Normal Bundle of Grassmannian $\mathbb{G}(k, n)$

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Definition

- $\mathbb{G}(k,n)$=*{k-dimensional subspaces of* $\mathbb{P}^n$*}*
- *Plücker embedding:* $\mathbb{G} = \mathbb{G}(k,n) \subseteq \mathbb{P} = \mathbb{P}(\bigwedge^{k+1} V)$

$$\Lambda = \begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix} \longmapsto \text{minors of order } k+1$$

- *Goal: to give a resolution for the conormal bundle* $N^*_{\mathbb{G}|\mathbb{P}}$ *of the Grassmannian* $\mathbb{G}(k,n)$ *.*

Universal Bundles

- *V={ linear forms on $\mathbb{P}^n$ }*
- $$S^* = \{(\Lambda, H) \in \mathbb{G} \times V | H \text{ vanishes in } \Lambda\} \subseteq \mathbb{G} \times V$$
  $$\downarrow q$$
  $$\mathbb{G}$$
- S^* *universal vector bundle of rank* $n - k$.
- *Universal exact sequence (of sheaves and vector bundles)*

$$0 \longrightarrow \mathcal{S}^* \longrightarrow V \otimes \vartheta_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Key diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
N^*_{\mathbb{G}|\mathbb{P}} & \longrightarrow & \Omega_{\mathbb{P}|\mathbb{G}} & \longrightarrow & \Omega_{\mathbb{G}} \cong S^* \otimes Q^* & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^* \otimes \bigwedge^k V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes \bigwedge^k V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^2 S^* \otimes \bigwedge^{k-1} V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes S^* \otimes \bigwedge^{k-1} V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & M_2 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^3 S^* \otimes \bigwedge^{k-2} V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes S^2 S^* \otimes \bigwedge^{k-2} V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & M_3 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^{k-1} S^* \otimes \bigwedge^2 V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes S^{k-2} S^* \otimes \bigwedge^2 V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & M_{k-1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^k S^* \otimes V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes S^{k-1} S^* \otimes V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & M_k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & S^{k+1} S^* \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes S^k S^* \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & M_{k+1} \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where $M_i = \bigwedge^{k+1-i} V \otimes F_i(S^*)$, *with* $F_i(S^*) = (S^* \otimes S^{i-1} S^* / S^i S^*) \otimes \vartheta_{\mathbb{G}}(-1)$

Key diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
N^*_{\mathbb{G}|\mathbb{P}} & \longrightarrow & \Omega_{\mathbb{P}|\mathbb{G}} & \longrightarrow & S^* \otimes Q^* \cong \Omega_{\mathbb{G}} & \longrightarrow & 0 \\
\uparrow & & \uparrow {\scriptstyle f'_{k+1}} & & \uparrow {\scriptstyle f_{k+1}} & & \uparrow \\
0 & \longrightarrow & S^* \otimes \bigwedge^k V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & S^* \otimes \bigwedge^k V \otimes \vartheta_{\mathbb{G}}(-1) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & kerf'_{k+1} & \longrightarrow & kerf_{k+1} & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

Snake lemma: $0 \longrightarrow kerf'_{k+1} \longrightarrow kerf_{k+1} \longrightarrow N^*_{\mathbb{G}|\mathbb{P}} \longrightarrow 0$

Therefore $0 \longrightarrow M_{k+1} \longrightarrow M_k \longrightarrow \cdots \longrightarrow M_3 \longrightarrow M_2 \longrightarrow N^*_{\mathbb{G}|\mathbb{P}} \longrightarrow 0$ *is exact.*

Main Theorem

Let $\mathbb{G} = \mathbb{G}(k,n)$, $\mathbb{P} = \mathbb{P}(\bigwedge^{k+1} V)$ *and* $N^*_{\mathbb{G}|\mathbb{P}}$ *the conormal bundle, we have the following resolution for the conormal bundle:*

$$0 \longrightarrow F_{k+1}(S^*) \longrightarrow V \otimes F_k(S^*) \longrightarrow \bigwedge^2 V \otimes F_{k-1}(S^*) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigwedge^{k-2} V \otimes F_3(S^*) \longrightarrow \bigwedge^{k-1} V \otimes F_2(S^*) \longrightarrow N^*_{\mathbb{G}|\mathbb{P}} \longrightarrow 0$$

where $F_i(S^*) = (S^* \otimes S^{i-1}S^* / S^i S^*) \otimes \vartheta_{\mathbb{G}}(-1)$

Particular Case (A.Tocino)

If k=1, $N_{\mathbb{G}|\mathbb{P}} = \bigwedge^2 S \otimes \vartheta_{\mathbb{G}}(1)$