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Homogeneity of gradient Ricci soliton metrics

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Trabajo conjunto con

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Lorentzian homogeneous spaces

Notation

Ricci solitons:
(M, g) Lorentzian manifold and $X \in \mathfrak{X}(M)$ vector field satisfying

$$\boxed{\mathcal{L}_X g + Ric = \lambda g}$$

where $\mathcal{L}$ denotes the Lie derivative, Ric is the Ricci tensor and $\lambda \in \mathbb{R}$.

Gradient Ricci solitons:
(M, g) Lorentzian manifold and $f : M \to \mathbb{R}$ potential function

$$\boxed{Hess_f + Ric = \lambda g}$$

where $Hess_f$ denotes the Hessian.

- A (gradient) Ricci soliton is *expanding*, *steady* or *shrinking* depending on $\lambda < 0$, $\lambda = 0$, $\lambda > 0$.
- Lorentzian Ricci solitons may have different causal character (timelike, null, spacelike).

$\rightarrow$ Nontrivial gradient Ricci solitons may exist with $\|\nabla f\| = 0$.

(Batat, Brozos-Vazquez, GR, Gavino-Fernandez 2011)

$\rightarrow$ Cahen-Wallach symmetric spaces are isotropic steady gradient Ricci solitons

Ricci solitons: motivation

- Motivation comes from the study of the Ricci flow

$$t \to g(t), \qquad \frac{d}{dt} g(t) = -2Ric(t)$$

(Hamilton 1982)

For any prescribed metric $g(0)$ on a compact manifold M, there exists a unique solution on a maximal interval $[0, T)$, where $0 < T \leq \infty$.

→ If the initial metric g_0 is Einstein with $Ric_0 = \lambda g_0$, then

$$g(t) = (1 - 2\lambda t) g_0$$

→ For any Ricci soliton (M, g, X, λ) the solution is given by

$$g(t) = (1 - 2\lambda t) \varphi_t^* g,$$

where φ_t is the one-parameter group of diffeomorphisms generated by X.

→ Lorentzian Ricci solitons are self-similar solutions of the Ricci flow

$$\boxed{\mathcal{L}_X g + Ric = \lambda g}$$

Plan of the talk

AIM:
To investigate homogeneous gradient Ricci solitons, both for Riemannian and Lorentzian metrics.

1. **Riemannian gradient Ricci solitons with constant scalar curvature: curvature homogeneity**
2. **Homogeneous Lorentzian gradient Ricci solitons**
3. **Three-dimensional homogeneous Walker metrics**
4. **Three-dimensional homogeneous Lorentz gradient Ricci solitons**

Constant scalar curvature – Riemannian case

Theorem (Petersen, Wylie; 2009)
If (M,g) has constant scalar curvature, then

(1) If $\lambda = 0$, then $Sc = 0$ and (M,g) is Ricci flat.
(2) If $\lambda > 0$, then $Sc \in [0, n\,\lambda]$.
(3) If $\lambda < 0$, then $Sc \in [n\,\lambda, 0]$.

Moreover, the extreme values are achieved only in the Einstein case.

Theorem (Fernandez-Lopez, G-R; 2013)
Let (M,g) be a non-steady complete gradient Ricci soliton with constant scalar curvature. Then the scalar curvature $Sc = k\lambda$, where $k = 1, \ldots, n-1$. Moreover, the extreme values $k = 1$ and $k = n-1$ are achieved only in the Einstein and the rigid cases, respectively.

$\rightarrow$ Scalar curvature depends on the dimension of the level set submanifolds of the potential function.

Constant scalar curvature – Riemannian case

A gradient Ricci soliton is said to be *rigid* if it is isometric to a quotient of $N \times \mathbb{R}^k$ where N is Einstein and $f = \frac{\lambda}{2}|x|^2$ on the Euclidean factor.

Theorem (Petersen, Wylie; 2009)
Let (M, g) be a complete shrinking or expanding gradient Ricci soliton.
If any of the following conditions holds, then the Ricci soliton is rigid

(1) Sc is constant and the radial curvature $K(\,\cdot\,, \nabla f)$ is nonnegative or nonpositive.
(2) Sc is constant and $0 \leq Ric \leq \lambda g$ or $\lambda g \leq Ric \leq 0$.
(3) $Ric \geq 0$ or $Ric \leq 0$ and the radial curvature $K(\,\cdot\,, \nabla f)$ vanishes.

Theorem (Fernandez-Lopez, G-R; 2011; Munteanu-Sesum; 2011)
A complete gradient shrinking Ricci soliton is rigid if and only if its Weyl tensor is harmonic.

$\rightarrow$ Complete locally conformally flat gradient shrinking Ricci solitons are rigid.

Constant scalar curvature – Riemannian case

Theorem (Petersen, Wylie; 2009)
Any homogeneous gradient Ricci soliton is rigid.

(M, g) is said to be k-*curvature homogeneous* if for each pair of points $p, q \in M$ there is a linear isometry $\Phi_{pq} : T_pM \to T_qM$ such that

$$\Phi_{pq}^* R(q) = R(p), \qquad \Phi_{pq}^* \nabla R(q) = \nabla R(p), \qquad \dots \qquad \Phi_{pq}^* \nabla^k R(q) = \nabla^k R(p).$$

Any locally homogeneous manifold is k-curvature homogeneous for all k and the converse holds true if k is sufficiently large.

Theorem (Fernandez-Lopez, G-R; 2013)
Let (M, g) be a 0-curvature homogeneous complete gradient Ricci soliton. Then it is rigid.

→ Complete gradient Ricci solitons with constant scalar curvatur are rigid in dimension ≤ 4.

Theorem (Fernandez-Lopez, G-R; 2013)
Let (M, g) be a non-steady complete gradient Ricci soliton with constant scalar curvature. Then (M, g) is rigid, provided that the Ricci operator has at most four distinct eigenvalues.

Homogeneous Lorentzian gradient Ricci solitons

- Let (M, g, f) be a gradient Ricci soliton with constant scalar curvature. If X is a Killing vector field, then $\operatorname{grad} X(f)$ is a parallel vector field. Moreover, if $\lambda \neq 0$, then $\operatorname{grad} X(f) = 0$ if and only if $X(f) = 0$.

$$\begin{aligned} 0 = \nabla f(\cdot\,) = \nabla f(X(f)) &= \nabla_{\nabla f} g(\nabla f, X) = g(\nabla_{\nabla f} \nabla f, X) + g(\nabla f, \nabla_{\nabla f} X) \\ &= \mathrm{Hes}_f(\nabla f, X) + \tfrac{1}{2}(\mathcal{L}_X g)(\nabla f, \nabla f) \\ &= -\operatorname{Ric}(\nabla f, X) + \lambda\, g(\nabla f, X) = \lambda \cdot\,, \end{aligned}$$

$\rightarrow$ If $\operatorname{grad} X(f)$ is timelike/spacelike, then (M, g) splits a 1-dimensional factor.

$\rightarrow$ If $\operatorname{grad} X(f)$ is null, then (M, g) is a Walker manifold.

Theorem

Let (M, g) be a homogeneous Lorentzian manifold. If (M, g, f) is a non-steady gradient Ricci soliton, then it splits as a product $M = N \times \mathbb{R}^k$ for some $k \geq 0$, where either

(1) (N, g_N) is a Lorentzian Einstein manifold and the soliton is rigid, or

(2) (N, g_N) is a Lorentzian Walker manifold admitting a parallel null vector field.

Homogeneous Lorentzian gradient Ricci solitons

- Some consequences of the gradient Ricci soliton equation

$$\mathrm{Hes}_f + \mathrm{Ric} = \lambda g$$

→ $\nabla \,\mathrm{Sc} = 2\hat{\mathrm{Ric}}(\nabla f)$, and hence $\mathrm{Ric}(\nabla f, \cdot\,) = 0$ if Sc is constant.

In the steady case $(\lambda = 0)$ $\mathrm{hes}_f(\nabla f) = 0$, and hence ∇f is a geodesic vector field if Sc is constant.

→ Bochner identity $\frac{1}{2}\Delta\, g(\nabla f, \nabla f) = \|\,\mathrm{hes}_f\,\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$ shows that $\lambda((n+2)\lambda - \tau) = \|\,\mathrm{hes}_f\,\|^2$ if Sc is constant.

In the steady case $(\lambda = 0)$ hes_f, and hence $\hat{\mathrm{Ric}}$, is isotropic if Sc is constant.

→ $\mathrm{Sc} + \|\nabla f\|^2 - 2\lambda f = \mathrm{cont.}$, and thus in the steady case $(\lambda = 0)$ f is a solution of the Eikonal equation $\|\nabla f\|^2 = \mu$, if Sc is constant.

We will consider separately the cases $\mu < 0$, $\mu = 0$ and $\mu > 0$.

Homogeneous Lorentzian gradient Ricci solitons

Theorem
Let (M, g, f) be a homogeneous steady gradient Ricci soliton such that $\|\nabla f\|^2 = \mu < 0$.
Then (M, g) splits isometrically as a product $(\mathbb{R} \times N, -dt^2 + g_N)$, where (N, g_N) is a flat Riemannian manifold and f is the projection on $\mathbb{R}$.

$\rightarrow$ Bochner identity

$$\frac{1}{2}\,\Delta\, g(\nabla f, \nabla f) = \|\operatorname{hes}_f\|^2 + \operatorname{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f).$$

Homogeneous Lorentzian gradient Ricci solitons

Theorem

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→ Bochner identity (and diagonalizability of $\mathrm{hes_f}$)

$$-g(\nabla \Delta f, \nabla f) = \| \, \mathrm{hes}_f \, \|^2 \geq \frac{1}{n}(\mathrm{trace}\,\mathrm{hes_f})^2 = \frac{1}{n}(\Delta f)^2.$$

→ Completeness of ∇f shows that $\Delta f = 0$ and hence $\mathrm{hes_f} = 0$

→ ∇f is a parallel timelike vector field. Hence $M = N \times \mathbb{R}$, with (N, g_N) Riemannian

→ (N, g_N) is Ricci flat homogeneous, and hence flat.

Homogeneous Lorentzian gradient Ricci solitons

- Let (M, g, f) be a homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 = 0$.
  Then the Ricci operator is two-step or three-step nilpotent. Moreover, if the Ricci operator is two-step nilpotent then there is a null parallel vector field on (M, g).

(Calvaruso, 2007)
A three-dimensional (complete and simply connected) homogeneous space is either symmetric or a Lie group.

- Let (M, g, f) be a three-dimensional homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 = 0$.
  Then (M, g) admits a null parallel vector field.

$\rightarrow$ Classification of three-dimensional homogeneous Walker manifolds.

Homogeneous Lorentzian gradient Ricci solitons

- Let (M,g,f) be a three-dimensional homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 > 0$.
  Then (M,g) admits a null parallel vector field.

(Calviño-Louzao, G-R, Vazquez-Abal, Vazquez-Lorenzo 2012)
A three-dimensional Lorentzian homogeneous manifold is Walker if and only if $\hat{\mathrm{Ric}}^2 = 0$.

→ If $\dim(\ker(\mathrm{Ric})) = 2$, then $\hat{\mathrm{Ric}}$ is 2-step nilpotent or diagonalizable.

If $\hat{\mathrm{Ric}}$ is diagonalizable $\hat{\mathrm{Ric}} = \mathrm{diag}[0,0,\mathrm{Sc}]$ and isotropic, then $\mathrm{Sc} = 0$ and $\mathrm{Ric} = 0$

→ If $\dim(\ker(\mathrm{Ric})) = 1$, then $X = \nabla f$ is left-invariant and $\hat{\mathrm{Ric}}^2 = 0$.

(Brozos-Vazquez, Calvaruso,G-R, Gavino-Fernandez 2012)
A three-dimensional Lorentzian Lie group admits a left-invariant Ricci soliton if and only if the Ricci operator has a single eigenvalue.

→ Classification of three-dimensional homogeneous Walker manifolds.

Three-dimensional gradient Ricci soliton Walker metrics

Walker manifold: a Lorentzian manifold admitting a parallel null vector field. There exist local coordinates $(x, y, \tilde{x})$ where the metric is given by

$$g(\partial_x, \partial_x) = -2\phi(x, y), \qquad g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_y) = 1\,.$$

$\rightarrow$ Three-dimensional Walker manifolds are pp-waves.

Let $\mathcal{M}_\phi$ be a three-dimensional non-trivial Walker gradient Ricci soliton. Then

(R.1) $\phi(x, y) = \frac{1}{b^2} e^{by}\alpha(x) + y\beta(x) + \gamma(x)$,
and the potential function of the soliton is given by $f(x, y, \tilde{x}) = \frac{b}{2}y + \hat{f}(x)$ where $\hat{f}_{xx} = \frac{b}{2}\beta(x)$.

(R.2) $\phi(x, y) = y^2\alpha(x) + y\beta(x) + \gamma(x)$,
and the potential function of the soliton is given by $f(x, y, \tilde{x}) = \hat{f}(x)$ where $\hat{f}_{xx} = -\alpha(y)$.

Moreover, in both cases the Ricci soliton is steady.

(R.1) The soliton vector field $\operatorname{grad} f = \frac{b}{2}\partial_y + \hat{f}_x(x)\partial_{\tilde{x}}$ is spacelike.

(R.2) The soliton vector field $\operatorname{grad} f = \hat{f}_x(x)\partial_{\tilde{x}}$ is lightlike.

Three-dimensional locally homogeneous Walker metrics

Let $\mathcal{M}_\phi$ be a three-dimensional Walker manifold.

Let $\mathcal{N}_b$ be defined by $\phi(x,y) = b^{-2}e^{by}$ for $b \neq 0$.
Let $\mathcal{P}_c$ be defined by $\phi(x,y) = \frac{1}{2}y^2\alpha(x)$ where $\alpha_x = c\alpha^{3/2}$, $\alpha > 0$.
Let $\mathcal{CW}_\varepsilon$ be defined by $\phi(x,y) = \varepsilon y^2$.

$\rightarrow$ $\mathcal{N}_b$ and $\mathcal{CW}_\varepsilon$ are geodesically complete.
$\mathcal{P}_c$ are geodesically incomplete.

Theorem (GR, Gilkey, Nikčević; 2012)

(1) The manifolds $\mathcal{CW}_\varepsilon$ are locally symmetric.
(2) The manifolds $\mathcal{N}_b$ and $\mathcal{P}_c$ are locally homogeneous.
(3) The manifolds $\{\mathcal{CW}_\varepsilon, \mathcal{N}_b, \mathcal{P}_c\}$ have non-isomorphic 1-curvature model.

Moreover, any locally homogeneous three-dimensional Walker metric is locally isometric to one of the above.

$\rightarrow$ $\mathcal{CW}_\varepsilon$ and $\mathcal{P}_c$ are plane waves.
$\mathcal{N}_b$ are not plane waves.

3D Homogeneous Lorentzian gradient Ricci solitons

Theorem

Let (M, g) be a three-dimensional homogeneous Lorentz gradient Ricci soliton. Then one of the following holds

(1) The soliton is trivial, i.e., $f = \text{const.}$ and (M, g) is a space of constant sectional curvature, where $\lambda = \frac{\text{Sc}}{3}$, or

(2) the soliton is rigid, i.e., $M = N(c) \times \mathbb{R}$, where (N, g_N) is a surface of constant curvature and $f(\,\cdot\,) = \frac{\lambda}{2}\pi_{\mathbb{R}}(\,\cdot\,)^2$, where $\lambda \neq 0$, or

(3) the gradient Ricci soliton is steady and (M, g) is a Walker manifold as in the following:

(3.i) (M, g) is locally isometric to $\mathcal{CW}_\varepsilon$, and the potential function of the soliton is given by $f(x, y, \tilde{x}) = -\frac{\varepsilon}{2}x^2 + \mu x + \nu$.

(3.ii) (M, g) is locally isometric to $\mathcal{P}_c$, and the potential function of the soliton is given by $f(x, y, \tilde{x}) = \hat{f}(x)$ where $\hat{f}_{xx} = -\frac{1}{2}\alpha(y)$.

(3.iii) (M, g) is locally isometric to $\mathcal{N}_b$, and the potential function of the soliton is given by $f(x, y, \tilde{x}) = \frac{b}{2}y + \mu x + \nu$.

$\rightarrow$ GRS are geodesic vector field and thus complete in $\mathcal{CW}_\varepsilon$ and $\mathcal{N}_b$.

$\rightarrow$ GRS in $\mathcal{CW}_\varepsilon$ and $\mathcal{P}_c$ are isotropic, while those in $\mathcal{N}_b$ are spacelike.

$\rightarrow$ $\mathcal{CW}_\varepsilon$ and $\mathcal{P}_c$ admit expanding, steady and shrinking RS, while $\mathcal{N}_b$ admits only steady RS.