Noncommutative L_p spaces and Calderón-Zygmund theory

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Functions as operators

Let (Ω, Σ, μ) be a measure space where μ is σ-finite. Given a function $f \in L_\infty(\Omega)$, it defines an operator acting on the Hilbert space $L_2(\Omega)$ in the following way:

$$M_f(g)(x) = f(x) \cdot g(x), \ \forall g \in L_2(\Omega).$$

M_f is linear and bounded with norm $\|f\|_{L_\infty}$. Its adjoint is $M_{\overline{f}}$.

- $L_\infty(\Omega)$ is a commutative Banach algebra (with the sum and the composition).

The inclusion

$$\begin{array}{rcl} \mathcal{J} : L_\infty(\Omega) & \hookrightarrow & B(L_2) \\ f & \longmapsto & M_f \end{array}$$

is an isometric $*$-homomorfism of Banach algebras.

- $\mathcal{J}(L_\infty(\Omega))$ is closed in the weak operator topology of $B(L_2)$.
- If $f = \chi_A$ is an indicator function, then $\mathcal{J}(f)$ is an orthogonal projection.

ALTERNATIVE DEFINITION OF L_p

- Assume you are given the algebra L_∞ and the integral defined over the set of simple functions $\mathcal{S}$.
- Define
$$L_p = \overline{\mathcal{S}}^{\|\cdot\|_p}, \ 1 \leq p < \infty.$$
- Idea: generalize this situation to some class of noncommutative algebras that are not necessarily L_∞ spaces.

Von Neumann Algebras

The correct notion for noncommutative L_∞ space is that of von Neumann algebras:

Definition

Let $\mathcal{H}$ be a Hilbert space. A subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ is called a von Neumann Algebra (VNA) if

1. $\mathbf{1} \in \mathcal{M}$.
2. $x \in \mathcal{M} \Leftrightarrow x^* \in \mathcal{M}$.
3. $\mathcal{M}$ is closed in the weak operator topology.

PROPERTIES OF VNAS

THEOREM (VON NEUMANN, 1929/1930, [2])

Let $\mathcal{M}$ a $*$-subalgebra of $B(\mathcal{H})$ cointaining $\mathbf{1}$. The following statements are equivalent:

1. $\mathcal{M}$ is closed in the weak operator topology.
2. $\mathcal{M}$ es closed in the strong operator topology.
3. $\mathcal{M} = \mathcal{M}''$, where $\mathcal{M}''$ stands for the bicommutant of $\mathcal{M}$.

If an operator x belongs to a VNA $\mathcal{M}$ then the following operators are also in $\mathcal{M}$:

- Its adjoint x^*.
- Its modulus $|x| = (x^*x)^{\frac{1}{2}}$.
- Its spectral projections.
- The operator $f(|x|)$, if $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable and bounded on $\sigma(|x|)$.

Normal semifinite faithful traces

The positive cone of a VNA is

$$\mathcal{M}_+ = \{x \in \mathcal{M} : \langle xh, h \rangle \geq 0 \; \forall h \in \mathcal{H}\} = \{x \in \mathcal{M} : x = y^* y\}.$$

A trace τ is a generalized linear positive functional on $\mathcal{M}$ such that $\tau(x^* x) = \tau(x x^*)$.

- τ is *normal* if for all increasing bounded net $\{x_j\}_{j \in \mathcal{J}} \subset \mathcal{M}_+$,

$$\tau(\sup_{j \in \mathcal{J}} x_j) = \sup_{j \in \mathcal{J}} \tau(x_j) \text{ (Monotone Convergence).}$$

- τ is *semifinite* if for all nonzero $x \in \mathcal{M}_+$, there exists $0 \neq y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$.
- τ is *faithful* if, for $x \in \mathcal{M}_+$, $\tau(x) = 0 \Leftrightarrow x = 0$.

Definition

Let $\mathcal{M}$ be a VNA, and assume τ is a n.s.f. trace on $\mathcal{M}$. The pair $(\mathcal{M}, \tau)$ is called a *noncommutative measure space*.

NONCOMMUTATIVE L_p SPACES

Set

$$S_+(\mathcal{M}) = \{x \in \mathcal{M}_+ : \tau(s(x)) < \infty\},$$

where the support $s(x)$ is defined as the minimal projection such that $s(x)xs(x) = x$, and $S(\mathcal{M}) = \text{span}\{S_+(\mathcal{M})\}$.

- For general operators x, we have left and right supports, which we denote $l(x)$ and $r(x)$.
- $S(\mathcal{M})$ is the set of operators whose supports are majorised by a projection of finite trace.
- $S(\mathcal{M})$ is a noncommutative analogue of the space of bounded functions whose support has finite measure.

Recall that, for $x \in \mathcal{M}$, we have the formula

$$|x|^p = \int_{\sigma(|x|)} \lambda^p de_{|x|}(\lambda) \ \in \ S(\mathcal{M}).$$

Definition

Define $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$, $0 < p < \infty$, and

$$L_p = \overline{S(\mathcal{M})}^{\|\cdot\|_p}.$$

We put for convenience $L_\infty(\mathcal{M}) = \mathcal{M}$.

- L_p is a Banach space, $1 \leq p < \infty$.
- For $0 < p < 1$, $\|\cdot\|_p$ is a p-norm on $L_p(\mathcal{M})$.

BASIC PROPERTIES OF L_p

Noncommutative L_p spaces inherit certain properties from classical ones:

- Minkowski inequality. If $p \geq 1$,

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p,\ x, y \in L_p(\mathcal{M}).$$

- Hölder inequality: if $1/r = 1/p + 1/q$,

$$\|xy\|_r \leq \|x\|_p \|x\|_q,\ x \in L_p(\mathcal{M}),\ y \in L_q(\mathcal{M}).$$

- Duality:

$$(L_p(\mathcal{M}))^* = L_{p'},\ p > 1,\ \frac{1}{p} + \frac{1}{p'} = 1.$$

$$(L_1(\mathcal{M}))^* = L_\infty(\mathcal{M}).$$

- Interpolation.

MEASURABLE OPERATORS

- A linear operator x (not necessarily bounded) is *affiliated* to $\mathcal{M}$ if it commutes with the operators in $\mathcal{M}'$:

$$xy = yx \text{ if } y \in \mathcal{M}'.$$

- An operator x is said to be τ-*measurable* if it is affiliated to $\mathcal{M}$ if for all $\delta > 0$, there exists a projection e such that $e(\mathcal{H}) \subset \mathcal{D}(x)$ (the domain of x) and $\tau(\mathbf{1} - e) < \delta$.
- The set of τ-measurable operators is called $L_0(\mathcal{M})$.

ALTERNATIVE DEFINITION OF L_p

$$L_p = \{x \in L_0(\mathcal{M}) : \|x\|_p < \infty\}.$$

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- $0 \leq x \leq y$ does not imply in general $x^\alpha \leq y^\alpha$ (it does if $\alpha \leq 1$).
- $x \mapsto x^p$ is not operator convex if $\alpha > 2$.

1- Commutative and semicommutative cases

- Classical L_p spaces are noncommutative L_p spaces.
- Let (Ω, Σ, μ) be a classical measure space, and $(\mathcal{M}, \tau)$ a noncommutative measure space. Consider

$$\mathcal{A} = L_\infty(\Omega, \mu) \otimes \mathcal{M},$$

the von Neumann algebra tensor product of $L_\infty(\Omega, \mu)$ and $\mathcal{M}$. $\mathcal{A}$ is a VNA which is isomorphical to the Bochner space $L_p(\Omega; L_p(\mathcal{M}))$. The trace $\int \otimes \tau$ of a vector valued function $f \in \mathcal{A}$ is defined as

$$\left(\int \otimes \tau \right)(f) = \int_\Omega \tau(f) d\mu.$$

2- Schatten classes

- $\mathcal{M} = B(\mathcal{H})$.
- Given an orthogonal basis $\{\psi_j\}_{j\in\mathcal{J}}$, we define the trace

$$\begin{array}{rcl} \tau : \mathcal{M}_+ & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & \mathrm{Tr}(x) = \sum_{j\in\mathcal{J}} < \psi_j, x\psi_j >, \end{array}$$

- Tr is a n.s.f. trace on $B(\mathcal{H})$.
- A positive operator x of finite trace is compact, i.e

$$x(h) = \sum_{n=1}^{\infty} \lambda_n(x)\langle\psi_n, h\rangle\psi_n, \; h \in \mathcal{H},$$

and so $\mathrm{Tr}(x) = \sum_{n=1}^{\infty} \lambda_n(x)$. Consequently,

$$L_p(\mathcal{M}) = \left\{ x \text{ compact} : \sum_{n=1}^{\infty} \lambda_n(|x|^p) = \sum_{n=1}^{\infty} (\lambda_n(|x|))^p \text{ is finite} \right\}.$$

NONCOMMUTATIVE TORUS

Let u and v be unitary operators acting on $\mathcal{H}$ and satisfying

$$uv = e^{2\pi i\theta} vu,$$

with θ irrational. Define

$$B_\theta = \{\text{Polynomials in variables } u,\ u^*,\ v,\ v^*\}.$$

Its closure in the weak operator topology is a VNA, $\mathcal{M}_\theta$, and is called *noncommutative torus assocciated to the rotation of angle* θ. It can be seen that $\mathcal{M}_\theta$ does not depend on u and v.

TRACE ON $\mathcal{M}_\theta$

- In B_θ we define the Fourier coeficients in the natural way: if $x = \sum_{j,k=1}^{n,m} \alpha_{jk} u^j v^k$, then $\hat{x}(j,k) = \alpha_{jk}$.
- The trace τ_θ is defined so that $(\mathcal{M}_\theta, \tau_\theta)$ generalizes the definition of the classical torus:

$$\begin{aligned} \tau_\theta : B_\theta &\longrightarrow \mathbb{C} \\ x &\longmapsto \hat{x}(0,0) \end{aligned}$$

- The definition can be extended to a n.s.f. trace such that $\tau_\theta(\mathbf{1}) = 1$.

Calderón-Zygmund operators

Work in the semicommutative case, $\mathcal{A} = L_\infty(\Omega) \otimes \mathcal{M}$.

Definition (Calderón-Zygmund scalar kernel)

A function $K : \mathbb{R}^{2n} \setminus \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\} \to \mathbb{R}$ is called a Calderón-Zygmund kernel if it verifies:

$$|K(x,y)| \leq \frac{C_K}{|x-y|^k}.$$

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \leq C_K \frac{|y-y'|^\delta}{|x-y|^{k+\delta}} \text{ if } |y-y'| \leq \frac{1}{2}|x-y|.$$

Let K be a Calderón-Zygmund kernel. The operator T is formally defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy.$$

Known results

- L_p boundedness of Calderón-Zygmund operators in the commutative case (Calderón-Zygmund , [1]).
- Calderón-Zygmund decomposition for semicommutative case (Parcet, [3]).
- Consequence: L_p boundedness of semicommutative Calderón-Zygmund operators with doubling measures (Parcet, [3]).

Open problems

- Semicommutative case with nondoubling measures.
- Fully noncommutative case.

References

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THANK YOU!