nabla R(p) = \cdots = Y.\nabla^k R(p) = 0\}.$$

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Theorem

Let (M, g) be a $4D$ conformally flat pseudo-Riemannian four-manifold. At any point $p \in M$, we have that $\mathfrak{g}(0,p) = \{0\}$ if and only if Q_p is non-degenerate.

Corollary

Let (M, g) be a $4D$ conformally flat homogeneous pseudo-Riemannian manifold. If the Ricci operator Q of (M, g) is non-degenerate, then (M, g) is locally isometric to a Lie group equipped with a left-invariant pseudo-Riemannian metric.

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Theorem

Let (M, g) be a conformally flat homogeneous Lorentzian four-manifold. If the Ricci operator Q of (M, g) is not diagonalizable and non-degenerate, then Q can only be of Segre type $[11, z\bar{z}]$.

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A *4D* conformally flat homogeneous Lorentzian manifold, with Q of Segre type $[11, z\bar{z}]$, is locally isometric to the unsolvable Lie group $SL(2,\mathbb{R}) \times \mathbb{R}$, equipped with a left invariant Lorentzian metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the Lie algebra, such that the Lie brackets take one of the following forms:

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$$\text{i)} \quad [e_1,e_2] = 2\alpha(\varepsilon e_3 - e_4), \quad [e_1,e_3] = -\varepsilon\alpha e_2, \quad [e_1,e_4] = \alpha e_2,$$
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$$\text{ii)} \quad \begin{array}{lll} [e_1,e_2] = 2\alpha(\varepsilon e_3 + e_4), & [e_1,e_3] = \varepsilon\alpha e_2, & [e_1,e_4] = \alpha e_2, \\ [e_2,e_3] = \varepsilon\alpha e_1, & [e_2,e_4] = \alpha e_1, & \end{array}$$

where $\alpha \neq 0$ is a real constant and $\varepsilon = \pm 1$.

For case i), in the new basis

$$\{\hat{e}_1 = e_1,\ \hat{e}_2 = e_2,\ \hat{e}_3 = e_3 - \varepsilon e_4,\ \hat{e}_4 = e_3 + \varepsilon e_4\}$$

of $\mathfrak{g}$, the nonvanishing Lie brackets are

$$[\hat{e}_1, \hat{e}_2] = 2\varepsilon\alpha\hat{e}_3,\ [\hat{e}_1, \hat{e}_3] = -2\varepsilon\alpha\hat{e}_2,\ [\hat{e}_2, \hat{e}_3] = -2\varepsilon\alpha\hat{e}_1.$$

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Similarly, for case ii), in the basis

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For case *i*), in the new basis

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Thus, G is indeed $SL(2, \mathbb{R}) \times \mathbb{R}$.

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Let (M, g) be a conformally flat homogeneous, not locally symmetric, Lorentzian 4-manifold, with degenerate Ricci operator Q. Then, Q is of Segre type either $[(11, 2)]$ or $[(1, 3)]$.

Segre type $[(1,3)]$ only occurs in cases with trivial isotropy. More precisely, (M,g) is locally isometric to the solvable Lie group $\mathbb{R} \ltimes E(1,1)$, whose Lie algebra $\mathfrak{g}$ is described by

$$
\begin{aligned}
[e_1,e_2] &= (c_1-\sqrt{2}c_2)e_2+\tfrac{1}{4c_2}e_3-(c_1-\sqrt{2}c_2)e_4,\\
[e_1,e_3] &= \tfrac{3}{4c_2}e_2-\sqrt{2}c_2e_3-\tfrac{3}{4c_2}e_4,\\
[e_1,e_4] &= (c_1+\sqrt{2}c_2)e_2+\tfrac{1}{4c_2}e_3-(c_1+\sqrt{2}c_2)e_4,\\
[e_2,e_4] &= -\tfrac{\phi}{c_2}e_2+\tfrac{\phi}{c_2}e_4,\\
[e_2,e_3] &= -[e_3,e_4] = -\tfrac{3\sqrt{2}\phi}{16c_2^3}e_2+\tfrac{\phi}{2c_2}e_3+\tfrac{3\sqrt{2}\phi}{16c_2^3}e_4,
\end{aligned}
$$

where $\phi = \pm\sqrt{4\sqrt{2}c_1c_2^3-1}$, for any real constants $c_1, c_2 \neq 0$, such that $4\sqrt{2}c_1c_2^3-1>0$.

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EXAMPLE: $M = G/H$,
$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h} = \mathrm{Span}(u_1, .., u_4) \oplus \mathrm{Span}(h_1, h_2, h_3)$, described by

$$\begin{array}{lll} [e_1, e_2] = -e_3, & [e_1, e_3] = e_2, & [e_1, u_2] = u_4, \\ [e_1, u_4] = -u_2, & [e_1, u_2] = u_1, & [e_2, u_3] = -u_2, \\ [e_3, u_3] = u_4, & [e_3, u_4] = -u_1, & \\ [u_1, u_3] = u_1, & [u_2, e_3] = pe_2 + u_2, & [u_3, u_4] = pe_3 - u_4, \ p \neq 0. \end{array}$$

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Thus, simply connected *4D* Lorentzian Lie groups coincide with the Riemannian ones. They are:

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- one of the following solvable Lie groups:

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- or degenerate **(???)**.

Proposition

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- with respect to $\{e_1, e_2, e_3, e_4\}$, the Lorentzian inner product takes one of the following forms:

$$(a)\ \mathrm{diag}(1,1,-1,1); \quad (b)\ \mathrm{diag}(1,1,1,-1); \qquad (c)\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

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- and recognizes the obtained Lie algebra, calculating the derived Lie algebra.

Einstein examples

(a) $\{e_i\}_{i=1}^4$ is a pseudo-orthonormal basis, with e_3 time-like.
In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:

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a1) $\mathbb{R} \ltimes H$, with 4 possible forms of the Lie brackets.
EXAMPLE:

$$[e_1, e_2] = \varepsilon A e_1, \quad [e_1, e_3] = A e_1,$$
$$[e_1, e_4] = \delta A e_1, \quad [e_3, e_4] = -2A\delta(\varepsilon e_2 - e_3), \quad \varepsilon, \delta = \pm 1.$$

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a2) $\mathbb{R} \ltimes \mathbb{R}^3$, with 7 possible forms of the Lie brackets.
EXAMPLE:

$$[e_1, e_4] = -(A + B)e_1, \quad [e_2, e_4] = Be_2 \pm \sqrt{A^2 + AB + B^2} e_3$$

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(c) $\{e_i\}_{i=1}^4$ is a basis, with the inner product g on $\mathfrak{g}$ completely determined by $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_4) = g(e_4, e_3) = 1$ and $g(e_i, e_j) = 0$ otherwise.

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c2) $\mathbb{R}\ltimes\mathbb{R}^3$, with 2 possible forms of the Lie brackets.
EXAMPLE:

$$[e_1,e_4]=Ae_2+Be_3,\quad [e_2,e_4]=-Ae_1+Ce_3.$$

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For any solution we found in case (b) ($\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{g}_3$ with $\mathfrak{g}_3$ Riemannian), the Lie algebra is also of the form $\mathfrak{g} = \mathfrak{r}' \ltimes \mathfrak{g}_3'$ with $\mathfrak{g}_3'$ Lorentzian, that is, of type (a).

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Thus, $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}(e_1, e_3)$, and TL vector e_4 acts as a derivation on the Riemannian Lie algebra $\mathfrak{g}_3 = \mathrm{span}(e_1, e_2, e_3)$.

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Thus, $[\mathfrak{g}, \mathfrak{g}] = \mathrm{span}(e_1, e_3)$, and TL vector e_4 acts as a derivation on the Riemannian Lie algebra $\mathfrak{g}_3 = \mathrm{span}(e_1, e_2, e_3)$. On the other hand, SL vector e_2 acts on the Lorentzian Lie algebra $\mathfrak{g}_3' = \mathrm{span}(e_1, e_3, e_4)$ (the Lie algebra of the Heisenberg group), so that this example is already included in case (a).

Ricci-parallel examples

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- For case (b) (a TL vector acting on a 3D Riemannian Lie algebra) we find non-Einstein solutions on $\mathbb{R} \times \widetilde{SL}(2, \mathbb{R})$.
- For case (c) (a null vector acting on a 3D degenerate Lie algebra) we find non-Einstein solutions on $\mathbb{R} \ltimes H$, $\mathbb{R} \ltimes \widetilde{E}(2)$ and $\mathbb{R} \ltimes E(1, 1)$.

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The only Ricci-parallel not locally symmetric examples occur

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These homogeneous Lorentzian manifolds are Walker manifolds, and their Ricci operator is two-step nilpotent.

Ricci-parallel examples

One can use the classification of Einstein and Ricci-parallel 4D Lorentzian Lie groups, to deduce several geometric properties for these examples. In particular:

Up to isomorphisms, the only (nontrivial) Ricci-parallel Ricci solitons occur on $\mathbb{R} \ltimes H$. There are both

- symmetric examples, like:

$$[e_1, e_2] = \varepsilon B e_1, \qquad [e_1, e_3] = B e_1,$$
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- and non-symmetric examples, like:

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$$[e_2, e_4] = A e_1 + F e_3, \quad C(C + 2A) \neq 0 \neq F.$$

¡Gracias por su atención!