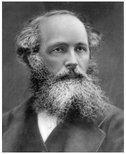
The homogeneity theorem in supergravity

José Miguel Figueroa O'Farrill



Workshop on homogeneous lorentzian manifolds
Madrid, 7 March 2013

Outline

1. A geometrical context

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2. Supergravity

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3. Homogeneity

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- they are special types of **twistor spinors**

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- Bär's cone construction reduces the determination of which riemannian manifolds admit real Killing spinors to a holonomy problem: which metric cones admit parallel spinors

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- Even today, all known simply-connected 6-dimensional riemannian manifolds admitting real Killing spinors (**nearly-Kähler 6-manifolds**) are homogeneous; although there are non-homogeneous quotients

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- $g, F, \ldots$ are subject to Einstein–Maxwell-like PDEs

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- Explicitly,

$$d \star F = \tfrac{1}{2} F \wedge F$$
$$\mathrm{Ric}(X, Y) = \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2$$

together with $dF = 0$

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- It is convenient to organise this information according to how much “supersymmetry” the background preserves.

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- such spinor fields are called **Killing spinors**

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- a background is said to be ν-**BPS**, where $\nu = \frac{n}{32}$

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where the second row are now known to be homogeneous!

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- It is a very useful invariant of a supersymmetric supergravity background

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- One gets either the compact or split real forms of the algebras
- This is the geometrization of Frank Adams's algebraic construction

JMF (2007)

1. A geometrical context
2. Supergravity
3. Homogeneity

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- This is the “right” working notion in supergravity

The homogeneity theorem

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

The homogeneity theorem

Homogeneity conjecture

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In fact, vector fields in the Killing superalgebra already span the tangent spaces to every point of M

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Every ν-BPS background of type IIB supergravity with $\nu > \frac{1}{2}$ is homogeneous.

Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is homogeneous.

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Every ν-BPS background of six-dimensional $(1,0)$ and $(2,0)$ supergravities with $\nu > \frac{1}{2}$ is homogeneous.

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The theorems actually prove the strong version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

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This actually only shows local homogeneity.

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The homogeneity theorem implies that classifying homogeneous supergravity backgrounds also classifies ν-BPS backgrounds for $\nu > \frac{1}{2}$.

This is **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

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subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

Skip technical details

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We raise and lower indices with γ_{ij}.

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where $U_{ijk} = f_{i(jk)}$

Homogeneous Ricci curvature

Finally, the Ricci tensor for a homogeneous (reductive) manifold is given by

$$R_{ij} = -\tfrac{1}{2} f_i{}^{k\ell} f_{jk\ell} - \tfrac{1}{2} f_{ik}{}^{\ell} f_{j\ell}{}^{k} + \tfrac{1}{2} f_{ik}{}^{a} f_{aj}{}^{k} + \tfrac{1}{2} f_{jk}{}^{a} f_{ai}{}^{k} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ij} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ji} + \tfrac{1}{4} f_{k\ell i} f^{k\ell}{}_{j}$$

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It is now a matter of assembling these ingredients to write down the supergravity field equations in a homogeneous Ansatz.

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- Solve the equations!

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Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper (i.e., inverse image of compact is compact). In particular, proper actions have compact stabilisers.

Homogeneous lorentzian manifolds II

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This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

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- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$
  JMF+Ungureanu (in preparation)

Summary and outlook

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- In particular, we can “dial up” a semisimple G and hope to solve the homogeneous supergravity equations with symmetry G
- Checking supersymmetry is an additional problem, perhaps it can be done at the same time by considering homogeneous supermanifolds

JMF+Santi+Spiro (in progress)