On the classification of Lorentzian r-th symmetric spaces

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Introduction

Aim of the talk:

- To classify the 2nd-symmetric Lorentzian manifolds, i.e.:

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- To classify the 2nd-symmetric Lorentzian manifolds, i.e.:

$$\nabla^2 R := \nabla(\nabla R) = 0$$

- To provide properties and open questions on the rth-symmetric case $\nabla^r R = 0$ and, in general on the implications of

$$\nabla^r T = 0$$

for any tensor field.

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Moreover,

- this is a natural generalization of symmetric spaces whose relation with homogeneous one must be clarified.

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- So, instead of $\nabla^2 R = 0$, semi-symmetric spaces were introduced (Cartan, Szabó):

$$
\begin{aligned}
\nabla^2 R(X,Y;\dots) \quad & -\nabla^2 R(Y,X;\dots) = \\
& = \nabla_X(\nabla_Y R) - \nabla_Y(\nabla_X R) - \nabla_{[X,Y]} R \\
& =: R(X,Y)\cdot R = 0
\end{aligned}
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- So, instead of $\nabla^2 R = 0$, semi-symmetric spaces were introduced (Cartan, Szabó):

$$\begin{aligned} \nabla^2 R(X, Y; \dots) \quad & -\nabla^2 R(Y, X; \dots) = \\ & = \nabla_X(\nabla_Y R) - \nabla_Y(\nabla_X R) - \nabla_{[X,Y]} R \\ & =: R(X, Y) \cdot R = 0 \end{aligned}$$

- Lorentzian and higher signatures: $\nabla^r R = 0 \not\Rightarrow \nabla R = 0$

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How hadn't 2nd-symmetry been studied before?

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Main result to be proven:

Theorem (Blanco, Senovilla, —)

Let (M, g) be a properly 2nd-symmetric Lorentzian n-manifold:

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- *(Local classification). (M, g) is locally isometric to a product*
  - *a (non-flat) symmetric Riemannian space (N, g_N)*
  - *a proper 2nd-order Cahen-Wallach space $(\mathbb{R}^{d+2}, g_A)$,*
    $g_A = -2du\left(dv + (\mathbf{a_{ij}u} + \mathbf{b_{ij}})x^i x^j du\right) + \delta_{ij} dx^i dx^j$
    with some $a_{ij} \neq 0$.

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    with some $a_{ij} \neq 0$.
- *(Global classification). Moreover, if (M, g) is 1-connected and geodesically complete, then it is globally isometric to $(\mathbb{R}^{d+2} \times N, g_A \oplus g_N)$.*

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References

- Blanco, Senovilla, — : J. Eur. Math. Soc. (2013)

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- Senovilla: Class. Quant. Grav. (2008)

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- Blanco, Senovilla, —, arxiv: 1001.3629, 1101.3438
- Senovilla: Class. Quant. Grav. (2008)
- Alekseevski, Galaev: J. Geom. Phys. (2011)
  Galaev, arxiv: 1110.1998, 1011.3977v2, 1211.5965.

Characterizations of local symmetry vs 2nd-symmetry

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Local symmetry

Proposition

For a (connected) semi-Riemannian manifold (N, h), they are equivalent:

(i) (N, h) *is locally symmetric, i.e.* $\nabla R = 0$.

(ii) *If* X, Y *and* Z *are parallel vector fields along a curve* γ*, then so is* $R(X, Y)Z$.

(iii) *The sectional curvature of non-degenerate planes is invariant under parallel transport*

(iv) *The local geodesic symmetry* s_p *is an isometry at any* $p \in N$.

(v) (N, h) *is locally isometric to a symmetric space.*

Characterizations of local symmetry vs 2nd-symmetry

Remark

"(N, h) is locally isometric to a symmetric space"
$\rightsquigarrow$ as a difference with the locally homogeneous case, as there exists *non-regular* ones (Kowalski'97)

Characterizations of local symmetry vs 2nd-symmetry

2nd symmetry

Lemma

For a semi-Riemannian (N, h), they are equivalent:

- *Skew symmetry of $\nabla^2 R$ in the derivatives slots.*
- *For any non-degenerate tangent plane $\Pi_p \subset T_pN$, its parallel transport Π_γ along any geodesic γ, the derivative of its sectional curvature $\frac{d}{d\tau}(K(\Pi_\gamma))$ is a constant along γ.*
- *For any parallelly propagated vector fields X, Y, Z along any geodesic γ, the vector field $(\nabla_{\gamma'} R)(X, Y)Z$ is itself parallelly propagated along γ.*

Characterizations of local symmetry vs 2nd-symmetry

Proposition

For a semi-Riemannian (N, h), they are equivalent:

(i) (N, h) *is 2nd-symmetric,* $\nabla\nabla R = 0$

(ii) (N, h) *is semi-symmetric ($R(X, Y)R = 0$) and satisfies any of the equivalent conditions to skew-symmetry in the lemma .*

(iii) *If V, X, Y, Z are parallelly propagated vector fields along any curve, then so is $(\nabla_V R)(X, Y)Z$.*

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Remark

Characterizations in terms of an analog of the *geodesic symmetry* or local isometries to a model space are conspicuously absent.

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Proof. Use de Rham decomposition

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  - g_R es reducible: $g_R = g_R^{(1)} \oplus g_R^{(2)} \oplus \ldots \oplus g_R^{(s)}$.
  - $L = \sum_{m=1}^{s} \lambda_m g_R^{(m)}$ for some $\lambda_m \in \mathbb{R}$.

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2. Holds even for homogeneous sp. (Alekseevsky, Kimelfeld '75) —and locally homogeneous with Ric≤ 0 are regular (Spiro '93)

Classification locally symmetric vs 2nd-symmetric

Lorentzian symmetric spaces

Theorem (Cahen, Wallach '70)

A complete 1-connected Lorentzian symmetric space (M, g) is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:

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Lorentzian symmetric spaces

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A complete 1-connected Lorentzian symmetric space (M, g) is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:

1. $(\mathbb{R}, -dt^2)$
2. *The universal cover of de Sitter or anti-de Sitter d-spaces,* $d \geq 2$,
3. *A Cahen-Wallach space* $CW^d(A) = (\mathbb{R}^d, g_A), d \geq 2$, *where* $A \equiv (A_{ij})$ *is a* $(d-2) \times (d-2)$ *matrix and* $g_A = -2du\left(dv + A_{ij}x^i x^j du\right) + \sum_{ij} \delta_{ij} dx^i dx^j$

Classification locally symmetric vs 2nd-symmetric

Remark

Choosing A with $\text{trace}(A) = 0$:
there are Ricci flat non-flat Lorentzian symmetric spaces.

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there are Ricci flat non-flat Lorentzian symmetric spaces.

Remark

Lorentzian symmetric space with a parallel lightlike v.f. $K \Rightarrow$:
Locally isometric to the product of a $CW^d(A), d > 2$ and Riemannian symmetric space.

Classification locally symmetric vs 2nd-symmetric

2nd-symmetric:

The theorem to be proven shows:

> *proper 2nd-symmetric spaces only appear generalizing the family of Cahen-Wallach spaces $CW^d(A), d > 2$:*
>
> - $\rightsquigarrow$ *allow an affine dependence of the matrix A in u*

Generalization of Cahen-Wallach family

Generalized Cahen-Wallach d-space of order r,
$CW_r^d(A)= (\mathbb{R}^d, g_A)$, $d \geq 2$: metric:

$$g_A = -2du\left(dv + \sum_{ij} A_{ij}(u)x^i x^j du\right) + \sum_{ij} \delta_{ij} dx^i dx^j$$

where $A \equiv (A_{ij}(u))$ is a $(d-2)\times(d-2)$ matrix:

$$A_{ij}(u) = A_{ij}^{(r-1)} u^{r-1} + \cdots + A_{ij}^{(1)} u + A_{ij}^0$$

for symmetric (constant) matrixes A_{ij}^k.

Generalization of Cahen-Wallach family

Proposition

Any generalized Cahen-Wallach space $CW_r^d(A)$ satisfies:

1. *If $A_{ij}^{(r-1)} \neq 0$ ($CW_r^d(A)$ is proper) then it is proper rth-symmetric*

1. Direct computation: in an appropriate basis $\{E_\alpha\} = \{E_0 = \partial_u - \sum A_{ij} x^i x^j \partial_v, E_1 = \partial_v, \partial_i\}$ the only non-vanishing components of $\nabla^l R$, $l \in \{0, \ldots r-1\}$ are:
$\nabla_0 \overset{(l)}{\ldots} \nabla_0 R^1_{i0j} = \frac{d^l A_{ij}}{du} = \sum_{k=l}^{r-1} \frac{k!}{(k-l)!} A_{ij}^{(k)} u^{k-l}$ □

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1. *If $A_{ij}^{(r-1)} \neq 0$ ($CW_r^d(A)$ is proper) then it is proper rth-symmetric*
2. *$K = \partial_v$ is a lightlike parallel vector field*
3. *It is analytic*
4. *it is geodesically complete*

Proof. 2,3: Trivial

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Proof. 2,3: Trivial
4. Direct computation or general results (Candela, Romero, — '13)

Generalization of Cahen-Wallach family

Corollary

A complete 1-connected Lorentzian manifold locally isometric to some $CW^d_r(A)$ is globally isometric too.

This will allow to go from the local to the global result.

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Remark

By the way:
Lafuente '88 proved that, for locally symmetric semi-Riemannian spaces, the three types of causal completeness (timelike, spacelike and lighlike) coincide. Does this hold for second/rth symmetric?

Must rth-symmetry imply local symmetry ?

This is a particular case of:

- When $\nabla^r T = 0 \Rightarrow \nabla T = 0$?

Riemannian case

Theorem

Let (M, g) be Riemannian and T a tensor field such that $\nabla^r T = 0$. Then $\nabla T = 0$ if either
(a) (Nomizu-Ozeki '62) g is complete and irreducible, or
(b) (Nomizu [unpub], Tanno '72) T is R, or Ric, Weyl, projective t.

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In particular, from (b), Riemmannian r-th symmetric implies locally symmetric.

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Proof (a) 1. Case $r = 2$ suffices (replace otherwise $\tilde{T} := \nabla^{r-2} T$).
2. Put $f := g(T, T)/2$. Using $\nabla^2 T = 0$:

$$\mathrm{Hess} f(X, Y) = g(\nabla_X T, \nabla_Y T) \quad \text{and} \quad \nabla \mathrm{Hess} f = 0$$

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3. By Eisenhart thm: $\mathrm{Hess} f = cg$, $c \in \mathbb{R}$. Thus $Z :=\mathrm{grad}(f)$ satisfies $\nabla_X Z = cX$ (in particular, is homothetic)
4. Under irreducibility + completeness homothetic vectors are Killing: $c = 0$ $g(\nabla_X T, \nabla_Y T) = 0$. As g is Riemannian $\nabla T = 0$.

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2. As before, one has $\nabla_X Z = cX$ and needs $c = 0$.
3. As Z is homothetic, it is affine. Thus $L_Z \nabla = 0 = L_Z T$ and:

$$0 = L_Z \nabla T = \nabla_Z(\nabla T) + (s+1)c\nabla T = (s+1)c\nabla T$$

(s: covar minus contrav slots for T). That is, if $c \neq 0$ directly $\nabla T = 0$. $\square$

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- The manifold is incomplete with a proper (non-Killing) homothetic vector field (necessarily without zeroes)

In the latter case the metric can be written locally as a *cone*: $M = I \times S, I \subset (0,\infty), (S, g_S)$ Riemannian

$$g = dt^2 + t^2 \pi_S^* g_S$$

being $Z = t\partial_t$ proper homothetic . In particular:

$$\nabla Z = 2 \cdot \mathrm{Id} (\neq 0) \qquad \nabla^2 Z = 0$$

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Difficulties for the semi-Riemannian extension

1. The (full, local) de Rham decomposition cannot be carried out when the subspaces invariant by local holonomy are degenerate
2. The conclusion $c = 0$ only means $g(T, T)$ constant and $g(\nabla T, \nabla T) = 0$ i.e. ∇T is a lightlike tensor
3. Even in the non-degenerate irreducible case, to apply Eisenhart one needs : if the restricted homogeneous holonomy group is irreducible and a symm. 2-cov tensor h is invariant by the group, then $h = cg$ for some function c, which is constant if h is parallel
   However, this holds in Lorentzian signature and others (Tanno'67, $n = 2$ or non-neutral signature)

Further properties: $\nabla^r T = 0$ in generic points

Definition

A point p is generic if the curvature endomorphism:

$$R : \Lambda^2(M) \to \Lambda^2(M) \qquad v^\flat \wedge w^\flat \mapsto 2R(v, w)$$

is an isomorphism when restricted to p.

Theorem

If there exists a generic point, $\nabla^r T = 0$ implies $\nabla T = 0$, for any semi-Riemannian metric.

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Proofs of increasing generality:

1. Simply, no conic metric (nor flat one) is generic.

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Proofs of increasing generality:

1. Simply, no conic metric (nor flat one) is generic.
   Remarks
   - Valid only for the Riemannian case
   - Extensible to generic (non-degenerate) Ric, as $\mathrm{Ric}(\partial_t, X) = 0$ in the conic metric

$\nabla^r T = 0$ in generic points

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Proofs of increasing generality:

2 (Tanno '72) As we had Z with $\nabla_X Z = cX$:
$0 = L_Z \nabla = \nabla^2 Z + R(Z, \cdot) = R(Z, \cdot)$
So R is not invertible except if $Z = 0$.

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If there exists a generic point, $\nabla^r T = 0$ implies $\nabla T = 0$, for any semi-Riemannian metric.

Proofs of increasing generality:

2 (Tanno '72) As we had Z with $\nabla_X Z = cX$:
$0 = L_Z \nabla = \nabla^2 Z + R(Z, \cdot) = R(Z, \cdot)$
So R is not invertible except if $Z = 0$.
Remarks:

- Also valid for Riemannian and extensible to generic Ric
- For Lorentz and non-neutral sign. + irreducibility, it applies, but then implies only $g(\nabla T, \nabla T) = 0$ and $g(T, T)$ =const.

$\nabla^r T = 0$ in generic points

Theorem

(Senovilla '08) *If there exists a generic point, $\nabla^r T = 0$ implies $\nabla T = 0$ on all M, for any semi-Riemannian metric.*

Proofs of increasing generality:

3. (Senovilla '08) Apply the Ricci identities to T and ∇T:
   The invertibility of R allows to clear $\nabla T = 0$.

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Proofs of increasing generality:

3. (Senovilla '08) Apply the Ricci identities to T and ∇T:
The invertibility of R allows to clear $\nabla T = 0$.
Remarks:
   - Independent of both, signature or previous computations
   - Extensible to: *all semi-symmetric spaces have constant curvature around generic points*

Limits of old techniques

A computation in the spirit of old papers:

Proposition

Let (M, g) be semi-Riemannian and r-th symmetric. If there exists a vector field Z:

$$\nabla_X Z = cX \qquad c \in \mathbb{R} \qquad \forall X \in \mathfrak{X}(M)$$

then either Z is parallel or $R = 0$.

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then either Z is parallel or $R = 0$.

Proof. As Z is homothetic, $L_Z \nabla = 0$, $L_Z \nabla^k R^l_{ijk} = 0$ and:

$$0 = L_Z(\nabla^{r-1} R) = \nabla_Z(\nabla^{r-1} R) + (1 + r)c\nabla^{r-1} R = (1 + r)c\nabla^{r-1} R$$

So, if $c \neq 0$, use induction. □

Limits of old techniques

Corollary

A proper rth-symmetric Lorentzian (M, g) either admits a parallel lightlike **direction** *or satisfies that $\nabla^{r-1}R$ is (parallel and)* **null** *and $g(\nabla^{r-2}R, \nabla^{r-2}R)$ is a constant.*

Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f.

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Proof. The first possibility occurs either when degenerately reducible or when admits a lightlike parallel v.f.
Otherwise, in each irreducible part, put again $T = \nabla^{r-2}R$, $f = g(T, T)$, $\mathrm{Hess}f(X, Y) = g(\nabla_X T, \nabla_Y T)$ and $Z = \mathrm{grad}f$
By previous Prop., necessarily $Z \equiv 0$. □

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Remark

Limit of "old" results: this **suggests** that at least 2nd-symmetric Lorentzian spaces must admit a parallel lightlike v.f. K.

Existence of a lightlike parallel vector field

Theorem

(Senovilla '08). Any proper 2nd-symmetric Lorentzian space admits a unique lightlike parallel vector field K.

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Uniqueness: a linear combination of $K_1 \pm K_2$ would be (parallel and) timelike in contradiction with no-decompsability/properness.

Existence of a lightlike parallel vector field

Theorem

(Senovilla '08). Any proper 2nd-symmetric Lorentzian space admits a unique independent lightlike parallel vector field K.

- Analyze the curvature concomitants showing that, either such a K exists, or they vanish:
  (a) 1-form concomitants of order m and degree up to $m+1$
  (b) scalar or 2-cov. concomitants of equal order and degree.
- Using Ricci identity, such restrictions force the existence of K

Brinkmann spaces

Definition

A Brinkmann space is any Lorentzian n-manifold endowed with a complete lightlike parallel vector field K.

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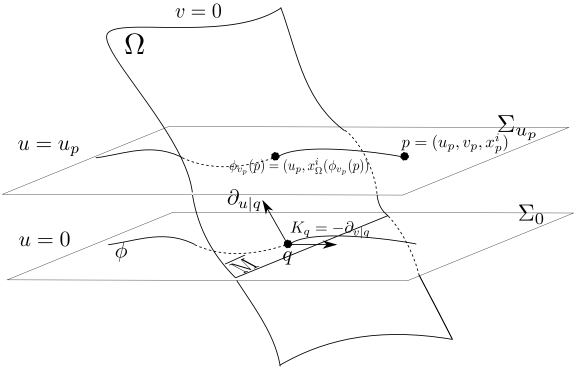
Brinkmann decomposition $\{u, v\}$:

1. K parallel: fix u (up to a constant) s.t.: $K = \text{grad} u$
2. K lightlike: $K^\perp$ degenerate totally geodesic integrable foliation with leaves Σ_u
3. Choose a hypersup. Ω transverse to K so that $\bar{M} := \Sigma_{u=0} \cap \Omega$ is spacelike a transverse
4. Let ϕ the flow of K, define v so that $\phi_{-v(p)}(p) \in \Omega$

Construction of the Brinkmann decomposition



Construction of a Brinkmann chart

- Brinkmann chart $\{u, v, x^i\}$: complete u, v to a chart by choosing $n-2$ coordinates x^i independent of u in Ω.

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- Expression of g in a Brinkmann chart:

$$g = -2du\left(dv + H(u,x^k)du + W_i(u,x^k)dx^i\right) + g_{ij}(u,x^k)dx^i dx^j$$

(natural sum in repeated indexes, $K \equiv -\partial_v$)

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Remark

Being more careful, one could get $H = 0$ and $W_i = 0$!
But it is preferred as above, as we wish to remove the u-dependence of $g_{ij}(u, x^i)$.

Geometric developments

- In general:

  Study of degenerate hypersurfaces
  $\rightsquigarrow$ *Transverse vector field* ξ

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- This happens in Brinkmann spaces too:

  degenerate hypersurfaces Σ_u *with transverse* ∂_u
  (non-univocally determined)

- Issues on Brinkmann spaces:
  - Relations between different choices of ∂_u (and Ω)
  - To introduce associated geometric objects with nice properties
  - Study potentially extensible to other degenerate cases

Geometric developments

- Foliations
  1. Spacelike $(n-2)$-foliation $\mathcal{M}$: $\{u = u_0, v = v_0\}$
  2. Timelike 2 foliation: $\mathcal{U}$: $\{x^i = x^i_0\}$

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  1. Non-orthogonal: $TM = T\mathcal{M} \oplus T\mathcal{U}$
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- Tangent bundle decompositions:
  1. Non-orthogonal: $TM = T\mathcal{M} \oplus T\mathcal{U}$
  2. Orthogonal: $TM = T\mathcal{U} \oplus (T\mathcal{U})^\perp$
- Natural bases:
  1. $T\mathcal{U} = \text{span}\{E_0 := \partial_u - H\partial_v, E_1 := \partial_v\}$
  2. $(T\mathcal{U})^\perp = \text{span}\{E_i := \partial_i - W_i\partial_v\}$
  3. $T\mathcal{M} = \text{span}\{\partial_i\}$

The spacelike foliation $\mathcal{M}$

Foliation $\mathcal{M}$: $\{u = u_0, v = v_0\}$
Metric induced bundle by the foliation:

$$\overline{g} = g_{ij}\overline{dx^i}\;\overline{dx^j}$$

(Notation: if dx^i, α on M, then $\overline{dx^i}, \overline{\alpha}$ on the foliation)

Exterior derivative $\overline{d}$

For any 1-form α on M:

$$\overline{d}\ \overline{\alpha} = \overline{d\alpha}.$$

Satisfies the properties of a derivation for $\omega, \tau \in \Lambda^q \mathcal{M}$:

1. Linearity plus $\overline{d}(\omega \wedge \tau) = \overline{d}\omega \wedge \tau + (-1)^s \omega \wedge \overline{d}\tau$.
2. $\overline{d}(\overline{d}\omega) = 0$.
3. If $\omega = \frac{1}{s!}\omega_{i_1 \dots i_s} \overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s}$, then
$\overline{d}\omega = \frac{1}{s!}\partial_k(\omega_{i_1 \dots i_s})\overline{d}x^k \wedge \overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s}$
4. Poincaré Lemma: $\overline{d}$-closed implies $\overline{d}$-exact.

Covariant derivative $\overline{\nabla}$ for $\mathcal{M}$

- Vector fields on $\mathcal{M}$ are naturally on M
- $\mathcal{M}$ is endowed with a Riemannian metric and then a natural $\overline{\nabla}$

$$\overline{\nabla}_X Y (\in \mathfrak{X}(\mathcal{M})) \qquad \forall X, Y \in \mathfrak{X}(\mathcal{M})$$

Extended to tensor fields on $\mathcal{M}$ satisfies

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Extended to tensor fields on $\mathcal{M}$ satisfies

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Defines a foliation curvature $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}}(X, Y)Z = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]})Z \in \mathfrak{X}(\mathcal{M}),\ \forall X, Y, Z \in \mathfrak{X}(\mathcal{M})$$

plus Ricci tensor $\overline{\mathcal{R}ic}$ and scalar curvature $\overline{\mathcal{S}}$.

Covariant derivative $\overline{\nabla}$ for $\mathcal{M}$

Definition

- $\mathcal{M}$ is flat (resp. locally symmetric) if $\overline{\mathcal{R}} = 0$ (resp. $\overline{\nabla}\ \overline{\mathcal{R}} = 0$)
- u-Einstein if $\overline{\mathcal{R}ic} = \mu\overline{g}$ for some μ s.t. $d\mu \wedge du = 0$ (Schur lemma Ric$= fg \Rightarrow f \equiv c$ does not apply to foliations) and:
  1. $\mathcal{M}$ is Einstein if $\mu = const.$,
  2. $\mathcal{M}$ is Ricci-flat if $\mu \equiv 0$.

Covariant derivative $\overline{\nabla}$ for $\mathcal{M}$

From Riemannian results:

Proposition

Let (M, g) be a Brinkmann space:

1. $\overline{\nabla}^r \overline{\mathcal{R}} = 0$ *(rth-symmetric)* $\Longrightarrow \overline{\nabla}\,\overline{\mathcal{R}} = 0$ *(locally symmetric).*
2. $\overline{\nabla}\,\overline{\mathcal{R}} = 0$ *(locally symmetric) and* $\overline{\mathcal{R}ic} = 0$ *(Ricci-flat)* $\Longrightarrow \overline{\mathcal{R}} = 0$ *(flat)*
3. *If $\mathcal{M}$ is flat, there exists a chart $\{u, v, y^i\}$ s.t.:* $g = -2du(dv + Hdu + W_i dy^i) + \delta_{ij} dy^i dy^j$. ($g_{ij} = \delta_{ij}$ *independent of u*)

Transverse operators for $\mathcal{M}$: dot derivative

For $T \in \Gamma(T^r_s\mathcal{M})$:

$$\dot{T} = \overline{\mathcal{L}_{\partial_u}\mathring{T}} \in \Gamma(T^r_s\mathcal{M})$$

That is, in the base $\{\partial_i\}$:

$$\dot{T}^{i_1\ldots i_r}_{j_1\ldots j_s} = \partial_u (T^{i_1\ldots i_r}_{j_1\ldots j_s})$$

Transverse operators for $\mathcal{M}$: D_0 derivative

Recall $E_0 = \partial_u - H\partial_v$

$$\begin{array}{rccc} D_0: & \Gamma(T^r_s\mathcal{M}) & \longrightarrow & \Gamma(T^r_s\mathcal{M}) \\ & T & \to & D_0 T = \overline{(\nabla_{E_0} \mathring{T})} \end{array}$$

Transverse operators for $\mathcal{M}$: D_0 derivative

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Properties:

1. Algebraic properties of a tensor derivation
2. $D_0\overline{g} = 0$

Lemma

Each vector field on a leave of $\mathcal{M}$ can be extended to a unique $K(= -\partial_v)$-invariant D_0-parallel vector field in $\mathfrak{X}(\mathcal{M})$.

Reducibility in $\mathcal{M}$

$T \in \Gamma(T^k_s\mathcal{M})$ is reducible if, there are foliations $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$ s.t., in a natural sense:

$$T\mathcal{M} = T\mathcal{M}^{(1)} \oplus T\mathcal{M}^{(2)} \qquad T = T^{(1)} \oplus T^{(2)}$$

i.e. there exists a Brinkmann chart $\{u, v, x^i\}$ and a partition of the indexes $I_1 = \{2, \ldots, d+1\}$, $I_2 = \{d+2, \ldots, n-1\}$ s.t.

$$T_{aa'} = 0 \quad \text{y} \quad \partial_{a'} T_{ab} = 0,$$

where a, b belong to some I_m and a', b' to the other one.

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$$T_{aa'} = 0 \quad \text{y} \quad \partial_{a'} T_{ab} = 0,$$

where a, b belong to some I_m and a', b' to the other one.
In particular, when $\overline{g} \in \Gamma(T_2\mathcal{M})$ is reducible the sum is orthogonal and we write $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$,

$$g = -2du(dv + Hdu + \mathring{W}) + \mathring{g}^{(1)} \oplus \mathring{g}^{(2)}$$

Extended Eisenhart theorem

Theorem

Let (M, g) be a Brinkmann space and $\{u, v, x^i\}$ a Brinkmann chart. If there exist a symmetric $\overline{L} \in \Gamma(T^0_2\mathcal{M})$, $\overline{L} \neq c\overline{g}$, which is v-invariant, $\overline{\nabla}$-parallel and D_0-parallel.

Extended Eisenhart theorem

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Then there exists a Brinkmann chart $\{u, v, y^i\}$ in the Brinkmann decomposition $\{u, v\}$ such that:

1. *$\overline{g}$ is reducible: $\overline{g} = \overline{g}^{(1)} \oplus \ldots \oplus \overline{g}^{(s)}$, $s \geq 2$ (u-dependent)*
2. *$\overline{L} = \sum_{m=1}^{s} \lambda_m \, \overline{g}^{(m)}$ for some $\lambda_m \in \mathbb{R}$ (u-independent, $\dot{\lambda}_m = 0$).*

Local version of the theorem

Aim:

Theorem

A properly 2nd-symmetric Brinkmann space is locally isometric to a product of:

- *a proper 2nd-order Cahen-Wallach space* $(\mathbb{R}^{d+2}, g_A)$,
  $g_A = -2du\left(dv + (\mathbf{a_{ij}u} + \mathbf{b_{ij}})x^i x^j du\right) + \delta_{ij}dx^i dx^j$
  with some $a_{ij} \neq 0$, *and*
- *symmetric Riemannian space* (N, g_N).

Step 1: define appropriate elements on $\mathcal{M}$

Express the non-trivial parts of $R, \nabla R$ in terms of tensors on $\mathcal{M}$

- Tensors for R: $A \in T_2\mathcal{M}$, $B \in T_3\mathcal{M}$, $\overline{R} \in T_3^1\mathcal{M}$
  - $A(X,Y) = \theta^1(R(E_0, \mathring{Y})\mathring{X})$, i.e. $A_{ij} = R^1{}_{i0j}$
  - $B(X,Y,Z) = \theta^1(R(\mathring{Y}, \mathring{Z})\mathring{X})$, i.e., $B_{ijk} = R^1{}_{ijk}$
  - $\overline{R}(X,Y)Z = \overline{R}(\mathring{X}, \mathring{Y})\mathring{Z}$, i.e., $\overline{R}^i{}_{jkl} = R^i{}_{jkl}$
- Tensors for ∇R: $\widetilde{A} \in T_2\mathcal{M}$, $\widehat{A}, \widetilde{B} \in T_3\mathcal{M}$, $\widehat{B}, \widetilde{R} \in T_3^1\mathcal{M}$

$$\widetilde{A}(X,Y) = \theta^1\left((\nabla_{E_0} R)(E_0, \mathring{Y})\mathring{X}\right), \qquad \widehat{A}(X,Y,Z) = \theta^1\left((\nabla_{\mathring{X}} R)(E_0, \mathring{Z})\mathring{Y}\right),$$
$$\widetilde{B}(X,Y,Z) = \theta^1\left((\nabla_{E_0} R)(\mathring{Y}, \mathring{Z})\mathring{X}\right), \qquad \widehat{B}(X,Y,Z,V) = \theta^1\left((\nabla_{\mathring{X}} R)(\mathring{Z}, \mathring{V})\mathring{Y}\right),$$
$$\widetilde{R}(X,Y)Z = \overline{\nabla_{E_0} R(\mathring{X}, \mathring{Y})\mathring{Z}}.$$

$$\widetilde{A}_{ij} = \nabla_0 R^1{}_{i0j}; \ \widehat{A}_{sij} = \nabla_s R^1{}_{i0j}$$
$$\widetilde{B}_{ijk} = \nabla_0 R^1{}_{ijk}; \ \widehat{B}_{sijk} = \nabla_s R^1{}_{ijk}; \ \widetilde{R}^i{}_{jkl} = \nabla_0 R^i{}_{jkl}$$

Step 2: simplification of chart-dependent elements

Proposition

For any 2nd-symmetric Brinkmann decomposition $\{u, v\}$:

(a) *All the (chart-dependent) elements for ∇R vanish but $\tilde{A}$, i.e.* $\widehat{B} = \widetilde{R} = \widehat{A} = \widetilde{B} = 0$.

(b) *$\widetilde{A}$ is independent of the chosen chart*

(c) *The equations of 2nd symmetry reduce to:*

$$\overline{\nabla}\widetilde{A} = 0, \qquad D_0\widetilde{A} = 0$$
$$\overline{\nabla}\,\overline{R} = 0, \qquad D_0\overline{R} = 0$$

with $\widehat{B} = 0$, $\widetilde{B} = 0$, $\widehat{A} = 0$.

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- Use the conditions of integrability of 2nd symmetry equations

$$(\overline{\nabla}_k D_0 - D_0 \overline{\nabla}_k) F^i{}_j = (H_{,k})(\partial_v F^i{}_j) + F^i{}_m B_{kj}{}^m - F^m{}_j B_{km}{}^i - t^m{}_k \overline{\nabla}_m F^i{}_j$$

$$(\overline{\nabla}_n \overline{\nabla}_m - \overline{\nabla}_m \overline{\nabla}_n) T^{i_1 \dots i_k}_{j_1 \dots j_s} = \sum_{b=1}^{s} \overline{R}^l{}_{j_b nm} T^{i_1 \dots i_k}_{j_1 \dots j_{b-1} l j_{b+1} \dots j_s} - \sum_{a=1}^{k} \overline{R}^{i_a}{}_{lnm} T^{i_1 \dots i_{a-1} l i_{a+1} \dots i_k}_{j_1 \dots j_s}$$

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- Use the equations derived from 2nd Bianchi identity
$\nabla_{[\alpha} R_{\beta\lambda]\nu\mu} = 0 \Longrightarrow \widetilde{R}_{ijkl} = -2\widehat{B}_{[ij]kl}, \quad \widetilde{B}_{kij} = 2\widehat{A}_{[ij]k}.$

Technical point: algebraic criteria for the vanishing of tensor fields are also introduced, as:

In an Euclidean vector space, T_{ijk} vanishes if
$T_{i[jk]} = T_{ijk}, \; T_{ijk} + T_{jki} + T_{kij} = 0$ and $T_{(ij)}{}^r T_{rnm} = 0$

Step 2: simplification of chart-dependent elements

Remark

- $\nabla R \neq 0$ iff $\widetilde{A} \neq 0$.
- The scalar curvature S (not only of $\mathcal{M}$ but also) of M is constant.

Step 3: Reducibility of $\tilde{A}$ and $\overline{\text{Ric}}$

From the equations of 2nd-symmetry:

$$\overline{\nabla}\widetilde{A} = 0, \qquad D_0\widetilde{A} = 0$$
$$\overline{\nabla}\ \overline{R} = 0, \qquad D_0\overline{R} = 0$$

$\widetilde{A}$ and $\overline{\text{Ric}}$ (and also $\overline{g}$) are D_0- $\overline{\nabla}$-invariant so that Extended Eisenhart theorem applies and:

Step 3: Reducibility of $\tilde{A}$ and $\overline{\mathrm{Ric}}$

- $\mathcal{M} = \mathcal{M}^{(1)} \times \mathcal{M}^{(2)}$ with $\mathcal{M}^{(1)}$ flat and $\mathcal{M}^{(2)}$ locally symmetric non Ricci-flat.

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For any Brinkmann decomposition $\{u, v\}$:

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Remark

For any Brinkmann decomposition $\{u, v\}$:

- $\widetilde{A}$, $\overline{\mathrm{Ric}}$ and $\overline{g}$ are simultaneously reducible
- The non-trivial part of $\widetilde{A}$ lies in $\mathcal{M}^{(1)}$ and the non-trivial one of Ricci on $\mathcal{M}^{(2)}$

Step 4: reduction to two independent Lorentzian problems

From previous result in a Brinkmann chart:

$$g = -2du(dv + H du + \mathring{W}) + \mathring{g}^{(1)} \oplus \mathring{g}^{(2)}$$

and one can check that H, W are also simultaneously reducible, so that in some new chart:

$$g = -2du(dv + (H^{(1)} + H^{(2)})du + \mathring{W}^{(1)} + \mathring{W}^{(2)}) + \mathring{g}^{(1)} \oplus \mathring{g}^{(2)}$$

Step 4: reduction to two independent Lorentzian problems

Now, define two lower dimensional Lorentzian spaces $M^{[m]} = \mathbb{R}^2 \times \overline{M}^{(m)}$, $m = 1, 2$:

$$g^{[m]} = -2du(dv + H^{(m)}du + W^{(m)}) + \mathring{g}^{(m)}.$$

Remark

- These two Lorentzian spaces are 2nd symmetric as so was the original one.
- So, the problem is reduced to the 2nd symmetry of two simple spaces

Step 4: reduction to two independent Lorentzian problems

- $(M^{[2]}, g^{[2]})$ 2nd symmetric with $\widetilde{A}^{[2]} = 0$:
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- $(M^{[1]}, g^{[1]})$ 2nd-symmetric with flat $\mathcal{M}^{[1]}$ ($\widetilde{A}^{[1]} \neq 0$): 2nd-symmetric plane wave: directly computable obtaining a generalized Cahen-Wallach of orden 2 :

$$g_A = -2du\left(dv + (a_{ij}u + b_{ij})x^i x^j du\right) + \delta_{ij}dx^i dx^j$$

□

Further open questions

Modest:

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Senovilla's:

1. Solve all the linear conditions for curvature:

$$\nabla^r R + t_1 \otimes \nabla^{r-1} R + t_2 \otimes \nabla^{r-2} R + \cdots + t_{r-1} \otimes \nabla R + t_r \otimes R = 0$$

for some m- covariant tensors t_m.