

An Introduction to Topological Groups

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Workshop of Young Researchers

- 1 Definition and first examples
- 2 Connectedness in topological groups
- 3 An introduction to Pontryagin duality theory
- 4 Free abelian topological groups

1 Definition and first examples

Definition

A pair (G, τ) is called **topological group** if G is a group and τ is a topology on G such that multiplication

$$G \times G \longrightarrow G, (x, y) \mapsto xy$$

and inversion

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- 4 topological vector spaces

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Proposition

*Let H be a normal subgroup of a topological group (G, τ) and let $\pi : G \rightarrow G/H$ denote the canonical projection. We endow G/H with the **quotient topology***

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Corollary

A group topology τ is determined by a neighborhood basis $\mathcal{U}(e)$ at the neutral element e .

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Example

Let (\mathbb{R}, τ) be the Sorgenfrey-line. Then addition $(\mathbb{R}, \tau) \times (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$ is continuous (not only all the translations!). However, inversion is discontinuous.

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yU is open, since $x \rightarrow yx$ is a homeomorphism and $BU = \bigcup_{b \in B} bU$ is open by the same reason.

Theorem

Let (G, τ) be a topological group with neighborhood basis \mathcal{U} at e . For $U \in \mathcal{U}$ we define

$$U_L := \{(x, y) \in G \times G : x^{-1}y \in U\} \quad \text{and}$$
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Corollary

Every T_1 topological group is $T_{3\frac{1}{2}}$.

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*In this case, there exists a left (right-) invariant metric which induced the topology. A metric d is called **left invariant** if $d(x, y) = d(ax, ay)$ for all $a \in G$.*

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Assume now, there exists a left and right invariant metric d on $Sl(2, \mathbb{R})$. Then

$$d(A_n B_n, E) \stackrel{li}{=} d(B_n, A_n^{-1}) \stackrel{ri}{=} d(B_n A_n, E),$$

which is impossible.

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Since connected components are always closed, the assertion follows. □

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Corollary

Let G be a totally disconnected compact group. Then G can be embedded into a product of finite groups.

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Let $(N_i)_{i \in I}$ be neighborhood basis at e consisting of open normal subgroups.

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$$\varphi(N_j) = \varphi(G) \cap \{e_{G/N_j}\} \times \prod_{i \in I \setminus \{j\}} G/N_i,$$

which shows that φ is an embedding.



3 An introduction to Pontryagin duality theory

Definition

Let (G, τ) be a topological group. By G^\wedge we denote the set of all continuous homomorphisms $G \rightarrow \mathbb{T}$ and call this group **dual group** or **character group** of G .

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Remark

Obviously, G^\wedge is an abelian group (under pointwise addition). The commutator subgroup of G is contained in the kernel of every continuous character.

Examples

The following mappings are topological isomorphisms:

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Theorem (Gel'fand, Raikov)

Let G be a locally compact abelian (LCA) group. For every $x \in G \setminus \{e\}$ there exists a continuous character $\chi \in G^\wedge$ such that $\chi(x) \neq 1$.

Since quotients of LCA groups w.r.t. closed subgroups are again LCA, we have:

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Corollary

A subgroup H of a LCA group G is dense iff $H^\perp := \{\chi \in G^\wedge : \chi(H) = \{1\}\} = \{1\}$.

Example (Kronecker)

Let $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{R}^n$ be elements of \mathbb{R}^n and put

$$H = \mathbb{Z}^n + \langle \{\mathbf{a}\} \rangle.$$

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Proposition

Let (G, τ) be a Hausdorff group. The sets of the form

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Question

Is a group G reflexive if G and $G^{\wedge\wedge}$ are topologically isomorphic?

Theorem (Pontryagin, van Kampen)

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If G is a connected LCA group then G is topologically isomorphic to $\mathbb{R}^n \times K$ where $n \in \mathbb{N}_0$ and K is a compact and connected group.

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Example

The character group \mathbb{Q}^\wedge is a connected (and divisible) compact group.

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The group \mathbb{Q} is not a L -group, hence \mathbb{Q}^\wedge is not locally connected.

Definition (Whitehead)

A torsion free abelian group G is called a **W -group** iff every homomorphism $\chi : G \rightarrow \mathbb{T}$ can be lifted over \mathbb{R} , i.e. there exists a homomorphism $\varphi : G \rightarrow \mathbb{R}$ such that $\exp(2\pi i\varphi) = \chi$.

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Every free abelian group is a W -group.

In ZFC it is undecidable whether every W -group is free.

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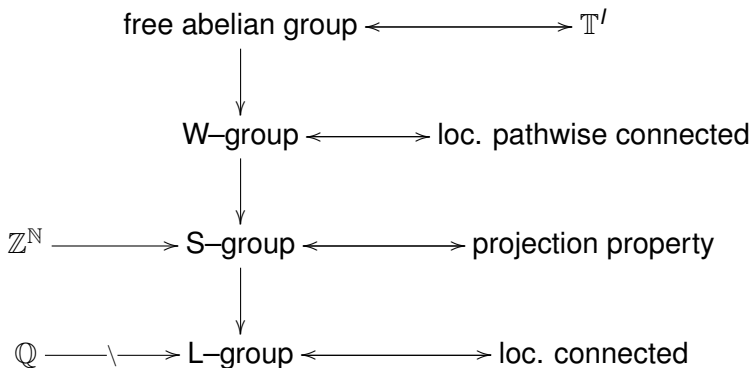
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Remark

Every W -group is an S -group. $\mathbb{Z}^{\mathbb{N}}$ is an S -group.



4 Free abelian topological groups

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- 2 There exists a (topological) embedding $\eta : X \rightarrow A(X)$ such that $\eta(X)$ is a basis for the free abelian topological group and $\eta(X)$ is closed in $(A(X), \tau)$.
- 3 For every continuous mapping $f : X \rightarrow H$ where H is an abelian topological group, the unique homomorphism $f' : A(X) \rightarrow H$ which satisfies $f' \circ \eta = f$ is continuous.

Theorem (Markov 1945)

For every completely regular space X , the free abelian topological group $(A(X), \tau)$ exists and is unique up to topological isomorphism. $(A(X), \tau)$ is a Hausdorff space.

Proposition (Pestov)

$A(X)^\wedge \rightarrow \mathcal{C}(X, \mathbb{T})$, $\chi \mapsto \chi \circ \eta$ is a continuous isomorphism
when $\mathcal{C}(X, \mathbb{T})$ is endowed with the compact–open topology.

Proposition (Pestov)

$A(X)^\wedge \rightarrow \mathcal{C}(X, \mathbb{T})$, $\chi \mapsto \chi \circ \eta$ is a continuous isomorphism when $\mathcal{C}(X, \mathbb{T})$ is endowed with the compact–open topology. If G is e.g. paracompact, then the above mapping is also open.

Theorem (Pestov, L.A., Galindo, Hernandez)

Let X be compact. Then $A(X)$ is reflexive iff X is punctiform, i.e. every connected compact subset of X is a singleton.

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Question

Is the character group of every abelian metrizable group reflexive?

Thank you for your attention!