Conformally flat homogeneous Lorentzian manifolds

Kazumi Tsukada
Ochanomizu University

Thank you the organizer for giving me an opportunity of a talk in this workshop. This is a joint work with Kyoko Honda.

Our problem is to classify conformally flat homogeneous semi-Riemannian manifolds.

Here a semi-Riemannian manifold (M_q^n, g) is said to be *conformally flat* if for any point p of M, there exist a coordinate neighborhood $(V, x_1, \cdots, x_n)$ of p and a positive smooth function $\rho > 0$ on V such that the following equation holds:

$$g = \rho^2(-dx_1^2 - \cdots - dx_q^2 + dx_{q+1}^2 + \cdots + dx_n^2) \quad \text{on } V .$$

That is, it is locally conformally equivalent to a semi-Euclidean space. A semi-Riemannian manifold (M_q^n, g) is said to be *homogeneous* if for any two points p, p' of M, there exists an isometry ϕ of M such that $\phi(p) = p'$.

For the Riemannian case, the following is known: Conformally flat homogeneous Riemannian manifolds were classified by H. Takagi [10].

Theorem. An n-dimensional simply connected conformally flat homogeneous Riemannian manifold is isometric to one of the following:
(1) $M^n(k)$,
(2) $M^m(k) \times M^{n-m}(-k)$, $k \neq 0, 2 \leq m \leq n-2$,
(3) $M^{n-1}(k) \times \mathbb{R}$, $k \neq 0$,
where $M^m(k)$ denotes the simply connected complete Riemannian manifold of constant curvature k.

Consequently, they are all symmetric spaces. In this talk we would like to show that in the semi-Riemannian case, there exist many interesting examples which are not symmetric spaces.

In this talk, I would like to explain the following:

<u>Plan of talk</u>

1. Characterizations of conformally flatness
2. A classification of conformally flat symmetric spaces

3. A method of the construction of examples I
4. A method of the construction of examples II
5. A classification of conformally flat homogeneous Lorentzian manifolds

First we show two kinds of the characterizations of conformally flatness. Next we explain a classification of conformally flat symmetric semi-Riemannian manifolds given by Cahen and Kerbrat [1]. And we show two methods of the construction of examples inspired by their results. Finally we show classification of conformally flat homogeneous Lorentzian manifolds.

§1 Characterizations of conformally flatness

We recall two kinds of the characterizations of conformally flatness. Let M_q^n be an $n(\geq 4)$-dimensional semi-Riemannian manifold equipped with a semi-Riemannian metric g of index q.

It is known that the following three conditions are equivalent:

(1) A semi-Riemannian manifold (M_q^n, g) is conformally flat.

(2) The curvature tensor R of M satisfies the following:

$$R(X,Y) = AX \wedge Y + X \wedge AY,$$

$$A = \frac{1}{n-2}\left(Q - \frac{S}{2(n-1)}Id\right),$$

where Q is the Ricci operator and S is the scalar curvature of M, respectively. Here the operator A is called *the Schouten tensor*.

(3) There exists an isometric immersion of M into the light cone $\Lambda \subset \mathbb{R}_{q+1}^{n+2}$, where the light cone is defined by

$$\Lambda = \{\boldsymbol{x} \in \mathbb{R}_{q+1}^{n+2} - \{0\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

The third condition is the key of our approach.

§2 A classification of conformally flat symmetric spaces

We explain a classification of conformally flat symmetric semi-Riemannian manifolds given by Cahen and Kerbrat [1] and its simplified arguments by K.Honda [4].

Let M_q^n be an $n(\geq 4)$-dimensional symmetric semi-Riemannian manifold of index q. Then its Schouten tensor A is parallel with respect to the Levi-Civita connection. So the derivation of A by the curvature tensor $R(X,Y)$ is equal to 0, i.e., $R(X,Y) \cdot A = 0$. Then we have $R(AX,X) = 0$ for any tangent vectors X. Now we assume that M_q^n is conformally flat. Then

$$A^2X \wedge X = 0.$$

So we have

$$A^2 = \lambda\, Id \qquad \text{for some real number } \lambda.$$

According to the sign of λ, we consider three cases.

Case 1. $\lambda > 0$: They are well-known ones. It is locally isometric to one of the following:
(1) $M^n_q(2\sqrt{\lambda}),\ \ M^n_q(-2\sqrt{\lambda})$,
(2) $M^m_{q'}(\sqrt{\lambda}) \times M^{n-m}_{q-q'}(-\sqrt{\lambda}),\ \ 2 \le m \le n-2$,
(3) $M^{n-1}_q(k) \times \mathbb{R}$ or $M^{n-1}_{q-1}(k) \times \mathbb{R}_1,\ \ k = \pm\sqrt{\lambda}$,
where $M^m_q(k)$ denotes the semi-Riemannian manifold of constant sectional curvature k .

Case 2. $\lambda = 0$: This case was classified by Cahen and Kerbrat. Here we show the examples in a slightly different way.

Let $(\mathbb{R}^{n+2}_{q+1}, \langle,\rangle)$ be an $(n+2)$-dimensional semi-Euclidean space with an inner product $\langle,\rangle$ of index $q+1$ and Λ be the light cone. Let F be a linear endomorphism of $\mathbb{R}^{n+2}_{q+1}$ which satisfies the following conditions:
(1) F is self-adjoint with respect to $\langle,\rangle$,i.e., $\langle F\boldsymbol{x}, \boldsymbol{y}\rangle = \langle \boldsymbol{x}, F\boldsymbol{y}\rangle$.
(2) $F^2 = 0$.
(3) There exists a point $\boldsymbol{x} \in \Lambda$ such that $\langle F\boldsymbol{x}, \boldsymbol{x}\rangle > 0$.

We consider the following subset of $\mathbb{R}^{n+2}_{q+1}$

$$\Lambda \cap \{\boldsymbol{x} \in \mathbb{R}^{n+2}_{q+1} \,|\, \langle \boldsymbol{x}, F\boldsymbol{x}\rangle = 1\}$$

and we define M by one of its connected components. Then M is a semi-Riemannian submanifold in $\mathbb{R}^{n+2}_{q+1}$ with codimension 2 and its index is q. Moreover we obtain the following.
(i) M is an extrinsic symmetric submanifold in $\mathbb{R}^{n+2}_{q+1}$. In particular it is a symmetric space.
(ii) M is a hypersurface of the light cone. Therefore it is conformally flat.
(iii) Its Schouten tensor A satisfies $A^2 = 0$.
Cahen and Kerbrat ([1]) have shown that every conformally flat symmetric space with $A^2 = 0$ is obtained by this construction.

Case 3. $\lambda < 0$: We define a complex sphere. Let $\mathbb{C}^{n+1}$ be an $(n+1)$-dimensional complex vector space with a complex inner product $(,)$:

$$(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n+1} z_i w_i, \quad \boldsymbol{z} = (z_1, \cdots, z_{n+1}), \boldsymbol{w} = (w_1, \cdots, w_{n+1})$$

and denote by $\langle,\rangle$ its real part,i.e,

$$\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \text{the real part of} \quad (\boldsymbol{z}, \boldsymbol{w}).$$

Then the signature of the inner product $\langle , \rangle$ is $(n+1, n+1)$. We define a complex hypersurface in $\mathbb{C}^{n+1}$ by the following equation

$$(\boldsymbol{z}, \boldsymbol{z}) = z_1^2 + \cdots + z_{n+1}^2 = c, \quad c \in \mathbb{C}, c \neq 0.$$

We denote it by $\mathbb{C}S^n(c)$ and call it *a complex sphere.* The complex sphere $\mathbb{C}S^n(c)$ with the induced metric from $\langle , \rangle$ is a semi-Riemannian symmetric space with the signature (n, n). In particular if $c = \sqrt{-1}b$ $(b \in \mathbb{R})$ is pure imaginary, then the complex sphere $\mathbb{C}S^n(\sqrt{-1}b)$ is contained in the light cone. Therefore it is conformally flat. We compute its Schouten tensor A and see that it has the form:

$$\frac{1}{2b}\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

with respect to an orthonormal basis $\{e_1, \cdots, e_{2n}\}$, $\langle e_{2i-1}, e_{2i-1}\rangle = -\langle e_{2i}, e_{2i}\rangle = 1$ $(i = 1, \cdots, n)$.
Cahen and Kerbrat ([1]) have shown that a conformally flat symmetric space with $A^2 = \lambda\, Id$, $\lambda < 0$ is isometric to a complex sphere $\mathbb{C}S^n(\sqrt{-1}b)$ for $\lambda = -\frac{1}{4b^2}$. A generalization of their results is shown in [4].

Inspired by these results, we obtain two kind of constructions of conformally flat homogeneous semi-Riemannian manifolds. We will show them in the next two sections.

§3 A method of the construction of examples I

In this section, we show a method of a construction of conformally flat semi -Riemannian manifolds with nilpotent Schouten tensor. In this construction, we show interesting relations between the semi-Riemannian geometry and the affine differential geometry of centro-affine hypersurfaces.

Let $(\mathbb{R}^{n+2}_{q+1}, \langle , \rangle)$ be an $(n+2)$-dimensional semi-Euclidean space with an inner product $\langle , \rangle$ of index $q+1$. It is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{k+1}\{x_i y_{k+1+i} + x_{k+1+i} y_i\} + \sum_{j=2(k+1)+1}^{n+2} \varepsilon_j x_j y_j, \quad \varepsilon_j = 1 \text{ or } -1.$$

We denote by h the semi-Riemannian metric on $\mathbb{R}^{n+2}$ induced from $\langle , \rangle$

Let $\pi : \mathbb{R}^{n+2} \to \mathbb{R}^{k+1}$ be the projection defined by

$$(x_1, \cdots, x_{k+1}, x_{k+2}, \cdots, x_{n+2}) \mapsto (x_1, \cdots, x_{k+1}).$$

We denote by $\bar{\pi}$ the restriction of π to $\Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\})$.
Then $\bar{\pi} : \Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\}) \to \mathbb{R}^{k+1} - \{0\}$ is a fibre bundle over $\mathbb{R}^{k+1} - \{0\}$ with the standard fibre diffeomorphic to $\mathbb{R}^{n-k}$.

Let N be a k-dimensional manifold and $F : N \to \mathbb{R}^{k+1} - \{0\}$ be a centro-affine hypersurface immersion. That is, it is an immersion such that for each point $p \in N$, the position vector $F(p)$ is transversal to the tangent space $F_*(T_pN)$. We consider the pull-back bundle of the fibre bundle $\bar{\pi}$ by the immersion F. We denote by M and f the total space of the pull-back bundle and the bundle homomorphism of M into $\Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\})$, respectively. That is, the following diagram holds:

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & \Lambda \cap \pi^{-1}(\mathbb{R}^{k+1} - \{0\}) \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ N & \xrightarrow[\ F\ ]{} & \mathbb{R}^{k+1} - \{0\} \end{array} .$$

Then M is an n-dimensional manifold diffeomorphic to the product manifold $TN \times \mathbb{R}^{n-2k}$ of the tangent bundle over N and $\mathbb{R}^{n-2k}$.

M with the induced metric by f has the following properties ([5]).

Theorem 1 (M, f^*h) is a conformally flat semi-Riemannian manifold whose Schouten tensor A satisfies $A^2 = 0$ (equivalently $Q^2 = 0$).

Theorem 2 There are interesting relations between (M, f^*h) and (N, F).
(1) M is locally symmetric if and only if there exists a symmetric bilinear form $\bar{b}$ on $\mathbb{R}^{k+1}$ such that $F(N)$ is contained in a hypersurface defined by $\bar{b}(x, x) = -1$.
(2) M is geodesically complete if and only if N is geodesically complete with respect to the induced affine connecton as a centro-affine hypersurface.
(3) If N is a homogeneous centro-affine hypersurface, then M is a homogeneous semi-Riemannian manifold.
Here "homogeneous " in (3) means that there exist a connected Lie group H which acts transitively on N and a Lie group homomorphism $\phi : H \to GL(k+1, \mathbb{R})$ such that

$$F(ap) = \phi(a)F(p) \qquad \text{for all } a \in H,\, p \in N.$$

There are many homogeneous centro-affine hypersurfaces and hence by our results we have many conformally flat homogeneous semi-Riemannian manifolds with nilpotent Schouten tensor.

Example When $k = 1$, it is easy to classify homogeneous non-degenerate centro-affine curves in $\mathbb{R}^2$. They are the follwing:

1. $y = x^\lambda$ $(\lambda > 1, x > 0)$,
2. $y = x^\lambda$ $(\lambda \leq -1, x > 0)$,
3. $\begin{cases} x = e^t \cos bt \\ y = e^t \sin bt \end{cases} \quad (b > 0)$,
4. $x^2 + y^2 = 1$,
5. $y = x \log x$ $(x > 0)$.

Let M be a semi-Riemannian manifold constructed from one of centro-affine curves in the above. Then M is a symmetric space if and only it is constructed from case 2 with $\lambda = -1$ and case 4. It is easily seen that the other are geodesically incomplete. By Theorem 2 (3), we see that M admits a large isometry group, that is, $\dim \mathrm{Isom}(M) = \frac{1}{2}n(n-1)+1$. It is known as a kind of an Egorov space (cf. [8]).

§4 A method of the construction of examples II

In this section, we explain the second method of the construction of examples.

We consider the linear isotropy representation of a semisimple symmetric space and find its orbits which are hypersurfaces in the light cone. At the present, we have no systematic way of investigations and I only show some examples. Now we show the examples which correspond to the following symmetric pairs:

$$(SO_+(p,q+2), SO_+(p,q)\times SO(2)),$$
$$(SO_+(p+1,q+1), SO_+(p,q)\times SO_+(1,1)).$$

We describe the linear isotropy representation of these symmetric spaces. We define an indefinite inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^m$ $(n = 2m-2)$ by

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{i=p+1}^{m} x_i y_i = {}^t\boldsymbol{x} I_{p,q} \boldsymbol{y} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m.$$

Here $I_{p,q}$ denotes the diagonal matrix

$$I_{p,q} = (\overbrace{-1,\cdots,-1}^{p}, \overbrace{1,\cdots,1}^{q}), \qquad (p+q=m).$$

We denote by $M(m,2:\mathbb{R})$ the linear space of real $m\times 2$ matrices. For two matrices $X, Y \in M(m,2:\mathbb{R})$, we define two kinds of inner products by

$$(X,Y)_1 = \mathrm{tr}({}^t\boldsymbol{X} I_{p,q}\boldsymbol{Y}) = \langle \boldsymbol{x}_1, \boldsymbol{y}_1\rangle + \langle \boldsymbol{x}_2, \boldsymbol{y}_2\rangle,$$
$$(X,Y)_2 = \mathrm{tr}(I_{1,1}{}^t\boldsymbol{X} I_{p,q}\boldsymbol{Y}) = -\langle \boldsymbol{x}_1, \boldsymbol{y}_1\rangle + \langle \boldsymbol{x}_2, \boldsymbol{y}_2\rangle.$$

Here we consider a matrix X as a pair of two column vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. These inner products $(\ ,\ )_i$ are nondegenerate and have the following signature:

The signature of $(\ ,\ )_1 = (2q, 2p)$, index $2p$
The signature of $(\ ,\ )_2 = (m, m)$, index m

We consider the product Lie groups:

$$K_1 = SO_+(p,q)\times SO(2),$$

$$K_2 = SO_+(p,q)\times SO_+(1,1).$$

Here $SO_+(p,q)$ denotes a connected component which contains the identity element of the group of orthogonal transformations of $(\mathbb{R}^m, \langle\ ,\ \rangle)$.

The action of K_i $(i=1,2)$ on $M(m,2:\mathbb{R})$ by

$$(k_1,k_2)\times X \mapsto k_1 X k_2^{-1} \qquad (k_1\in SO_+(p,q), k_2\in SO(2) \text{ or } SO_+(1,1)).$$

Then K_i acts as the group of orthogonal transformations with respect to $(\ ,\ )_i$ $(i=1,2)$, respectively. If we find a lightlike vector in $M(m,2:\mathbb{R})$ whose orbit by K_i is a hypersurface in the light cone and has a nondegenerate induced metric, we obtain a conformally flat homogeneous semi-Riemannian manifold of signature $(2q-1,2p-1)$, index $2p-1$ or of signature $(m-1,m-1)$, index $m-1$.

Now we show some examples of lightlike vectors:

$$X_1=(\boldsymbol{e}_1,\boldsymbol{e}_{p+1}),\ \ X_2=(\boldsymbol{e}_{p+1},\boldsymbol{e}_{p+2}),\ \ X_3=(\boldsymbol{e}_1+\sqrt{2}\boldsymbol{e}_{p+1},\boldsymbol{e}_{p+1}),$$

where $\boldsymbol{e}_i$ is a column vector with ${}^t\boldsymbol{e}_i=(0,\cdots,0,\overset{i\text{-th}}{1},0,\cdots,0)$. Then X_1 is lightlike with respect to $(\ ,\ )_1$ and X_2, X_3 are lightlike with respect to $(\ ,\ )_2$. We denote by M_1 the K_1 -orbit through X_1 and by M_2, M_3 the K_2 -orbits through X_2, X_3,respectively. Then they are hypersurfaces in the light cones and hence conformally flat homogeneous semi-Riemannian manifolds.

The Schouten tensors A of M_1 and M_2 have the form:

$$A=\frac{1}{2}\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & -I_{m-2} & \\ & & & I_{m-2} \end{pmatrix}$$

with respect to some semi-orthonormal basis.
The Schouten tensor A of M_3 has the form:

$$A=\frac{1}{2}\begin{pmatrix} -1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}$$

with respect to some orthonormal basis.

To find more examples, we need to investigate "semisimple symmetric spaces of rank 2. "

§5 A classification of conformally flat homogeneous Lorentzian manifolds

Finally we explain our classification results of conformally flat homogeneous Lorentzian manifolds ([6],[7]).

The key of our approach is to determine the form of the Schouten tensor A. We classify the possible forms of the operator A. For this purpose, we show the useful identity of the eigenvalues of A. We assume that M is a homogeneous semi-Riemannian manifold. Then evidently, the —possibly complex— eigenvalues of A and their algebraic multiplicities are constant on M. It is a similar situation to the shape operators for isoparametric hypersurfaces in the semi-Riemannian space form. Hahn obtained the basic identity concerning principal curvatures of an isoparametric hypersurface ([2] Theorem 2.9). We have the same result for the eigenvalues of A.

Theorem 3 Let M^n_q be a conformally flat homogeneous semi-Riemannian manifold and $\lambda_1, \cdots, \lambda_r$ be the distinct eigenvalues of the Schouten tensor A on M with algebraic multiplicities $m_1, \cdots, m_r$, respectively. If for some $i \in \{1, \cdots, r\}$, the eigenvalue λ_i is real and the dimension of its eigenspace coincides with its algebraic multiplicity, then we have

$$\sum_{j \neq i} m_j \frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i} = 0.$$

Here the sum runs over all j which are not equal to i.

As an application of Theorem 3 we consider the case that the Schouten tensor A is diagonalizable with real eigenvalues. By the identity above we see that A has at most two distinct eigenvalues. In this case, the classification is same as that of the Riemannian case.

Theorem 4 Let M^n_q be a conformally flat homogeneous semi-Riemannian manifold whose Schouten tensor A is diagonalizable with real eigenvalues. Then M^n_q is locally isometric to one of the following: (1)$M^n_q(k)$, (2)$M^m_{q'}(k) \times M^{n-m}_{q-q'}(-k), k \neq 0, 2 \leq m \leq n-2$, (3)$M^{n-1}_q(k) \times \mathbb{R}$ or $M^{n-1}_{q-1}(k) \times \mathbb{R}_1$, $k \neq 0$, where $M^m_{q'}(k)$ denotes a semi-Riemannian manifold of constant curvature k and index q'.

As a second application of Theorem 3, we give a classification of possible candidates for the Schouten tensor A of a conformally flat homogeneous Lorentzian manifold. From now on we assume that M is an $n(\geq 4)$ dimensional conformally flat homogeneous Lorentzian manifold whose Schouten tensor A is not diagonalizable with real eigenvalues.

As our main result, we have the following.

Theorem 5 Under the assumption above, the Schouten tensor A has exactly one of the following three forms:

$$\text{Case 1.} \quad \begin{pmatrix} a & -b & & & & & & \\ b & a & & & & & & \\ & & \lambda & & & & & \\ & & & \ddots & & & & \\ & & & & \lambda & & & \\ & & & & & -\lambda & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda \end{pmatrix} \quad \begin{array}{l} a^2 + b^2 = \lambda^2 \\ b \neq 0 \\ \dim T_\lambda = \dim T_{-\lambda} \end{array}$$

In case 1, the tensor A has the complex eigenvalues $a \pm \sqrt{-1}b$ and real eigenvalues $\pm\lambda$. And the dimensions of the eigenspaces of $\pm\lambda$ coincide. Here T_λ and $T_{-\lambda}$ denote the eigenspaces of eigenvalues λ and $-\lambda$, respectively.

$$\text{Case 2.} \quad \begin{pmatrix} \lambda & \varepsilon & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix} \quad \begin{array}{l} \varepsilon = 1 \text{ or } -1 \\ \lambda \leq 0 \end{array}$$

In case 2, the tensor A has the only one nonpositive real eigenvalue λ.

$$\text{Case 3.} \quad \begin{pmatrix} \lambda & 0 & 0 & & & & & & \\ 0 & \lambda & 1 & & & & & & \\ 1 & 0 & \lambda & & & & & & \\ & & & \lambda & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda & & & \\ & & & & & & -\lambda & & \\ & & & & & & & \ddots & \\ & & & & & & & & -\lambda \end{pmatrix} \quad \begin{array}{l} \lambda < 0 \\ \dim T_{-\lambda} \leq \dim T_\lambda - 2 \end{array}$$

In case 3, the tensor A has one negative eigenvalue λ and the dimension of its eigenspace does not coincide with its algebraic multiplicity. In this case it is possible that A does not have the eigenvalue $-\lambda$.

Here our expressions of matrices are those with respect to a semi-orthonormal basis $\langle e_1, e_2 \rangle = 1$, $\langle e_i, e_j \rangle = \delta_{ij}$ $(i, j \geq 3)$.

Now we discuss case by case.

Case 1.

The local classification of case 1 is given by the following.

Theorem 6 Let M be an $n(\geq 4)$-dimensional conformally flat homogeneous Lorentzian manifold whose Schouten tensor A has the form of Case 1. Then M is locally isometric to M_1 constructed in section 4 up to homothety.

Therefore the local classification of case 1 is complete.

Case 2 with $\lambda = 0$.

Next we consider Case 2 with $\lambda = 0$. These conformally flat homogeneous Lorentzian manifolds are constructed in section 3. Here we show another description, i.e., the structures of Lie algebras:

Example Let $\mathfrak{k}$ be a real linear space with the basis E_i $(3 \leq i \leq n)$, F_{ij} $(3 \leq i < j \leq n)$, X_i $(1 \leq i \leq n)$. We define a bracket operation $[\ ,\ ]$ on $\mathfrak{k}$ as follows:

$$
\begin{array}{ll}
[E_i, E_j] = 0 & [F_{ij}, F_{kl}] = -\delta_{ik}F_{jl} + \delta_{jk}F_{il} + \\
[E_i, F_{jk}] = \delta_{ij}E_k - \delta_{ik}E_j & \qquad\qquad \delta_{il}F_{jk} - \delta_{jl}F_{ik} \\
[E_i, X_1] = 0 & [F_{ij}, X_1] = 0 \\
[E_i, X_2] = -X_i - cE_i & [F_{ij}, X_2] = 0 \\
[E_i, X_j] = \delta_{ij}X_1 & [F_{ij}, X_k] = -\delta_{ik}X_j + \delta_{jk}X_i \\
 & \\
[X_1, X_2] = -cX_1 & \\
[X_1, X_j] = 0 & [X_i, X_j] = 0 \\
 & \\
[X_2, X_j] = -\varepsilon E_j & \\
 & \\
i, j, k, l \geq 3 & c \in \mathbb{R} \qquad \varepsilon = 1 \text{ or } -1
\end{array}
$$

The space spanned by F_{ij} $(3 \leq i < j \leq n)$ is the Lie algebra of skew-symmetric endomorphisms. We define the action of the Lie algebra on the space spanned by E_i $(3 \leq i \leq n)$ and the space spanned by X_i $(1 \leq i \leq n)$. We remark that the constant c is contained in these brackets $[E_i, X_2] = -X_i - cE_i$ and $[X_1, X_2] = -cX_1$. Then $[\ ,\ ]$ satisfies the Jacobi identity and $\mathfrak{k}$ becomes a Lie algebra. Let $\mathfrak{h}$ be a linear subspace of $\mathfrak{k}$ spanned by $\{E_i, F_{ij}\}$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{k}$. And the dimension of the Lie algebra of $\mathfrak{k}$ is equal to $\frac{1}{2}n(n-1) + 1$

From this pair $(\mathfrak{k}, \mathfrak{h})$ of Lie algebra and its Lie subalgebra, we construct a homogeneous Lorentzian manifold by the standard method. Let K be a simply connected Lie group corresponding to $\mathfrak{k}$ and H be the connected Lie subgroup of K which corresponds to $\mathfrak{h}$. Then H is a closed subgroup of K and hence we obtain the homogeneous space $M = K/H$. Let $\pi : K \longrightarrow K/H = M$ be the projection. We put $\pi(H) = o$. The differential of π at the unit element $e \in K$ defines the projection $\pi : \mathfrak{k} \longrightarrow T_oM$. Let $\mathfrak{p}$ be the subspace spanned by $\{X_i\ (1 \leq i \leq n)\}$. We identify $\mathfrak{k}/\mathfrak{h}$ with $\mathfrak{p}$. Under this identification, we define an inner product on $\mathfrak{k}/\mathfrak{h}$ by

$$\langle X_1, X_2 \rangle = 1, \quad \langle X_i, X_j \rangle = \delta_{ij}\ (3 \leq i, j \leq n), \quad \text{otherwise } 0.$$

This inner product $\langle\ ,\ \rangle$ is invariant by the adjoint representation of $\mathfrak{h}$ on $\mathfrak{k}/\mathfrak{h}$. Therefore we can define the K-invariant Lorentzian metric g on M such that

$$g(\pi(X), \pi(Y)) = \langle X, Y\rangle \qquad X, Y \in \mathfrak{p}$$

at $o \in M$. Thus we obtain the homogeneous Lorentzian manifold (M, g). Moreover we see that it is conformally flat and its Schouten tensor A has the form:

$$A = \begin{pmatrix} 0 & \varepsilon & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

Furthermore we have

Theorem 7 Let M be an $n(\geq 4)$-dimensional conformally flat homogeneous Lorentzian manifold whose Schouten tensor A has the form of case 2 with $\lambda = 0$. Then M is locally isometric to the model constructed in Example above.

Remark Let c be the parameter in the Lie algebra $\mathfrak{k}$. Then we see that (M, g) in Example is a Lorentzian symmetric space if and only if $c = 0$.

Case 2 $\lambda < 0$.

In this case, we can construct examples similarly to Example in the above and characterize them.

Case 3.

In this case, we can construct examples of Lie algebras similarly to Example in the above. However we cannot solve the classification problem for this case at the present. So our classification problem for Lorentzian manifolds is still open.

Our classification is local. In this talk our approach is a classification by local isometric classes. We think that a global classification is a difficult problem. It may be complicated, compared with the Riemannian case. For example, the homogeneity of a semi-Riemannian metric does not imply the geodesically completeness.

Our method of a classification is due to the theory of infinitesimally homogeneous spaces by Singer. We explain this theory quickly. The essential local invariants of a semi-Riemannian manifold M are the curvature tensor R and its covariant derivatives $\nabla R, \nabla^2 R, \cdots$. In particular, if M is conformally flat, then the Schouten tensor A and its covariant derivatives $\nabla A, \nabla^2 A, \cdots$ are essential local invariants. Singer's theory discusses the relation between the homogeneity and the curvature tensor and its covariant derivatives .

We recall the notion of a curvature homogeneous space introduced by Singer [9]. For a non-negative integer l, we consider the following condition:

$P(l)$: for every $p, q \in M$ there exists a linear isometry $\phi : T_pM \to T_qM$ such that

$$\phi^*(\nabla^i R)_q = (\nabla^i R)_p \quad i = 0, 1, ..., l.$$

That is, ϕ preserves the curvature tensors and their higher covariant derivatives up to order l. If M is locally homogeneous, then it satisfies $P(l)$ for any l. A semi-Riemannian manifold which satisfies $P(l)$ is said to be *curvature homogeneous up to order* l .

We denote by $\mathfrak{so}(T_pM)$ the Lie algebra of the endomorphisms of T_pM which are skew-symmetric with respect to $\langle , \rangle$. For a non-negative integer l, we define a Lie subalgebra $\mathfrak{g}_l(p)$ of $\mathfrak{so}(T_pM)$ by

$$\mathfrak{g}_l(p) = \{ X \in \mathfrak{so}(T_pM) \mid X \cdot (\nabla^i R)_p = 0, \quad i = 0, 1, ..., l \ \},$$

where X acts as a derivation on the tensor algebra on T_pM. Since $\mathfrak{g}_l(p) \supseteq \mathfrak{g}_{l+1}(p)$, there exists a first integer $s(p)$ such that $\mathfrak{g}_{s(p)}(p) = \mathfrak{g}_{s(p)+1}(p)$. Namely we have

$$\mathfrak{so}(T_pM) \supseteq \mathfrak{g}_0(p) \supsetneq \mathfrak{g}_1(p) \supsetneq \mathfrak{g}_2(p) \supsetneq \cdots \supsetneq \mathfrak{g}_{s(p)}(p) = \mathfrak{g}_{s(p)+1}(p).$$

Following Singer we say that $(M, \langle , \rangle)$ is *infinitesimally homogeneous* if M satisfies $P(s(p) + 1)$ for some point $p \in M$. If M is infinitesimally homogeneous, $s(q)$ does not depend on $q \in M$. We put $s_M = s(p)$ for some point $p \in M$ and call it *the Singer invariant* of an infinitesimally homogeneous semi-Riemannian manifold M. The remarkable result by Singer is the following.

Theorem S.1 A connected infinitesimally homogeneous semi-Riemannian manifold is locally homogeneous.

The proof of Theorem S.1 implies the following.

Theorem S.2 Let M and M' be two locally homogeneous semi-Riemannian manifolds and $p \in M$ and $p' \in M'$. Suppose that there exists a linear isometry $\phi : T_pM \to T_{p'}M'$ such that

$$\phi^*(\nabla^i R')_{p'} = (\nabla^i R)_p \quad i = 0, 1, ..., s_M + 1,$$

where s_M denotes the Singer invariant of M. Then there exists a local isometry φ of a neighborhood of p onto a neighborhood of p' which satisfies $\varphi(p) = p'$ and $\varphi_{*p} = \phi$.

As a corollary, the following holds. It gives us a fundamental method for our classification problems .

Corollary Let M and M' be two conformally flat locally homogeneous semi-Riemannian manifolds and $p \in M$ and $p' \in M'$. Suppose that there exists a linear isometry $\phi : T_pM \to T_{p'}M'$ such that

$$\phi^*(\nabla^i A')_{p'} = (\nabla^i A)_p \quad i = 0, 1, ..., s_M + 1.$$

Then there exists a local isometry φ of a neighborhood of p onto a neighborhood of p' which satisfies $\varphi(p) = p'$ and $\varphi_{*p} = \phi$.

Finally we show the Singer invariants of conformally flat homogeneous Lorentzian manifolds:

	the Singer invariant
Case 1 $dim = 4$	0
Case 1 $dim \geq 6$	1
Case 2 $\lambda < 0$	1
Case 2 $\lambda = 0$	0
Case 3	?

We think that there are examples of higher Singer invariants in case 3. So it is one of reasons why the classification problem in case 3 is difficult.

References

[1] M.Cahen and Y.Kerbrat:*Domaines symetriques des quadriques projectives,* J. Math. pures et appl., 62(1983), 327-348.

[2] J.Hahn:*Isoparametric hypersurfaces in the pseudo-Riemannian space forms,* Math. Z., 187(1984), 195-208.

[3] J.Hahn:*Isotropy representations of semisimple symmetric spaces and homogeneous hypersurfaces,* J. Math. Soc. Japan ,40(1988), 271-288.

[4] K.Honda:*Conformally flat semi-Riemannian manifolds with commuting curvature and Ricci operators*, Tokyo J. Math. , **26**(2003), 241-260.

[5] K.Honda and K.Tsukada:*Conformally flat semi-Riemannian manifolds with nilpotent Ricci operators and affine differential geometry*, Annals of Global Analysis and Geometry, **25**(2004), 253-275.

[6] K.Honda and K.Tsukada:*Three-dimensional conformally flat homogeneous Lorentzian manifolds*, J. of Phys. A: Math.Theor.,**40**(2007),831-851.

[7] K.Honda and K.Tsukada:*Conformally flat homogeneous Lorentzian manifolds*, in "Recent trends in Lorentzian geometry" , edited by M.Sánchez, M.Ortega, A.Romero, Springer Proceedings in Mathematics and Statistics, **26**(2013), 295-314.

[8] V. Patrangenaru:*Lorentz manifolds with the three largest degrees of symmetry*, Geom. Dedicata, **102**(2003), 25-33.

[9] I.M.Singer:*Infinitesimally homogeneous spaces*, Comm. Pure Appl. Math., 13(1960), 685-697.

[10] H.Takagi:*Conformally flat Riemannian manifolds admitting a transitive group of isometries*, Tôhoku Math. J., 27(1975), 103-110.