ON A CLASS OF ORTHOGONAL POLYNOMIALS WITH RESPECT TO A JACOBI OPERATOR

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DEFINITION*

Let $\mu(x)$ be a positive Borel measure on the real line and $\{\rho_j(x)\}_{j=0}^{M}, \quad \rho_M \equiv 1$ be a set of functions such that $\rho_j(x)d\mu(x)$ has finite moments, $j = 0, \ldots, M$. Denote

$$\mathcal{L}^{(M)} = \sum_{j=0}^{M} \rho_j(x) \frac{d^j}{dx^j}$$

We say that $\{Q_n\}$, $n \in \mathbb{N}$, is a sequence of orthogonal polynomials with respect to the differential operator $\mathcal{L}^{(M)}$ if $deg(Q_n) \leq n$ and

$$\int \mathcal{L}^{(M)}[Q_n]P(x)d\mu(x) = 0 \qquad (1)$$

for any polynomial $P(x)$ such that $deg(P) \leq n - 1$.

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$$Q_n(z) \perp (\mathcal{L}^{(M)}, \mu)$$

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$$\mathcal{L}[f] = (1 - x^2)f'' + (\beta - \alpha - (\alpha + \beta + 2)x)f', \quad f \in \mathbb{P},$$

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THEOREM

Let n be a fixed positive integer number. Equation (1) have a unique, except an additive constant, monic polynomial solution Q_n of degree n if and only if

$$\int_R L_n(x) d\mu_{\alpha,\beta}(x) = 0.$$

where $L_n \perp \mu$ and $d\mu_{\alpha,\beta}(x) = (1 - x)^\alpha \, (1 + x)^\beta \, dx$

COROLLARY

Let $d\mu_{\alpha,\beta}(x) = (1-x)^{\alpha}\,(1+x)^{\beta}dx$ and μ be a finite nonnegative Borel measures on R, such that $d\mu_{\alpha,\beta}(x) = \rho(x)d\mu(x)$ with $\rho \in L^2(\mu)$ μ-a.e. on R.
If $m \in \mathbb{Z}_+$, the sequence $\{Q_n\}$ exists for $n > m \Leftrightarrow \rho(x)$ is a non negative polynomial on $[-1, 1]$ of degree m.

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μ *belongs to the class* $\mathcal{P}_m(\mu_{\alpha,\beta})$ where $m \in \mathbb{Z}_+$, if

- $\mu(x)$ is a finite nonnegative Borel measure supported on R
- there exist a polynomial $\rho(x)$ of degree m , such that $d\mu_{\alpha,\beta}(x) = \rho(x)d\mu(x)$.

EXAMPLE

- $d\mu(x) = (1-x)^{\alpha-k_\alpha}(1+x)^{\beta-k_\beta}dx$, where k_α and k_β are nonnegative integers such that $\alpha - k_\alpha > -1$ and $\beta - k_\beta > -1$. Hence $\mu \in \mathcal{P}_{k_\alpha+k_\beta}(\alpha,\beta)$.

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- $d\mu(x) = \dfrac{(1-x)^\alpha(1+x)^\beta}{x^2+1}\,dx$, note that in this example μ is not a Jacobi measure. Hence $\mu \in \mathcal{P}_2(\alpha,\beta)$.

If $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$ and $n > m$ then

$$L_n(z) = P_n(z) + \sum_{k=1}^{m} b_{(n,n-k)} P_{n-k}(z),$$

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Let $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$ and $\{\zeta_n\}$ a sequence of complex numbers. We define $Q_n \perp (\mathcal{L}, \mu, \{\zeta_n\})$ for each $n > m$ as the monic polynomial solution of the initial value problem

$$\begin{cases} \mathcal{L}[y] &= \lambda_n L_n, \quad n > m, \\ y(\zeta_n) &= 0, \end{cases}$$

and we say that $\{Q_n\}$, for $n > m$, is the sequence of monic orthogonal polynomials with respect to the pair $(\mathcal{L}, \mu, \{\zeta_n\})$ such that $Q_n(\zeta_n) = 0$.

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We have then $Q_n(z) = \widehat{Q}_n(z) - \widehat{Q}_n(\zeta_n)$

THEOREM

Let $m \in \mathbb{Z}_+$, $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$. Then if $R(z)$ is any primitive of $\rho(z)$, for each $n > (2m+1)$ the sequence of polynomials $Q'_n(z)$ satisfy the relation

$$R(z)Q'_n(z) = \sum_{k=-m-1}^{m+1} \theta(R, n, n-k)\, Q'_{n-k}(z)$$

THEOREM

Let $m \in \mathbb{Z}_+$ and $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$, for each $n > 3m+1$ the sequence $\{\widehat{Q}_n(z)\}$ satisfies that

$$H(z)\widehat{Q}_n(z) = \sum_{k=-2m-1}^{2m+1} \vartheta_{(n,n-k)}\widehat{Q}_{n-k}(z)$$

where $H(z)$ is any primitive of the function $\rho^2(z)$, $R(z)$ is any primitive of the function $\rho(z)$

THEOREM

Let $m \in \mathbb{Z}_+$ and $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$. Then

1. $$\frac{\widehat{Q}_n(z)}{P_n^{(\alpha,\beta)}(z)} \underset{n\to\infty}{\rightrightarrows} \phi_m^2\,\Phi(\rho,z),$$

   uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1,1]$ where $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ and

   $$\Phi(\rho,z) = \prod_{k=1}^{m} \frac{z-\nu_i}{\varphi(z)-\varphi(\nu_i)},$$
   $$\phi_m = 2^m \exp\left(\frac{1}{2\pi}\int_{-1}^{1} \frac{\log(\rho(t))}{\sqrt{1-t^2}}\,dt\right).$$

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2. The set of accumulation points of the zeros of the sequence of polynomials $\{\widehat{Q}_n\}$ is $[-1, 1]$

COROLLARY

Let $m \in \mathbb{Z}_+$ and $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$. Then

$$\lim_{n\to\infty} \left|\widehat{Q}_n(z)\right|^{\frac{1}{n}} = \frac{|z+\sqrt{z^2-1}|}{2},$$

uniformly on closed subsets of $\overline{\mathbb{C}} \setminus [-1,1]$.

COROLLARY

Let $m \in \mathbb{Z}_+$ and $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$. If $\{\zeta_n\}$ is a sequence of complex number with limit $\zeta \in \mathbb{C} \setminus [-1,1]$. Then

- $$\frac{2^n\, Q_n(z)}{\varphi(z)^n} \underset{n\to\infty}{\rightrightarrows} \phi_m^2 \sqrt{\frac{\varphi'(z)}{2}} \left(\frac{\varphi(z)-1}{2(z-1)}\right)^{\alpha} \left(\frac{\varphi(z)+1}{2(z+1)}\right)^{\beta} \prod_{k=1}^{m} \frac{z-\nu_i}{\varphi(z)-\varphi(\nu_i)},$$

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- $$\frac{Q_{n+1}(z)}{Q_n(z)} \underset{n\to\infty}{\rightrightarrows} \frac{\varphi(z)}{2},$$
- $$\sqrt[n]{|Q_n(z)|} \underset{n\to\infty}{\rightrightarrows} \frac{|\varphi(z)|}{2}$$

uniformly on compact subsets of $\{z \in \mathbb{C} : |z| > \Delta(\zeta) + 1\}$, where for $z \in \mathbb{C}$ we denote $\Delta(z) = \sup_{x\in[-1,1]} |z - x|$.

THEOREM

Let $m \in \mathbb{Z}_+$ and $\mu \in \mathcal{P}_m(\mu_{\alpha,\beta})$, where $\alpha, \beta > -1$. If $\{\zeta_n\}$ is a sequence of complex number with limit $\zeta \in \mathbb{C} \setminus [-1, 1]$, then the accumulation points of zeros of $\{Q_n\}$ are located on the set $E = \mathcal{E}(\zeta) \bigcup [-1, 1]$, where $\mathcal{E}(\zeta)$ is the ellipse

$$\mathcal{E}(\zeta) := \left\{ z \in \mathbb{C} : z = \cosh(\eta_\zeta + i\theta), 0 \leq \theta < 2\pi \right\},$$

and $\eta_\zeta := \ln|\varphi(\zeta)| = \ln|\zeta + \sqrt{\zeta^2 - 1}|$.

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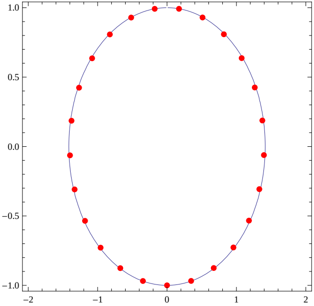

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Let $w_i \in \mathbb{C}$ be given. We say that w_i is a *source* point with strength $m_i > 0$ if the complex potential is given by the expression

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$$\mathcal{V}(z) := \sum_{i=1}^{n-1} \log \frac{1}{(z - w_i)} + \alpha \log \frac{1}{z - 1} + \beta \log \frac{1}{z + 1}.$$

Problem. *Let* $\{x_1, x_2, \ldots, x_n\}$ *be the set of zeros of the nth orthogonal polynomial* (L_n) *with respect to a nonnegative Borel measure* $\mu \in \mathcal{P}_1$ *with* $n > 1$. *Suppose given a flow, with complex potential equal to* $\mathcal{V}(z)$. *Build a* $n-1$ *system (location of the source points* $w_1, \ldots, w_{n-1}$*) such that the stagnation points are attained at the points* $z = x_i$, *with* $i = 1, 2, \ldots, n$.

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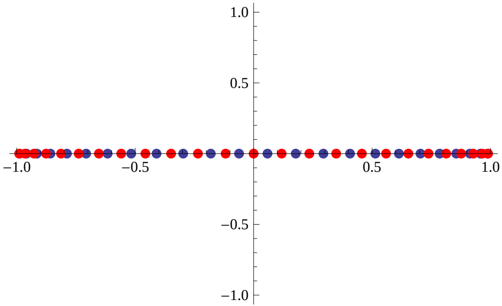
Answer. *The flow of an incompressible two–dimensional fluid, due to a* $(n-1)$ *system located in the critical points of the n-th orthogonal polynomial with respect to* $(\mathcal{L}, \mu)$ *with* $\mu \in \mathcal{P}_1$ *and complex potential equal to* $\mathcal{V}(z)$ *has its n stagnation points in the* $n-1$ *critical points of the nth orthogonal polynomial w.r.t.* μ.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Thanks you