An Introduction to Topological Groups

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Workshop of Young Researchers

1 Definition and first examples

Definition

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$$G \times G \longrightarrow G,\ (x, y) \mapsto xy$$

and inversion

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4. topological vector spaces

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3. $\mathbb{R}^n \times \mathbb{Z}^m$

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Let H be a normal subgroup of a topological group (G,τ) and let $\pi : G \to G/H$ denote the canonical projection. We endow G/H with the **quotient topology** $\{\widetilde{O} \subseteq G/H : \ \pi^{-1}(\widetilde{O}) \in \tau\}$.

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Every open subgroup H of a topological group G is closed. Indeed: $G \setminus H = \bigcup_{x \in G \setminus H} xH$ is open.

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Corollary

A group topology τ *is determined by a neighborhood basis* $\mathscr{U}(e)$ *at the neutral element* e.

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Example

Let $(\mathbb{R}, \tau)$ be the Sorgenfrey-line. Then addition $(\mathbb{R}, \tau) \times (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ is continuous (not only all the translations!). However, inversion is discontinuous.

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yU is open, since $x \to yx$ is a homeomorphism and $BU = \bigcup_{b \in B} bU$ is open by the same reason.

Theorem

Let (G, τ) be a topological group with neighborhood basis $\mathscr{U}$ at e. For $U \in \mathscr{U}$ we define

$$\begin{array}{ll} U_L := \{(x,y) \in G \times G : \ x^{-1}y \in U\} & \text{and} \\ U_R := \{(x,y) \in G \times G : \ yx^{-1} \in U\} & . \end{array}$$

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Corollary

Every T_1 topological group is $T_{3\frac{1}{2}}$.

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In this case, there exists a left (right-) invariant metric which induced the topology. A metric d is called **left invariant** *if $d(x,y) = d(ax,ay)$ for all $a \in G$.*

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Assume now, there exists a left and right invariant metric d on $\mathrm{Sl}(2,\mathbb{R})$. Then

$$d(A_nB_n, E) \overset{li}{=} d(B_n, A_n^{-1}) \overset{ri}{=} d(B_nA_n, E),$$

which is impossible.

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Since connected components are always closed, the assertion follows. □

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Corollary

Let G be a totally disconnected compact group. Then G can be embedded into a product of finite groups.

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$\ker(\varphi) = \bigcap_{i\in I} N_i = \{e\}$.

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$$\varphi : G \longrightarrow \prod_{i\in I} G/N_i,\ x \longmapsto xN_i.$$

Since $G \to G/N_i,\ x \mapsto xN_i$ is continuous for every i, φ is continuous.
$\ker(\varphi) = \bigcap_{i\in I} N_i = \{e\}$.

$$\varphi(N_j) = \varphi(G) \cap \{e_{G/N_j}\} \times \prod_{i\in I\setminus\{j\}} G/N_i,$$

which shows that φ is an embedding.

☐

3 An introduction to Pontryagin duality theory

Definition

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Remark

Obviously, $G^\wedge$ is an abelian group (under pointwise addition). The commutator subgroup of G is contained in the kernel of every continuous character.

Examples

The following mappings are isomorphisms:

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Theorem (Gel’fand, Raikov)

Let G be a locally compact abelian (LCA) group. For every $x \in G \setminus \{e\}$ there exists a continuous character $\chi \in G^\wedge$ such that $\chi(x) \neq 1$.

Since quotients of LCA groups w.r.t. closed subgroups are again LCA, we have:

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Corollary

A subgroup H of a LCA group G is dense iff $H^{\perp} := \{\chi \in G^{\wedge} : \ \chi(H) = \{1\}\} \ = \ \{1\}$.

Example (Kronecker)

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be elements of $\mathbb{R}^n$ and put

$$H = \mathbb{Z}^n + \langle \{\mathbf{a}\} \rangle.$$

Then H is dense in $\mathbb{R}^n$ iff $(1, a_1, \ldots, a_n)$ are linearly independent over $\mathbb{Q}$.

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We want to endow the character group with the compact–open topology:

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Proposition

Let (G,τ) *be a Hausdorff group. The sets of the form*

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Proposition

$$\textit{If } G \textit{ is } \begin{cases} \textit{compact abelian} \\ \textit{locally compact abelian} \\ \textit{discrete abelian} \end{cases} \textit{ then } G^\wedge \textit{ is}$$

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Question

Is a group G reflexive if G and $G^{\wedge\wedge}$ are topologically isomorphic?

Theorem (Pontryagin, van Kampen)

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Let (G, τ) be a LCA group. Then G is topologically isomorphic to $\mathbb{R}^n \times H$ where H is an abelian group which has a compact and open subgroup H and $n \in \mathbb{N}_0$.
If G is a connected LCA group then G is topologically isomorphic to $\mathbb{R}^n \times K$ where $n \in \mathbb{N}_0$ and K is a compact and connected group.

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Example

The character group $\mathbb{Q}^\wedge$ is a connected (and divisible) compact group.

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Example

The group $\mathbb{Q}$ is not a L–group, hence $\mathbb{Q}^\wedge$ is not locally connected.

Definition (Whitehead)

A torsion free abelian group G is called a W**–group** iff every homomorphism $\chi : G \to \mathbb{T}$ can be lifted over $\mathbb{R}$, i.e. there exists a homomorphism $\varphi : G \to \mathbb{R}$ such that $\exp(2\pi i\varphi) = \chi$.

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The discrete group $\mathbb{Z}^{\mathbb{N}}$ is an L–group, but not a W–group.
Every free abelian group is a W–group.
In ZFC it is undecidable whether every W–group is free.

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Remark

Every W–group is an S–group. $\mathbb{Z}^{\mathbb{N}}$ is an S–group.

$$\begin{tikzcd}
 & \text{free abelian group} \arrow[r, leftrightarrow] \arrow[d] & \mathbb{T}^{I} \\
 & \text{W–group} \arrow[r, leftrightarrow] \arrow[d] & \text{loc. pathwise connected} \\
\mathbb{Z}^{\mathbb{N}} \arrow[r] & \text{S–group} \arrow[r, leftrightarrow] \arrow[d] & \text{projection property} \\
\mathbb{Q} \arrow[r, "\backslash" marking] & \text{L–group} \arrow[r, leftrightarrow] & \text{loc. connected}
\end{tikzcd}$$

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1. $A(X)$ is algebraically the free abelian topological group generated by the set X.
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2. There exists a (topological) embedding $\eta : X \to A(X)$ such that $\eta(X)$ is a basis for the free abelian topological group and $\eta(X)$ is closed in $(A(X), \tau)$.
3. For every continuous mapping $f : X \to H$ where H is an abelian topological group, the unique homomorphism $f' : A(X) \to H$ which satisfies $f' \circ \eta = f$ is continuous.

Theorem (Markov 1945)

For every completely regular space X, the free abelian topological group $(A(X), \tau)$ exists and is unique up to topological isomorphism. $(A(X), \tau)$ is a Hausdorff space.

Proposition (Pestov)

$A(X)^\wedge \to \mathscr{C}(X,\mathbb{T}),\ \chi \mapsto \chi\circ\eta$ is a continuous isomorphism when $\mathscr{C}(X,\mathbb{T})$ is endowed with the compact–open topology.

Proposition (Pestov)

$A(X)^\wedge \to \mathscr{C}(X,\mathbb{T})$, $\chi \mapsto \chi \circ \eta$ is a continuous isomorphism when $\mathscr{C}(X,\mathbb{T})$ is endowed with the compact–open topology. If G is e.g. paracompact, then the above mapping is also open.

Theorem (Pestov, L.A.,Galindo, Hernandez)

Let X be compact. Then $A(X)$ is reflexive iff X is punctiform, i.e. every connected compact subset of X is a singleton.

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Question

Ist the character group of every abelian metrizable group reflexive?

Thank you for your attention!