Aspects of higher order differential equations.

Pablo Álvarez-Caudevilla

Departamento de Matemáticas, Universidad Carlos III de Madrid, Spain

Joint work with Victor A. Galaktionov – University of Bath (UK)

September 21, 2011

Contents

Second order elliptic problem

$$\begin{cases} -\Delta u = \lambda u + f, & \text{in} \Omega, \\ u = 0, & \text{on} \partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^N$ smooth bounded domain.

Second order elliptic problem

$$\begin{cases} -\Delta u = \lambda u + f, & \text{in}\Omega, \\ u = 0, & \text{on}\partial\Omega, \end{cases}$$

$\Omega \subset \mathbb{R}^N$ smooth bounded domain.

Then, there is λ_1 such that for all $f \geq 0$ and $f \neq 0$

- If $\lambda < \lambda_1$, then there is a positive solution $u > 0$;
- If $\lambda = \lambda_1$ there is no solution u;
- If $\lambda > \lambda_1$ then $u \neq 0$ and either there is no solution, or if there is any then,

$$\exists\, x \in \Omega, \quad \text{such that} \quad u(x) < 0.$$

Basically due to the Maximum Principle.

Second order elliptic problem

References

- Protter–Weinberger.
- For general non–smooth domains and general second order operators Berestycki–Nirenberg–Varadhan.
- Julián López–Gómez (UCM).

Higher order equations

Partial results

Boggio (1905) obtained a positive Green function in the unit ball $B \subset \mathbb{R}^N$

$$\begin{cases} (-\Delta)^m u = f, & \text{in} \quad B, \\ D_m u = 0, & \text{on} \quad \partial B, \end{cases}$$

where D_m is the Dirichlet boundary condition

$$D_m u = (D^\alpha u)_{|\alpha|<m-1}.$$

Green function in the unit ball given by Boggio

$$G_{m,N}(x,y) = K_{m,N}|x-y|^{2m-N} \int_1^{A(x,y)} \frac{(v^2-1)^{m-1}}{v^{N-1}} dv,$$

Solution of the problem $\quad u(x) = \int_{y \in B} G_{m,N}(x,y) f(y) dy.$

Conjecture for general domains

Hadamard (1908)

$$f \geq 0 \Rightarrow u \geq 0,$$

Hadamard (1908)

false conjecture in annuli with small inner radius.

Further improvements

- Counter examples for non-positivity of the Green function. Duffin (1949), Garabedian (1951), Loewner (1953) and Szegö (1953).

  Garabedian (1951). Nice domains such as an ellipse in $\mathbb{R}^2$ implies Green function changes sign.

- Osher (1973), Green function for the bi-harmonic in a wedge.

- Coffman (1982) and Coffman & Duffin (1980). Green function for the bi–harmonic on rectangles. Infinitely many oscillations on the corners.

- Grunau & Sweers (1990's), $(-\Delta)^m$ replaced by

$$(-\Delta)^m + \sum_{|\alpha| \leq 2m-1} a_\alpha(\cdot) D^\alpha.$$

Linear parabolic bi-harmonic equation

Unique solution for the bi-harmonic equation

Parabolic bi-harmonic equation

$$u_t = -\Delta^2 u.$$

$$\tilde{u}(x,t) = b(t) * u_0 \equiv t^{-\frac{N}{4}} \int_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{4}}) u_0(z)\, \mathrm{d}z.$$

Linear parabolic bi-harmonic equation

Unique solution for the bi-harmonic equation

Parabolic bi-harmonic equation

$$u_t = -\Delta^2 u.$$

$$\tilde{u}(x,t) = b(t) * u_0 \equiv t^{-\frac{N}{4}} \int_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{4}}) u_0(z)\, \mathrm{d}z.$$

Fundamental solution

$$b(x,t) = t^{-\frac{N}{4}} F(y), \quad y := \frac{x}{t^{1/4}} \quad (x \in \mathbb{R}^N).$$

Linear elliptic problem

$$\mathbf{B}F \equiv -\Delta_y^2 F + \frac{1}{4} y \cdot \nabla_y F + \frac{N}{4} F = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \int_{\mathbb{R}^N} F(y)\, \mathrm{d}y = 1.$$

B non-symmetric operator

$$\mathbf{B} \equiv -\Delta^2 + \frac{1}{4}\, y \cdot \nabla + \frac{N}{4} \;:\; H^4_\rho(\mathbb{R}^N) \longrightarrow L^2_\rho(\mathbb{R}^N)\,,$$

$$\text{where} \quad \rho(y) = \mathrm{e}^{a|y|^{4/3}}, \quad a > 0 \;\text{ small.}$$

Exponential decay of the kernel $F(y)$

$$a \in (0, 2d), \quad \text{such that} \quad d > 0,$$

$$|F(y)| \le D\,\mathrm{e}^{-d|y|^{4/3}} \quad \text{in} \quad \mathbb{R}^N \quad \big(D > 0,\;\; d = 3 \cdot 2^{-\frac{11}{3}}\big).$$

Spectral properties of the linear rescaled operators

Y. Egorov, V. A. Galaktionov, V. A. Kondratiev, and S. I. Pohozaev, *Advances in Differential Equations*, 2004.

Linear eigenvalue problem

$$\mathbf{B}\psi = \lambda\psi \quad \text{in} \quad \mathbb{R}^N, \quad \psi \in L^2_\rho(\mathbb{R}^N).$$

Theorem. Egorov et al.

(i) The spectrum of **B** comprises real eigenvalues only with the form

$$\sigma(\mathbf{B}) := \{\lambda_\beta = -\frac{|\beta|}{4}\,,\ |\beta| = 0, 1, 2, ...\}.$$

Eigenvalues λ_β have finite multiplicity with eigenfunctions,

$$\psi_\beta(y) := \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv \frac{(-1)^{|\beta|}}{\sqrt{\beta!}} (\frac{\partial}{\partial y_1})^{\beta_1} \cdots (\frac{\partial}{\partial y_N})^{\beta_N} F(y).$$

(ii) The subset of eigenfunctions $\Phi = \{\psi_\beta\}$ is complete.
(iii)For any $\lambda \notin \sigma(\mathbf{B})$, the resolvent $(\mathbf{B} - \lambda I)^{-1}$ is a compact operator.

Similar for the adjoint operator

$$\mathbf{B}^* \equiv -\Delta^2 - \frac{1}{4}\, y \cdot \nabla \;:\; H^4_{\rho^*}(\mathbb{R}^N) \to L^2_{\rho^*}(\mathbb{R}^N),$$

so

$$\langle \mathbf{B} v, w \rangle = \langle v, \mathbf{B}^* w \rangle \,, \;\; v \in H^4_{\rho}(\mathbb{R}^N), \;\; w \in H^4_{\rho^*}(\mathbb{R}^N),$$

with

$$\rho^*(y) \equiv \frac{1}{\rho(y)} = \mathrm{e}^{-a|y|^{\alpha}} > 0,$$

$$\sigma(\mathbf{B}^*) = \sigma(\mathbf{B}),$$

$$\psi^*_{\beta}(y) := \frac{1}{\sqrt{\beta!}} \Big[y^{\beta} + \sum_{j=1}^{[|\beta|/4]} \frac{1}{j!}\, \Delta^{2j} y^{\beta} \Big].$$

Fourth–order evolution equation

Thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1\,, \quad n \in (0,1)\,.$$

Fourth–order evolution equation

Thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1\,, \quad n \in (0,1)\,.$$

Boundary and initial conditions

$$\begin{cases} u = 0, & \text{zero-height,} \\ \nabla u = 0, & \text{zero contact angle,} \\ -\mathbf{n} \cdot \nabla (|u|^n \Delta u) = 0, & \text{conservation of mass (zero-flux)} \end{cases}$$

at the singularity surface (interface) $\Gamma_0[u]$,

$$\text{supp}\; u \subset \mathbb{R}^N \times \mathbb{R}_+, \quad N \geq 1\,.$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Gamma_0[u] \cap \{t = 0\}.$$

Motivations

Physical applications

Free boundary problems with non-negative solutions

Motivations

Physical applications

Free boundary problems with non-negative solutions

Bernis–Friedman 1990

Pioneering work, existence of weak solutions, $u \geq 0$ if $u_0 \geq 0$

Solutions of changing sign

Recent interest

Solutions of changing sign

Recent interest

Cauchy Problem

Very little is known.

concept of proper solution?

Solutions of changing sign

Recent interest

Cauchy Problem

Very little is known.

concept of proper solution?

For $0 < n < 1.7587...$

oscillatory sign changing solutions are related to the CP

while

nonnegative solutions appear for the standard FBP

Aim

Homotopy approach

From the thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u), \quad n \longrightarrow 0^+,$$

to the bi-harmonic equation

$$u_t = -\Delta^2 u.$$

Aim

Homotopy approach

From the thin film equation

$$u_t = -\nabla \cdot (|u|^n \nabla \Delta u), \quad n \longrightarrow 0^+,$$

to the bi-harmonic equation

$$u_t = -\Delta^2 u.$$

- Qualitative information;
- Multiplicity results.

Second order equation

Porous medium equation

$$u_t = \Delta(|u|^n u)$$

Homotopy transformation

to the heat equation $\quad u_t = \Delta u \quad$ as $\quad n \longrightarrow 0^+$

Methodology

Homotopy approach

- Rescaled equations,

$$u(x,t) := t^{-\alpha} f(y), \quad \text{with} \quad y := \frac{x}{t^{\beta}}, \quad \beta = \frac{1-\alpha n}{4}.$$

  Nonlinear eigenvalue problem

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \frac{1-\alpha n}{4} y \cdot \nabla f + \alpha f = 0, \quad f \in C_0(\mathbb{R}^N),$$

- n-branching;
- Lyapunov-Schmidt reduction.

Contents

Continuity/homotopy deformation approach

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \frac{1 - \alpha n}{4} y \cdot \nabla f + \alpha f = 0$$

Continuity/homotopy deformation approach

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \frac{1 - \alpha n}{4} y \cdot \nabla f + \alpha f = 0$$

$$\mathbf{B} F \equiv -\Delta^2 F + \frac{1}{4} y \cdot \nabla F + \frac{N}{4} F = 0$$

Continuity/homotopy deformation approach

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \frac{1 - \alpha n}{4} y \cdot \nabla f + \alpha f = 0$$

$$\downarrow \quad n \to 0^+$$

$$\mathbf{B}F \equiv -\Delta^2 F + \frac{1}{4} y \cdot \nabla F + \frac{N}{4} F = 0$$

Main result

$$-\nabla \cdot (|f|^n \nabla \Delta f) + \frac{1-\alpha n}{4}\, y \nabla \cdot f + \alpha f = 0, \quad f \in C_0(\mathbb{R}^N),$$

possesses a countable set of eigenfunction/value pairs $\{f_k,\, \alpha_k\}_{|\sigma|=k\geq 0}$

"nonlinear eigenfunctions" $f(y)$,

"nonlinear eigenvalues" $\{\alpha_k\}_{k\geq 0}$.

One branch if the dimension of the kernel is one.

Two branches if the dimension of the kernel is two.

At most four branches when the dimension is three.

Contents

Difficulties

Non-existence of entirely rigorous results

$$f \in C_0(\mathbb{R}^N) \quad \text{or} \quad f \in L^2_\rho(\mathbb{R}^N)$$

Difficulties

Non-existence of entirely rigorous results

$$f \in C_0(\mathbb{R}^N) \quad \text{or} \quad f \in L^2_\rho(\mathbb{R}^N)$$

Geometry and topology of zeros close to interfaces

Desired structure of transversal zeros.

Difficulties

Non-existence of entirely rigorous results

$$f \in C_0(\mathbb{R}^N) \quad \text{or} \quad f \in L^2_\rho(\mathbb{R}^N)$$

Geometry and topology of zeros close to interfaces

Desired structure of transversal zeros.

$$\frac{|f|^n - 1}{n} \rightharpoonup \ln|f| \quad \text{as} \quad n \to 0^+ \quad \text{in} \quad L^\infty_{\mathrm{loc}}.$$

equivalently

$$n \ln^2|f| \rightharpoonup 0, \quad \text{as} \quad n \downarrow 0^+, \quad \text{since} \quad \frac{|f|^n - 1}{n} - \ln|f| = \frac{1}{2} n \ln^2|f| + \dots$$

Difficulties

Integral form

$$f = -(\mathcal{L}(\alpha, n) + aI)^{-1}(\mathcal{N}(n, f) + af),$$

with

$$\mathcal{L}(\alpha, n) := -\Delta^2 + \frac{1 - \alpha n}{4} y \cdot \nabla + \alpha I,$$

and

$$\mathcal{N}(n, f) := \nabla \cdot ((1 - |f|^n)\nabla \Delta f).$$

Difficulties

Integral form

$$f = -(\mathcal{L}(\alpha, n) + aI)^{-1}(\mathcal{N}(n, f) + af),$$

with

$$\mathcal{L}(\alpha, n) := -\Delta^2 + \frac{1-\alpha n}{4} y \cdot \nabla + \alpha I,$$

and

$$\mathcal{N}(n, f) := \nabla \cdot ((1 - |f|^n)\nabla\Delta f).$$

Easier limit

$$(|f|^n - 1)\nabla\Delta f = (n \ln|f| + \dots)\nabla\Delta f.$$

Difficulties

Hammerstein–Uryson compact integral operator

$$f \sim (\nabla\Delta)^{-1}[(|f|^n - 1)\nabla\Delta f].$$

$$\mathcal{P} = \{f = f(\cdot, n) : \ f \in H^4_\rho(\mathbb{R}^N)\},$$

for which:

$$\mathcal{P} : \quad (\nabla\Delta)^{-1}\Big(\frac{|f|^n - 1}{n}\nabla\Delta f\Big) \to (\nabla\Delta)^{-1}(\ln|f|\nabla\Delta f) \quad \text{as} \quad n \to 0^+.$$

Difficulties

Hammerstein–Uryson compact integral operator

$$f \sim (\nabla\Delta)^{-1}[(|f|^n - 1)\nabla\Delta f].$$

$$\mathcal{P} = \{f = f(\cdot, n) : \ f \in H^4_\rho(\mathbb{R}^N)\},$$

for which:

$$\mathcal{P} : \quad (\nabla\Delta)^{-1}\Big(\frac{|f|^n - 1}{n}\nabla\Delta f\Big) \to (\nabla\Delta)^{-1}(\ln|f|\nabla\Delta f) \quad \text{as} \quad n \to 0^+.$$

Analysis not available for non-small n

Different approach.

Contents

Homotopy deformation

$$u_t = -\nabla \cdot (\phi_\epsilon(u)\nabla\Delta u) \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N,$$

such that

$$\phi_1(u) = 1, \quad \text{and} \quad \phi_\epsilon(u) \to |u|^n \quad \text{as} \quad \epsilon \to 0$$

uniformly on compact subsets.

Possible analytic homotopy

$$\phi_\epsilon(u) := \epsilon^n + (1-\epsilon)(\epsilon^2 + u^2)^{\frac{n}{2}}, \quad \epsilon \in (0,1].$$

Family of solutions

$$\mathcal{P}_\phi = \{u_\epsilon(x,t)\}_{\epsilon \in (0,1]}.$$

Family of solutions

$$\mathcal{P}_\phi = \{u_\epsilon(x,t)\}_{\epsilon\in(0,1]}.$$

$$u_\epsilon(x,t) \to u(x,t) \quad \text{as} \quad \epsilon \to 0^+.$$

Multiplying by a test function $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T])$ and integrating by parts in $\Omega \times [0,T]$

$$-\int_0^T \int_\Omega \varphi_t u_\epsilon - \int_0^T \int_\Omega \nabla\varphi \cdot (\phi_\epsilon(u) \nabla \Delta u_\epsilon) = 0.$$

Not convincing to have uniqueness.

Consider a double limit ϵ, $n \to 0^+$

$$u(x,t) \to \tilde{u}(x,t) \quad \text{as} \quad n \to 0^+.$$

with

$$\tilde{u}(x,t) = b(t) * u_0 \equiv t^{-\frac{N}{4}} \int_{\mathbb{R}^N} F((x-z)t^{-\frac{1}{4}}) u_0(z)\, \mathrm{d}z.$$

This allows us to obtain a proper solution for the Cauchy problem of the TFE.

Then, we claim that the fundamental solutions for $n = 0$ and $n > 0$ small exhibit similar properties.

Branching through homotopy as $n \to 0$ and $\epsilon \to 0$

$$u_\epsilon : \quad u_t = -\nabla \cdot ((\epsilon^2 + u^2)^{\frac{n}{2}} \nabla \Delta u), \quad u(x,0) = u_0(x).$$

$n \to 0^+$ as the main deformation parameter, and $\epsilon = \epsilon(n) \to 0$.

Perturbed version

$$u_t = -\Delta^2 u + \nabla \cdot \left([1 - (\epsilon^2 + u^2)^{\frac{n}{2}}] \nabla \Delta u \right),$$

Perturbed version

$$u_t = -\Delta^2 u + \nabla \cdot \left([1 - (\epsilon^2 + u^2)^{\frac{n}{2}}]\nabla \Delta u\right),$$

Equivalent integral form

$$u_\epsilon : \quad u(t) = b(t) * u_0 + \int_0^t \nabla b(t-s) * F_{n,\epsilon}(u(s))\nabla \Delta u(s)\, \mathrm{d}s,$$

$$\text{where} \quad F_{n,\epsilon}(u) := 1 - (\epsilon^2 + u^2)^{\frac{n}{2}}.$$

Expansion for the branching analysis

$$F_{n,\epsilon}(u) \equiv 1 - (\epsilon^2 + u^2)^{\frac{n}{2}} = -\frac{n}{2} \ln(\epsilon^2 + u^2)(1 + o(1)) \quad \text{as} \quad n \to 0^+.$$

Expansion for the branching analysis

$$F_{n,\epsilon}(u) \equiv 1 - (\epsilon^2 + u^2)^{\frac{n}{2}} = -\frac{n}{2} \ln(\epsilon^2 + u^2)(1 + o(1)) \quad \text{as} \quad n \to 0^+.$$

Key assumption for the parameter $\epsilon = \epsilon(n) > 0$ for $u \approx 0$

$$n \,|\ln \epsilon(n)| \to 0 \quad \text{as} \quad n \to 0.$$

Expansion for the branching analysis

$$F_{n,\epsilon}(u) \equiv 1 - (\epsilon^2 + u^2)^{\frac{n}{2}} = -\frac{n}{2} \ln(\epsilon^2 + u^2)(1 + o(1)) \quad \text{as} \quad n \to 0^+.$$

Key assumption for the parameter $\epsilon = \epsilon(n) > 0$ for $u \approx 0$

$$n \, | \ln \epsilon(n) | \to 0 \quad \text{as} \quad n \to 0.$$

Lyapunov-Schmidt method

$$u(t) = b(t) * u_0 - \frac{n}{2} \int_0^t \nabla b(t-s) * \ln(\epsilon^2 + u^2) \nabla \Delta u(s) \, \mathrm{d}s + o(n) \quad \text{as} \quad n \to 0.$$

Main result

Assume the condition on $\epsilon = \epsilon(n) > 0$ holds. Then, the solution of the perturbed problem

$$u_t = -\nabla \cdot ((\epsilon^2 + u^2)^{\frac{n}{2}} \nabla \Delta u), \quad u(x,0) = u_0(x),$$

converge continuously to the solution of the Cauchy problem of the bi-harmonic equation

$$u_t = -\Delta^2 u, \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N,$$

in the double limit $\epsilon \to 0$ and $n \to 0$. Hence, there exists a branching of solutions of the TFE-4 at $n = 0^+$ from the unique solution of the parabolic bi-harmonic equation, deforming the solutions of the TFE-4 as $n \to 0^+$ via an analytic path and inheriting the oscillatory and changing sign properties of the linear flow.

Thank you for your attention!!