

Motivic Invariants of Hypersurface Singularities

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1 Singular points of hypersurfaces

2 Motivic Invariants

3 Results

Singular points

$f : \mathbb{A}_{\mathbb{C}}^{d+1} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ algebraic or analytic morphism, $x \in f^{-1}\{0\}$
singular point.

- Milnor fibration
- Embedded resolution of singularities (Hironaka)

Topology of $F_{f,x}$

- Betti numbers $b_i(F_{f,x}) := \dim_{\mathbb{C}} H_c^i(F_{f,x}, \mathbb{C})$
- Euler characteristic $\chi(F_{f,x}) := \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H_c^i(F_{f,x}, \mathbb{C})$

If x is an **isolated** singular point of $f^{-1}\{0\}$,

$$b_i(F_{f,x}) = \dim_{\mathbb{C}} H_c^i(F_{f,x}, \mathbb{C}) = \begin{cases} 1 & \text{if } i = 0, \\ \mu(f, x) & \text{if } i = d, \\ 0 & \text{if } i \neq 0, d \end{cases}$$

where $\mu(f, x) = \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_{d+1}]]}{\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{d+1}}\right)}$ is the **Milnor number**.

$$\implies \chi(F_{f,x}) = 1 + (-1)^d \mu(f, x).$$

Geometric Monodromy The diffeomorphism of $F_{f,x}$ corresponding to going once around the boundary of \mathbf{D}_δ .

Algebraic Monodromy Operator of the cohomology ring induced by the geometric monodromy

$$M_{f,x}^\bullet : H_c^\bullet(F_{f,x}, \mathbb{C}) \rightarrow H_c^\bullet(F_{f,x}, \mathbb{C}).$$

Theorem (Monodromy Theorem)

The endomorphism $M_{f,x}$ is quasi-unipotent: $\exists A, B \in \mathbb{N}$ such that

$$(M_{f,x}^A - I)^B = 0.$$

\implies *The eigenvalues of $M_{f,x}$ are roots of unit.*

Monodromy Zeta Function

$$\xi_{M_{f,x}}(s) := \prod_{i \geq 0} (\det(I - sM_{f,x}^i))^{(-1)^{i+1}}.$$

$$\implies \deg \xi_{M_{f,x}}(s) = \chi(F_{f,x}).$$

Theorem (Steenbrink, Saito, Navarro-Aznar)

$H_c^i(F_{f,x}, \mathbb{Q})$ enjoys a mixed Hodge structure compatible with the operator $M_{f,x}$:

$$[H_c^i(F_{f,x}, \mathbb{Q})] := \sum_m [Gr_m^W H_c^i(F_{f,x}, \mathbb{Q})] \in K_0(HS^{mon}).$$

Hodge-Steenbrink spectrum of f at a singular point

$$hsp : K_0(HS^{mon}) \rightarrow \mathbf{Z}[t^{1/N}] := \bigcup_{n \geq 1} \mathbf{Z}[t^{1/n}, t^{-1/n}]$$

$$H_c^d(F_{f,x}, \mathbb{Q}) \mapsto \sum_{\alpha \in h \cap [0,1[} t^\alpha \left(\sum_{p,q \in \mathbb{Z}} h_\alpha^{p,q} t^p \right),$$

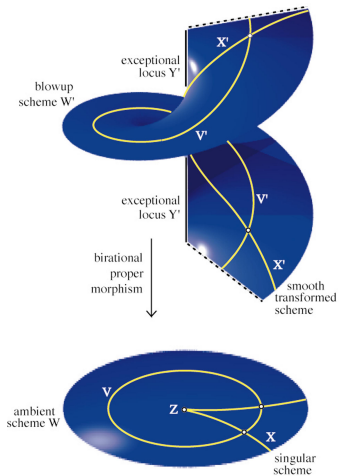
where $h_\alpha^{p,q} = \dim_{\mathbb{C}} H_\alpha^{p,q}$ corresponds to the eigenvalue $e^{-2\pi\sqrt{-1}\alpha}$.

Embedded Resolution of singularities of $f^{-1}\{0\}$

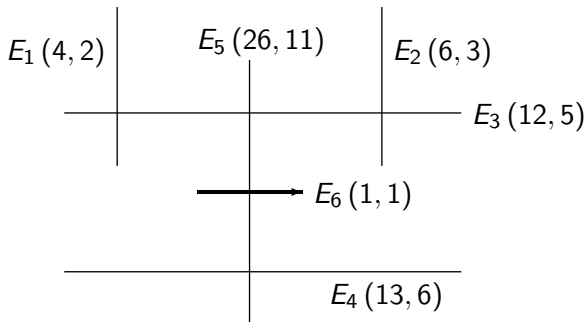
$h : Y \rightarrow \mathbb{A}_{\mathbb{C}}^{d+1}$ proper birational morphism with Y smooth complex algebraic variety such that:

- $h^{-1}(f^{-1}\{0\})$ is a normal crossing divisor (locally $x_1 \cdots x_s = 0$)
- $h|_Y : Y - (f \circ h)^{-1}\{0\} \rightarrow \mathbb{A}_{\mathbb{C}}^{d+1} - f^{-1}\{0\}$ is an isomorphism.

Blow-up of the plane at the origin:

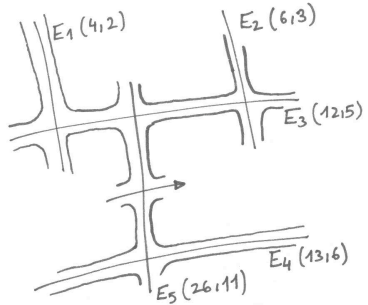
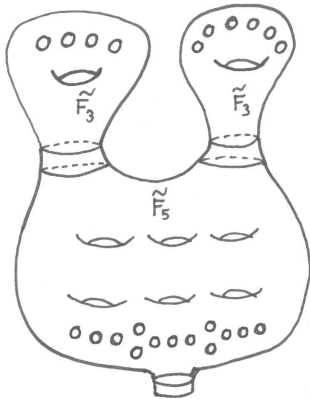


Example: $f = (y^2 - x^3)^2 - x^5y$.



$F_{f,0}$ is a genus 8 surface with a hole (due to the intersection with the strict transform E_6 of $f^{-1}\{0\}$).

$$\mu(f, 0) = b_1(F_{f,0}) = 16 \quad \text{and} \quad \chi(F_{f,0}) = -15$$



A'Campo Formula for the Monodromy Zeta Function:

$$\xi_{M_{f,x}}(s) = \frac{(1-s^4)(1-s^6)(1-s^{13})}{(1-s^{12})(1-s^{26})}$$

Formulas by Schrauwen, Steenbrink & Stevens (resolution) + Saito & Nemethi (Puiseux pairs)

$$hsp(f, x) = t^{5/12} + t^{11/12} + t^{13/12} + t^{19/12} + \sum_{i=0}^{11} t^{\frac{15+2i}{26}}$$

Idea (Denef-Loeser):

To substitute the Milnor fibre $F_{f,x}$ and its monodromy operator $M_{f,x}$ by the **motivic Milnor fibre** $S_{f,x} \in K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$.

$K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})$ is the **Grothendieck ring** of complex algebraic varieties endowed with a good action of the profinite group of roots of unity; i.e.

- $[X] = [Y]$ if X is isomorphic to Y ,
- $[X] = [X - Y] + [Y]$ if $Y \subset X$ closed,
- $[X] \cdot [Y] = [X \times_{\mathbb{C}} Y]$,
- Compatibility with the $\hat{\mu}$ -actions.

Notation: $\mathbb{L} := [\mathbb{A}_{\mathbb{C}}^1]$, $\mu_n := \text{Spec} \mathbb{C}[x]/(x^n - 1)$ and $\hat{\mu} := \varprojlim \mu_n$.

Theorem (Deligne)

If X is a complex algebraic variety, $H_c^i(X, \mathbb{Q})$ has a mixed Hodge structure. Furthermore, if X had a good $\hat{\mu}$ -action, then the mixed Hodge structure is endowed with a quasi-unipotent homomorphism.

Hodge Characteristic

$$\chi_h^{mon} : K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}}) \rightarrow K_0(\text{HS}^{mon})$$

$$[X, \hat{\mu}] \mapsto \sum_i (-1)^i [H_c^i(X, \mathbb{Q})]$$

and spectrum

$$Sp([X, \hat{\mu}]) := (hsp \circ \chi_h)([X, \hat{\mu}])$$

How to define motivic invariants like $S_{f,x}$? Use a motivic measure in the arc space (Kontsevitch, Denef & Loeser)

$\mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0$ **arcs of $\mathbb{A}_{\mathbb{C}}^{d+1}$ centered at 0**

$\varphi \equiv (\varphi_1(t), \dots, \varphi_{d+1}(t)) \in (\mathbb{C}[[t]])^{d+1}$ such that
 $\varphi_i(0) = 0$ for all $1 \leq i \leq d+1$

$\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0$ **n-jets of $\mathbb{A}_{\mathbb{C}}^{d+1}$ centered at 0:**

$\varphi \equiv (\varphi_1(t), \dots, \varphi_{d+1}(t)) \in (\mathbb{C}[t]/(t^{n+1}))^{d+1}$ such that
 $\varphi_i(0) = 0$ for all $1 \leq i \leq d+1$

Truncation map $\pi_n : \mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0 \rightarrow \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0$

$f : \mathbb{A}_{\mathbb{C}}^{d+1} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ morphism with $f(0) = 0$ and $\varphi \in \mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0$

$$\implies f \circ \varphi = a_s t^s + a_{s+1} t^{s+1} + \dots, a_s \neq 0$$

- $\text{ord}_t f \circ \varphi := s \quad \text{ac}(f \circ \varphi) := a_s$

Lemma

The set $\mathcal{X}_{n,1} := \{\varphi \in \mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0 \mid \text{ord}_t f \circ \varphi = n, \text{ac}(f \circ \varphi) = 1\}$ is a **constructible** subset of $\mathcal{L}_n(\mathbb{A}_{\mathbb{C}}^{d+1})_0$ and μ_n acts on it.

Denote $\mathcal{Z}_{n,1}$ the preimage $\mathcal{X}_{n,1}$ in $\mathcal{L}(\mathbb{A}_{\mathbb{C}}^{d+1})_0$; i.e. $\pi_n(\mathcal{Z}_{n,1}) = \mathcal{X}_{n,1}$.

The **motivic measure** is given by

$$\mu_{mot}(\mathcal{Z}_{n,1}) := [\mathcal{X}_{n,1}] \cdot \mathbb{L}^{-n(d+1)} \in K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$$

Motivic Zeta Function

$$Z(f, T) := \sum \mu_{mot}(\mathcal{Z}_{n,1}) T^n.$$

The motivic zeta function has two connections with monodromy:

- monodromy conjecture
- motivic Milnor fibre

which are based in the following rationality result:

Theorem (Denef-Loeser)

These zeta functions are rational functions w.r.t. T . There are expressions like

$$Z(f, T) = \sum_{I \subset J} (\mathbb{L} - 1)^{|I|-1} [\tilde{E}_I^\circ, \tilde{\mu}] \prod_{i \in I} \frac{\mathbb{L}^{-\nu_i} T^{N_i}}{1 - \mathbb{L}^{-\nu_i} T^{N_i}},$$

with help of embedded resolution of the singularity $(f^{-1}\{0\}, 0)$.

Monodromy Conjecture

If $\frac{\nu_i}{N_i}$ is a “pole” of $\chi_{top}(Z^{naive}(f, \mathbb{L}^{-s}))$ then $\exp(-2\pi \frac{\nu_i}{N_i})$ is an eigenvalue of the algebraic monodromy at some point $x \in f^{-1}\{0\}$.

Note: “Naive” means to forget the condition $ac(f \circ \varphi) = 1$.

The **Motivic Milnor Fibre** is defined as

$$S_{f,x} := -\lim_{T \rightarrow \infty} Z(f, T) \in K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}}).$$

Theorem (Denef-Loeser)

$$\chi_h(F_{f,x}) = \chi_h(S_{f,x}) \in K_0(HS^{mon})$$

$\implies S_{f,x}$ recovers cohomological invariants of $F_{f,x}$; e.g.

$$hsp(f, x) = Sp((-1)^d(S_{f,x} - 1))$$

Example: $f = (y^2 - x^3)^2 - x^5y$.

Formulas by Loeser and Veys

$$\chi_{top}(Z^{naive}(f, \mathbb{L}^s)) = \frac{10}{5 + 12s} - \frac{1}{15} \left(14 \frac{11}{11 + 26s} + \frac{1}{1 + s} \right)$$

Formula by Guibert

$$S_{f,0} = [\{(x, y) \in \mathbb{C}^2 \mid (y^2 - x^3)^2 = 1\}] \\ + [\{(x, y) \in \mathbb{C}^2 \mid y^2 - x^{13} = 1\}] - [\mu_2]\mathbb{L} - \mathbb{L} + 1$$

Theorem (González-Pérez & —)

The naive and motivic zeta functions $Z(f, T)$, the motivic Milnor fibre $S_{f,0}$ and the spectrum $\text{hsp}(f, 0)$ of an irreducible quasi-ordinary hypersurface singularity are determined by the embedded topological type.

Corolary (Budur, González-Pérez & —)

The log canonical threshold of an irreducible quasi-ordinary hypersurface singularity is also determined by the embedded topological type.

Explicit formulas for all these invariants in terms of the characteristic exponents.

Definition

A germ $(S, 0) \subset (\mathbb{A}_{\mathbb{C}}^{d+1}, 0)$ of complex hypersurface is **quasi-ordinary** if there exists a proper and finite morphism onto $(\mathbb{A}_{\mathbb{C}}^d, 0)$ whose discriminant lies on $x_1 \cdots x_d = 0$.

Data associated to an irreducible quasi-ordinary hypersurface:

- 1 a root $\zeta \sum c_{\lambda} x^{\lambda} \in \mathbb{C}[[x_1^{1/n}, \dots, x_d^{1/n}]]$ of f (Abhyankar-Jung).
- 2 a finite sequence of partially ordered **characteristic exponents**: $\lambda_1 < \dots < \lambda_g$ (equiv. to the embedded topological type).
- 3 a finite sequence of **semiroots**: $f_0, f_1, \dots, f_g = f$.
- 4 a d -dimensional **characteristic fan** Θ in \mathbf{R}^{d+g+1} (determined by the characteristic exponents).

Embedded resolution of irreducible quasi-ordinary hypersurfaces (González-Pérez) has two steps:

$$W \xrightarrow{\pi_2} U \xrightarrow{\pi_1} \mathbb{A}_{\mathbb{C}}^{d+1}$$

- 1 π_1 **canonical partial resolution**, sequence of g toric morphisms determined by the characteristic exponents.
- 2 π_2 standard/combinatorial **toric embedded resolution** of U .

Subdivision of the arc space:

Given an arc φ , measure its **contact order** with the coordinate hyperplanes $\mathbf{x} = (x_1, \dots, x_d)$ and the semi-roots $\mathbf{f} = (f_0, \dots, f_g)$.

$$H_{\mathbf{k},1} := \{\varphi \mid \mathbf{ord}(\mathbf{x}, \mathbf{f})(\varphi) = \mathbf{k}, ac(f \circ \varphi) = 1\} \text{ with } \mathbf{k} \in \mathbb{Z}_{\geq 0}^{d+g+1}$$

Show that the indices \mathbf{k} such that $H_{\mathbf{k},1} \neq \emptyset$ are the integer points of the interior of the **characteristic fan** $\Theta \subset \mathbb{R}_{\geq 0}^{d+g+1}$.

With help of the second step π_2 of the embedded resolution and the **Change of Variable Theorem** of Kontsevitch and Denef & Loeser compute the **motivic measure** of $H_{k,1}$

Show $\mu(H_{k,1})$ does not depend on the choice of π_2 .

References:

P.D. González-Pérez & —, *Motivic Milnor fiber of a Quasi-Ordinary Hypersurface*, arXiv:1105.2480

N. Budur, P.D. González-Pérez & —, *Log Canonical Thresholds of Quasi-Ordinary Hypersurface Singularities*, arXiv:1105.2794

Thank you for your attention!!