Analyticity in the Calderón problem with partial data

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Outline

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The Calderón problem with partial data

Our contribution (joint work with A. Ruiz and D. Dos Santos Ferreira)

An idea of the proof

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Applications: medical imaging and geophysical prospection.

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- Uniqueness:
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- Partial data:
  Applications in geophysical prospection requires to know γ from partial knowledge of Λ_γ.

Auxiliary problem and first partial data result

Auxiliary problem when γ is smooth:

- Associated equation

$$\mathrm{div}(\gamma \nabla u) = 0 \quad \Leftrightarrow \quad -(\Delta + q)v = 0$$
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Given $q_1, q_2 : \Omega \longrightarrow \mathbf{C}$. If $(\Lambda_{q_1} f)|_\Gamma = (\Lambda_{q_2} f)|_\Gamma$ for all f, then $q_1 = q_2$. Γ is the *half* of the boundary.

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Since $q_1 - q_2$ has compact support, $\mathcal{F}(q_1 - q_2)$ is analytic. Then $q_1 = q_2$.

Second partial data result

Theorem (Kenig, Sjöstrand & Uhlmann, **Ann. of Math.** 2007)

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- What kind of *analyticity*?

Analytic wave front set

- Wave packet transform (Córdoba & Fefferman, **Comm. PDEs** 1978): $h \in (0,1]$

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Theorem (Micro-local Holmgren Theorem, Kashiwara)

If $x_0 \in \mathrm{supp}\, u \subset \{\omega_0 \cdot (x - x_0) \leq 0\}$, then $(x_0, \omega_0) \in WF_a(u)$.

Radon transform & $WF_a(u)$

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- $\mathcal{R}(q_1 - q_2)(H) = 0$ for all $H \in \mathrm{neigh}\{\omega_0 \cdot (x - x_0) = 0\}$. Then $(x_0, \omega_0) \notin WF_a(q_1 - q_2)$.

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- Proofs of the Micro-local Helgason Theorem base on a different characterization of WF_a (as far as we know), which does not seem promising to quantify.

Main results

Theorem (C, Ruiz & Dos Santos Ferreira)

$M \geq 1$, $L > 0$ and $H_0 = \{\omega_0 \cdot (x - y_0) = 0\}$. There exists $C = C(n, M, L)$, such that

$$\|q_1 - q_2\|_{L^\infty(\mathrm{neigh}(x_0))} \leq \frac{C}{|\log \|\mathcal{R}(q_1 - q_2)\|_{\mathrm{neigh}(H_0)}|^{1/4}},$$

with q_1, q_2 satisfying:

(a) $y_0 \in \operatorname{supp} q_j$ *and* $\operatorname{supp} q_j \subset \{x \in \mathbf{R}^n : (x - y_0) \cdot \omega_0 \leq 0\}$ *for* $j = 1, 2$.

(b) $\|q_j\|_{L^\infty(E)} + \|\mathbf{1}_E q_j\| < M$ *for* $j = 1, 2$. $E = \{x \in H \in \mathrm{neigh}(H_0)\}$.

(c) $\inf\{K > 0 : |q_j(x) - q_j(y)| \leq K|x - y| \, \forall x, y\} < L$, *for* $j = 1, 2$.

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Theorem (C, Ruiz & Dos Santos Ferreira)

Given q_1, q_2 as above and Λ_1, Λ_2. The following estimate holds:

$$\|q_1 - q_2\|_{L^\infty(\mathrm{neigh}(x_0))} \leq F\left(\|(\Lambda_1 - \Lambda_2)|_{\Gamma_1}\|\right).$$

F is a $\log\log$-type modulus of continuity.

Segal-Bargmann and Wave packet transforms:

$$\mathcal{T}u(\zeta) = \int_{\mathbf{R}^n} e^{-\frac{1}{2h}(\zeta - x)^2} u(x)\,\mathrm{d}x, \quad \mathcal{T}u(\zeta) = e^{\frac{1}{2h}|\mathrm{Im}\,\zeta|^2}\mathcal{W}u(\mathrm{Re}\,\zeta, -\mathrm{Im}\,\zeta).$$

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$$\mathcal{T}u(\zeta) = \frac{h^{\frac{n-1}{2}}(1 + (-1)^{n-1})}{4(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{R}} \int_{\mathbf{S}^{n-1}} D_s^{n-1} e^{-\frac{1}{2h}(\omega\cdot(\zeta - y_0) - s)^2}\, \mathcal{R}_{y_0} u(s, \omega)\,\mathrm{d}A\,\mathrm{d}s$$

$$+ \frac{h^{\frac{n-1}{2}}(1 + (-1)^{n})}{4(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{R}} \int_{\mathbf{S}^{n-1}} D_s^{n-1} \mathcal{H}\left(e^{-\frac{1}{2h}(\omega\cdot(\zeta - y_0) - \cdot)^2}\right)(s)\, \mathcal{R}_{y_0} u(s, \omega)\,\mathrm{d}A\,\mathrm{d}s.$$

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$$\mathcal{T}u(\zeta) = \frac{h^{\frac{n-1}{2}}(1 + (-1)^{n-1})}{4(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{R}} \int_{\mathbf{S}^{n-1}} D_s^{n-1} e^{-\frac{1}{2h}(\omega\cdot(\zeta - y_0) - s)^2}\, \mathcal{R}_{y_0} u(s,\omega)\,\mathrm{d}A\,\mathrm{d}s$$
$$+ \frac{h^{\frac{n-1}{2}}(1 + (-1)^{n})}{4(2\pi)^{\frac{n-1}{2}}} \int_{\mathbf{R}} \int_{\mathbf{S}^{n-1}} D_s^{n-1} \mathcal{H}\left(e^{-\frac{1}{2h}(\omega\cdot(\zeta - y_0) - \cdot)^2}\right)(s)\, \mathcal{R}_{y_0} u(s,\omega)\,\mathrm{d}A\,\mathrm{d}s.$$

Quantitative version of Micro-local Helgason Theorem (complex integration to count oscillations):

$$|\mathcal{T}q(\zeta)| \le \frac{C}{h^{\frac{n}{2}}}(1 + |\mathrm{Re}\,\zeta - y_0| + |\mathrm{Im}\,\zeta|)^n\, e^{\frac{1}{2h}|\mathrm{Im}\,\zeta|^2} \left[\|\mathcal{R}q\|_{\mathrm{neigh}(H_0)} + \|q\|\, e^{-\frac{c}{2h}}\right],$$

for all $\zeta \in \mathrm{neigh}_{\mathbf{C}^n}(y_0 + i\omega_0)$ with $H_0 = \{\omega_0 \cdot (x - y_0) = 0\}$.

Segal-Bargmann and Wave packet transforms:

$$\mathcal{T}u(\zeta) = \int_{\mathbf{R}^n} e^{-\frac{1}{2h}(\zeta - x)^2} u(x)\,\mathrm{d}x, \quad \mathcal{T}u(\zeta) = e^{\frac{1}{2h}|\mathrm{Im}\,\zeta|^2} \mathcal{W}u(\mathrm{Re}\,\zeta, -\mathrm{Im}\,\zeta).$$

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Oscillations might make $\mathcal{T}$ vanish for non-small q. Restriction of $\mathcal{T}$ to real values gives convolution with a Gaussian.

Using the maximum principle for sub-harmonic functions, we can enlarge the controlled range of $\mathcal{T}$.

$$e^{-\frac{1}{2h}|\mathrm{Im}\,\zeta|^2}|\mathcal{T}q(\zeta)| < C\,\|\mathcal{R}q\|^{\kappa}_{\mathrm{neigh}(H_0)}\,,$$

for all $\zeta \in \mathrm{neigh}_{\mathbf{C}^n}(y_0 + i0)$. Here $0 < \kappa < 1$ and

$$h = \frac{c}{\log(\|\mathcal{R}q\|^{\kappa}_{\mathrm{neigh}(H_0)})^{-1}}.$$

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$$\|q\|_{L^\infty(G)} \le C\left(h^{-\frac{n}{2}}\,\|\mathcal{T}q|_{\mathbf{R}^n}\|_{L^\infty(G)} + L_q h^{\frac{\lambda}{4}} + M_q\, e^{-\frac{1}{4h^{1/2}}}\right),$$

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Hence

$$\|q_1 - q_2\|_{L^\infty(\mathrm{neigh}(x_0))} \leq \frac{C}{(\log\|\mathcal{R}(q_1 - q_2)\|_{\mathrm{neigh}(H_0)})^{1/4}}.$$

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- **Thank you for your attention!**