Introduction to Game Theory

Anna Khmelnitskaya

Saint-Petersburg State University, Russia

Workshop of Young Researchers in Mathematics 2011

School of Mathematics, Universidad Complutense de Madrid

September 21, 2011

Common features of all games:

1. there is a set of at least two players;
2. players follow some set of rules;
3. interests of different players are different.

Game theory (GT) is a theory of ***rational*** behavior of people with ***nonidentical*** interests.

Game theory can be defined as the theory of mathematical models of conflict and cooperation between intelligent rational decision-makers.

Its area of applications extends considerably beyond games in the usual sense.

Game theory is applicable whenever at least two individuals – people, companies, political parties, or nations – confront situations where the outcome for each depends on the behavior of all.

The models of game theory are highly abstract representations of classes of real-life situations.

By the term ***game*** we mean any such situation, defined by some set of ***rules***.

The term ***play*** refers to a particular occurrence of a game.

Modern game theory may be said to begin with the work of Zermelo (1913), Borel (1921), von Neumann (1928), and the great seminal book ***"Theory of Games and Economic Behavior"*** of von Neumann and Morgenstern (1944).

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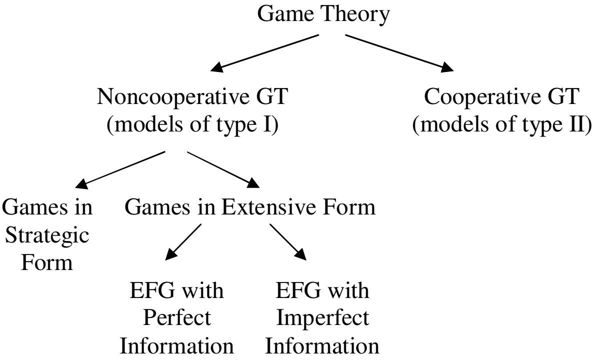
In all GT models the basic entity is a ***player***.

Once we defined the set of players we may distinguish between two types of models:

- primitives are the sets of possible actions of ***individual*** players;
- primitives are the sets of possible ***joint*** actions of ***groups*** of players.



A *strategic-form* game is $\Gamma = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$, where

$N = \{1, \dots, n\}$, $n \geq 2$, is a set of *players*,

S_i is a nonempty set of possible *strategies* (or *pure strategies*) of player i. When game Γ is played, each player i must choose $s_i \in S_i$.

Strategy profile $s = (s_1, \dots, s_n)$ is an *outcome* of the game Γ.

Let $S = \{s = (s_1, \dots, s_n) | s_i \in S_i\}$, the set of all possible outcomes.

$u_i \colon S \to \mathbb{R}$,

The number $u_i(s)$ represents the expected utility *payoff* of player i if the outcome of the game is s.

Equilibrium:

All players in n are happy to find such $s^* \in S$ that

$$u_i(s) \leq u_i(s^*), \qquad \text{for all } \ i \in N, \ s \in S.$$

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Notation:

Let $s \in S$, $s = (s_1, \ldots, s_n)$, $s_i \in S_i$.
$s \| t_i = (s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$, i.e. player i replaces his strategy s_i by t_i.

Nash Equilibrium (1950)

An outcome $s^* \in S$ is Nash equilibrium if for all $i \in N$,

$$u_i(s^*) \geq u_i(s^* \| s_i), \qquad \text{for all } s_i \in S_i.$$

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There is a convenient representation of a two-person ($N = \{1, 2\}$) strategic game in which each player has a finite set of strategies.

Let $S_1 = X = \{x_1, \dots, x_n\}$, $S_2 = Y = \{y_1, \dots, y_m\}$,

$$a_{ij} = u_1(x_i, y_j), \; b_{ij} = u_2(x_i, y_j).$$

	y_1	$\dots$	y_m
x_1	(a_{11}, b_{11})	$\dots$	(a_{1m}, b_{1m})
	$\dots$		$\dots$
x_n	(a_{n1}, b_{n1})	$\dots$	(a_{nm}, b_{nm})

Battle of the Sexes

This game models a situation in which two players wish to coordinate their behavior but have conflict interests - the wife wants to go to the concert but the husband prefers soccer. But in any case they prefer to spend evening together.

The game has two Nash equilibria: (c,c) and (s,s).

	concert	soccer
concert	2,1	0,0
soccer	0,0	1,2

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Two suspects in a crime are put into separate cells. If they both confess, each will be sentenced to five years in prison. If only one of them confesses, he will be freed and used as a witness against the other, who will receive a sentence of eight years. If neither confesses, they will both be convicted of a minor offence and spend one year in prison.

The best outcome for the players is that neither confesses, but each player has an incentive to be a "free rider"...

Whatever one player does, the other prefers *confess* to *don't confess*, so the game has unique Nash equilibrium (c,c).

	don't confess	confess
don't confess	-1,-1	-8,0
confess	0,-8	-5,-5

The Prisoner's Dilemma

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Hawk-Dove

Two animals are fighting over some prey. Each can behave like a dove or like a hawk. the best outcome for each animal is that in which it acts like a hawk while the other acts like a dove; the worst outcome is that in which both animals act like hawks. Each animal prefers to be hawkish if its opponent is dovish and dovish if its opponent is hawkish.

The game has two Nash equilibria, (d,h) and (h,d), corresponding to two different conventions about the player who yields.

	dove	hawk
dove	3,3	1,4
hawk	4,1	0,0

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Matching Pennies

Each of two people chooses either Head or Tail. If the choices differ, person 1 pays person 2 one euro; if they are the same, person 2 pays person 1 one euro. Each person cares only about the amount of money that he receives.

The game has **no** Nash equilibria.

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head	1,-1	-1,1
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Definition

A strategic game $\Gamma = \langle \{1, 2\}; S_1, S_2; u_1, u_2 \rangle$ is *strictly competitive* if for any outcome $s \in S$, $s = (s_1, s_2)$, $s_1 \in S_1$, $s_2 \in S_2$, we have $u_2(s) = -u_1(s)$.

Another name is a *zero-sum* game.

In what follows we denote $X = S_1$, $Y = S_2$, and $u(s) = u_1(s)$.

If an outcome (x^*, y^*), $x^* \in X$, $y^* \in Y$, is a Nash equilibrium, then

$$u(x, y^*) \le u(x^*, y^*) \le u(x^*, y), \qquad \text{for all } x \in X, y \in Y,$$

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i.e., Nash equilibrium is a *saddle point*.

If player 2 chooses strategy $y \in Y$, then player 1 can get at most

$$\max_{x \in X} u(x, y).$$

Similarly, if player 1 fixes strategy $x \in X$, then player 2 looses at least

$$\min_{y \in Y} u(x, y).$$

Definition

A strategy $x^* \in X$ is a *best guaranteed outcome for player 1* if

$$\min_{y \in Y} u(x^*, y) \geq \min_{y \in Y} u(x, y), \qquad \text{for all } \; x \in X;$$

similarly, $y^* \in Y$ is a *best guaranteed outcome for player 2* if

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In general always

$$\max_{x \in X} \min_{y \in Y} u(x, y) \leq \min_{y \in Y} \max_{x \in X} u(x, y).$$

MinMax Theorem

An outcome (x^*, y^*) is a Nash equilibrium in a strictly competitive game $\Gamma = \langle \{1, 2\}; X, Y; u \rangle$ if and only if

$$\max_{x \in X} \min_{y \in Y} u(x, y) = \min_{y \in Y} \max_{x \in X} u(x, y) = u(x^*, y^*),$$

where x^* is the best outcome for player 1 while y^* is the best outcome for player 2.

Corollary:

All Nash equilibria of any game yield the same payoffs.

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Any finite strictly competitive strategic game admits simple and convenient representation in the matrix form.

Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$,

$$a_{ij} = u_1(x_i, y_j),\ \ u_2(x_i, y_j) = -u_1(x_i, y_j) = -a_{ij}.$$

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{matrix}$$

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\downarrow & \downarrow & \dots & \downarrow & & \\
max & max & \dots & max & &
\end{array}$$

Strictly Competitive Games

Any finite strictly competitive strategic game admits simple and convenient representation in the matrix form.

Let $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_m\}$,

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$$\left.\begin{array}{cccccc} a_{11} & a_{12} & \dots & a_{1m} & \to & min \\ a_{21} & a_{22} & \dots & a_{2m} & \to & min \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} & \to & min \end{array}\right\} \stackrel{max}{\Longrightarrow} m$$

$$\begin{array}{cccc} \downarrow & \downarrow & \dots & \downarrow \end{array}$$

$$\underbrace{max \;\; max \;\; \dots \;\; max}$$

$$\Downarrow \; min$$

$$M$$

Let $\Gamma = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a strategic game.

A *mixed strategy* of player i is a probability distribution σ_i over the set S_i of its pure strategies.

$\sigma_i(s_i)$ is the probability that player i chooses strategy $s_i \in S_i$.

We assume that mixed strategies of different players are independent, i.e., the set of probability distributions over S is given by $\Sigma = \times_{i \in N} \Sigma_i$.

Definition

The *mixed extension* of the strategic game $\Gamma = \langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ is the strategic game $\Gamma^* = \langle N, \{\Sigma_i\}_{i \in N}, \{U_i\}_{i \in N} \rangle$ in which Σ_i is the set of probability distributions over S_i, and U_i is the expected value of u_i under the lottery over S that is induced by $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i \in \Sigma_i$, i.e.,

$$U_i(\sigma) = \sum_{s \in S} u_i(s)\sigma(i).$$

Mixed Strategy Nash Equilibrium

Definition

A *mixed strategy Nash equilibrium of a strategic game* is a Nash equilibrium of its mixed extension.

Theorem (Nash, 1950)

Every finite strategic game has a mixed strategy Nash equilibrium.

Remark:

For matrix games this result was obtained by von Neumann in 1928.

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$N = \{1, \ldots, n\}$ is a finite set of $n \geq 2$ players.

A subset $S \subseteq N$ (or $S \in 2^N$) of s players is a *coalition*.

$v(S)$ presents the *worth* of the coalition S.

$v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, is a *characteristic function*.

A *cooperative TU game* is a pair $\langle N, v \rangle$.

$\mathcal{G}_N$ is the class of TU games with a fixed N.

A game v is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.

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Every $x \in \mathbb{R}^n$ can be considered as a *payoff vector* to N.

$x \in \mathbb{R}^n$ is *efficient in the game v* if $x(N) = v(N)$.

For any $x \in \mathbb{R}^n$ and any $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$.

The *imputation set* of a game $v \in \mathcal{G}_N$ is
$I(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \ \ x_i \geq v(i), \forall i \in N\}$.

Definition

The *core* (Gillies, 1959) of a game $v \in \mathcal{G}_N$ is

$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \, x(S) \geq v(S), \text{ for all } S \subseteq N, \ S \neq \emptyset\}.$$

Bondareva (1963), Shapley (1967)

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$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \; x(S) \geq v(S), \text{ for all } S \subseteq N, \; S \neq \emptyset\}.$$

Bondareva (1963), Shapley (1967)

Every $x \in \mathbb{R}^n$ can be considered as a *payoff vector* to N.

$x \in \mathbb{R}^n$ is *efficient in the game v* if $x(N) = v(N)$.

For any $x \in \mathbb{R}^n$ and any $S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$.

The *imputation set* of a game $v \in \mathcal{G}_N$ is
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The most reasonable approach to the choice of a solution concept is the *axiomatic approach* that allows choosing a solution satisfying a number of a priori chosen properties stated as axioms reflecting reasonable under the circumstances criteria, such as social efficiency, fairness, marginality, simplification of computational aspects etc,.

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A player i is a *null-player* in the game $v \in \mathcal{G}$ if
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Shapley value

Theorem (Shapley, 1953)

There is a unique value defined on the class $\mathcal{G}_N$ that satisfies efficiency, symmetry, null-player property, and additivity, and for all $v \in \mathcal{G}_N$, for every $i \in N$, it is given by

$$Sh_i(v) = \sum_{s=0}^{n-1} \frac{s!(n-s-1)!}{n!} \sum_{\substack{S \subseteq N \setminus \{i\} \\ |S|=s}} \big(v(S \cup \{i\}) - v(S)\big).$$

A value ξ is *marginalist* if, for all $v \in \mathcal{G}$, for every $i \in N$,

$$\xi_i(v) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N \setminus i}),$$

where $\phi_i \colon \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

Theorem (Young, 1985)

The only efficient, symmetric, and marginalist value defined on the class $\mathcal{G}_N$ is the Shapley value.

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Let Π be a set of all $n!$ permutations $\pi : N \to N$ of N.
Denote by $\pi^i = \{j \in N \,|\, \pi(j) \leq \pi(i)\}$ the set of players with rank number not greater than the rank number of i, including i itself.

The *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of a game v and a permutation π is given by

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In general, $Sh(v)$ is not a core selector.

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Definition

A game v is *convex* (Shapley, 1971) if for all $i \in N$ and $S \subseteq T \subseteq N \setminus i$,

$$v(S \cup i) - v(S) \leq v(T \cup i) - v(T).$$

In a convex game v

- every $m^\pi(v) = \{m_i^\pi(v)\}_{i \in N} \in C(v)$, $\pi \in \Pi$,

  $\{m_i^\pi(v)\}_{i \in N}$ creates a set of extreme points for $C(v)$,

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A *bankruptcy problem* $(E; d)$ is defined by a set of claimants N, an estate $E \in \mathbb{R}_+$ and a vector of claims $d \in \mathbb{R}^n_+$ assuming that the total claim of the creditors exceeds the estate,

$$d(N) = \sum_{i \in N} d_i > E.$$

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One Mishnah in the Babylonian Talmud discusses three bankruptcy problems of the division of the estate E of the died person, $E = 100,\ 200,$ and 300 respectively, among his three widows that according to his testament should get $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$ correspondingly. The Mishnah prescribes the following division

		Estate		
		100	200	300
Claim	d_1=100	33.33	50	50
	d_2=200	33.33	75	100
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$$x_1 \le x_2 \le \ldots \le x_n$$
$$(d_1 - x_1) \le (d_2 - x_2) \le \ldots \le (d_n - x_n)$$

The *bankruptcy game* $v_{E;d} \in G_N$ corresponding to bankruptcy problem $(E; d)$ is defined by Aumann and Mashler (1985) as

$$v_{E;d}(S) = \begin{cases} \max\{0,\ E - d(N \backslash S)\}, & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset. \end{cases}$$

		Estate		
		100	200	300
S	1	0	0	0
	2	0	0	0
	3	0	0	0
	12	0	0	0
	13	0	0	100
	23	0	100	200
	123	100	200	300

For a game v, the *excess* of a coalition $S \subseteq N$ with respect to a payoff vector $x \in \mathbb{R}^n$ is

$$e^v(S, x) = v(S) - x(S).$$

The *nucleolus* of a game v (Schmeidler, 1969) is a minimizer of the lexicographic ordering of components of the excess vector of a given game v arranged in decreasing order of their magnitude over the imputation set $I(v)$:

$$\nu(v) = x \in I(v): \;\; \theta(x) \preceq_{lex} \theta(y), \, \forall y \in I(v),$$

where $\theta(x) = (e(S_1, x), e(S_2, x), ..., e(S_{2^n-1}, x))$,
while $e(S_1, x) \geq e(S_2, x) \geq ... \geq e(S_{2^n-1}, x)$.

If $C(v) \neq \emptyset$ then $\nu(v) \in C(v)$.

Nucleolus

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For a game v we consider a *marginal worth vector* $m^v \in \mathbb{R}^n$ equal to the vector of marginal contributions to the grand coalition,

$$m_i^v = v(N) - v(N \backslash \{i\}), \qquad \text{for all } \ i \in N,$$

and the *gap* vector $g^v \in \mathbb{R}^{2^N}$,

$$g^v(S) = \begin{cases} \sum_{i \in S} m_i^v - v(S), & S \subseteq N, S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases}$$

that for every coalition $S \subseteq N$ measures the total coalitional surplus of marginal contributions to the grand coalition over its worth.

For any game v, the vector m^v provides upper bounds of the core: for any $x \in C(v)$,

$$x_i \leq m_i^v, \qquad \text{for all } \ i \in N.$$

In particular, for an arbitrary game v, the condition

$$v(N) \leq \sum_{i \in N} m_i^v$$

is a necessary (but not sufficient) condition for non-emptiness of the core, i.e., a strictly negative gap of the grand coalition $g^v(N) < 0$ implies $C(v) = \emptyset$.

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Definition

A game v is *1-convex* (Driessen, Tijs (1983), Driessen (1985)) if

$$0 \leq g^v(N) \leq g^v(S), \qquad \text{for all } S \subseteq N,\ S \neq \emptyset.$$

In a 1-convex game v,

- every 1-convex game has a nonempty core $C(v)$;
- for every efficient vector $x \in \mathbb{R}^n$,

$$x_i \leq m_i^v, \ \text{for all } i \in N \implies x \in C(v);$$

in particular, the characterizing property of a 1-convex game is:

$$\bar{m}^v(i) = \{\bar{m}_j^v(i)\}_{j \in N} \in C(V),$$

$$\bar{m}_j^v(i) = \begin{cases} v(N) - m^v(N \backslash i) = m_i^v - g^v(N), & j = i, \\ m_j^v, & j \neq i, \end{cases} \qquad \text{for all } j \in N;$$

moreover, $\{\bar{m}^v(i)\}_{i \in N}$ is a set of extreme points of $C(v)$, and $C(v) = co(\{\bar{m}^v(i)\}_{i \in N})$;

- the nucleolus coincides with the barycenter of the core vertices, and is given by

$$\nu_i(v) = m_i^v - \frac{g^v(N)}{n}, \qquad \text{for all } i \in N,$$

i.e., the nucleolus defined as a solution to some optimization problem that, in general, is difficult to compute, appears to be linear and thus simple to determine.

To a cost game $\langle N, c\rangle$ the associated (surplus) game $\langle N, v\rangle$ is

$$v(S) = \sum_{i\in S} c(i) - c(S), \qquad \text{for all } S \subseteq N.$$

The *core* of a cost game $c \in \mathcal{G}_N$ is

$$C(c) = \{x \in \mathbb{R}^n \mid x(N) = c(N),\ x(S) \leq c(S), \forall S \subseteq N, S \neq \emptyset\}.$$

A cost game c is *concave* if for all $i \in N$ and $S \subseteq T \subseteq N \setminus i$,

$$c(S \cup i) - c(S) \geq c(T \cup i) - c(T).$$

A cost game c is *1-concave* if

$$0 \geq g^v(N) \geq g^v(S), \qquad \text{for all } S \subseteq N, S \neq \emptyset.$$

N is a set of n players (universities)

G is a set of m goods (electronic journals)

$D = (d_{ij})_{\substack{i \in N \\ j \in G}}$ is a demand $(n \times m)$-matrix
$d_{ij} \geq 0$ is the number of units of jth journal in the historical demand of ith university

$c_j \geq 0$ is the cost per unit of jth journal based on the price of the paper version in the historical demand

$\alpha \in [0, 1]$ is the common discount percentage for goods that were never requested in the past;
in applications usually $\alpha = 10\%$.

The library cost game $\langle N, c^l \rangle$ is given by

$$c^l(S) = \begin{cases} \sum_{j \in G} \left[\sum_{i \in S} d_{ij} \right] c_j + \sum_{\substack{j \in G \\ \sum_{i \in S} d_{ij} = 0}} \alpha\, c_j, & S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases} \quad \text{for all } S \subseteq N.$$

Theorem

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The library game is a sum of games, one for each journal.

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The library game is a sum of games, one for each journal.

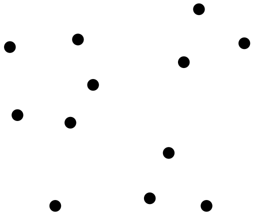
The library cost game $\langle N, c^l \rangle$ is given by

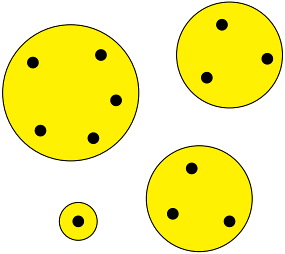
$$c^l(S) = \begin{cases} \sum_{j \in G} \Big[\sum_{i \in S} d_{ij} \Big] c_j + \sum_{\substack{j \in G \\ \sum_{i \in S} d_{ij} = 0}} \alpha\, c_j, & S \neq \emptyset, \\ 0, & S = \emptyset, \end{cases} \quad \text{for all } S \subseteq N.$$

Theorem

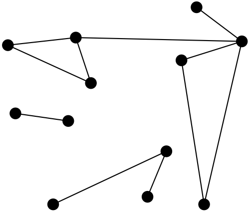
The library game c^l is 1-concave.

The library game is a sum of games, one for each journal.

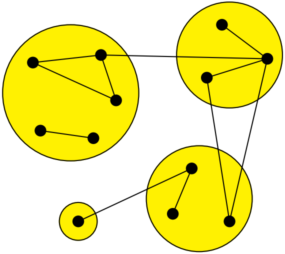




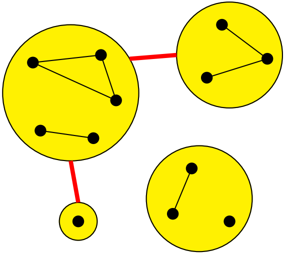
Aumann and Drèze (1974), Owen (1977)



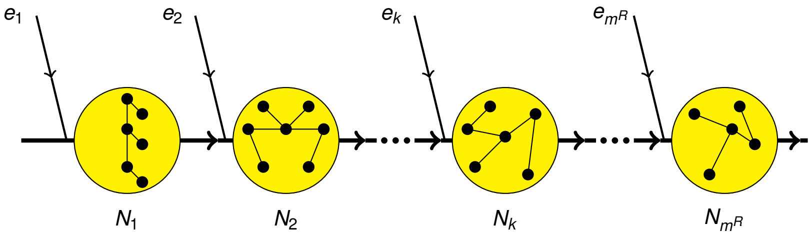
Myerson (1977)



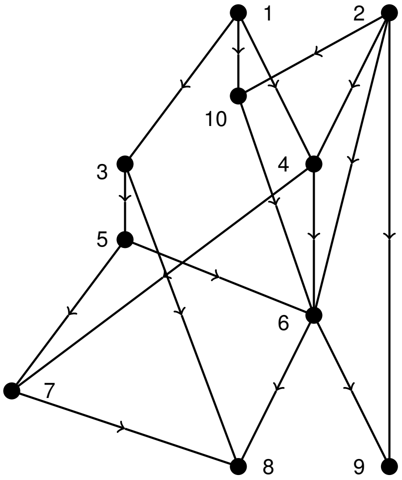
Vázquez-Brage, García-Jurado, and Carreras (1996)



Khmelnitskaya (2007)

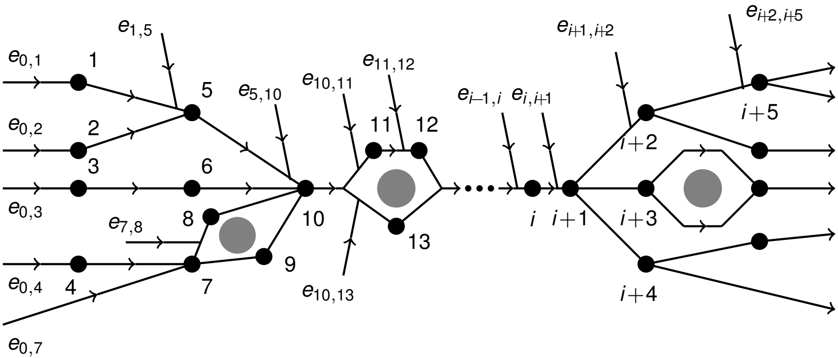


sharing an international river among multiple users without international firms



Khmelnitskaya, Talman (2010)

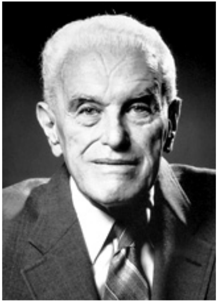
Games with Limited Cooperation



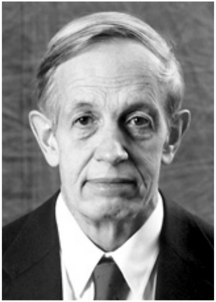
A river with multiple sources, a delta, and several islands along the river bed

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994

"for their pioneering analysis of equilibria in the theory of non-cooperative games"



John C. Harsanyi
(1920-2000)

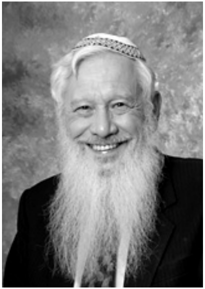


John F. Nash Jr.
b. 1928

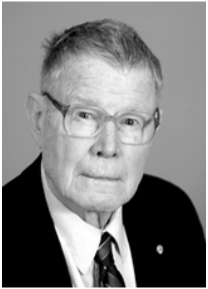
Reinhard Selten
b. 1930

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005

"for having enhanced our understanding of conflict and cooperation through game-theory analysis"



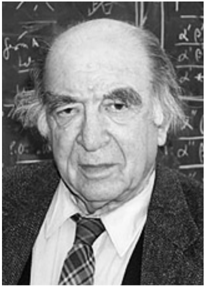
Robert J. Aumann
b. 1930



Thomas C. Schelling
b. 1921

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2007

"for having laid the foundations of mechanism design theory"



Leonid Hurwicz
b. 1917

Eric S. Maskin
b. 1950

Roger B. Myerson
b. 1951

Thank You!

Literature

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- R.B. Myerson, *Game theory. Analysis of conflict*, 1991.
- H. Peters, *Game theory. A multi-leveled approach*, 2008.
- B. Peleg and P. Südholter, *Introduction to the theory of cooperative games*, 2003 (1st ed.), 2007 (2nd ed.)
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