Analysis on graph-like spaces

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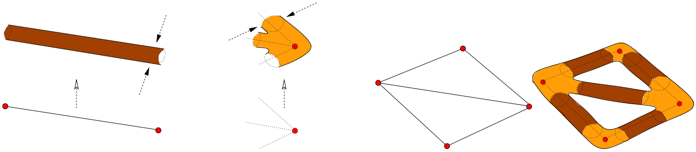
Graph-like manifolds

- Simplest example: ε-tubular neighbourhood of a metric graph X_0 embedded in $\mathbb{R}^d$ (example with boundary ∂X_ε): thick (=fat) graph

- (abstract) construction from building blocks (with or without boundary like the interior of a pipeline network or its surface): graph-like manifold



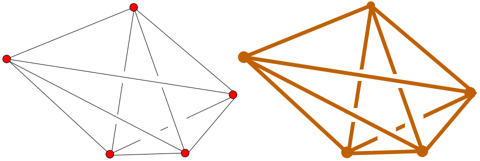
Where are graph-like spaces used?

Graph-like spaces I: Spectral Geometry

Colin de Verdière ['86, Comm. Math. Helv.]: M closed manifold, $\dim M \geq 3$, $n \in \mathbb{N}$, then there exists a metric g_n on M such that the first non-zero eigenvalue of $\Delta_{(M,g_n)}$ has multiplicity n

Idea:

- embed complete metric graph K_{n+1} into M

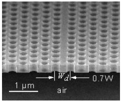


- use ε-neighbourhood of K_{n+1} in M ("thick graph") and show convergence of (Neumann-)eigenvalues to Kirchhoff on K_{n+1} (spectral convergence: graph-like mfd $\rightarrow$ metric graph, see below)
- K_{n+1} has eigenvalue of mult. n, use enough parameters in order to preserve multiplicity

Where are graph-like spaces used?

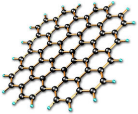
Graph-like spaces II: Physical and other models

- Models for micro- or opto-electronic devices on nanometer level (quantum mechanical description)
- Periodic structures: nature of spectrum, gaps?

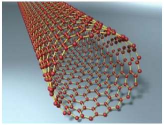


spie.org/x13545.xml

Carbon nanostructures:



graphene

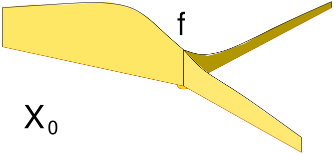


carbon nanotube

- Solvable models: one can calculate spectral properties
- Accoustic waves: [P. Joly et al'10]
- Model for a lung: (fractal) [Pinchover, Wolansky, Zelig'08]
- Model for proteins: "fatgraph models" [R. C. Penner et al'10]

Graph-like manifold converging to metric graph

H_0 Kirchhoff Laplacian on metric graph X_0, i.e.,
$(H_0 f) = -f_e''$, f continuous at v, $\sum_{e \sim v} f_e'(v) = 0$



First convergence result:

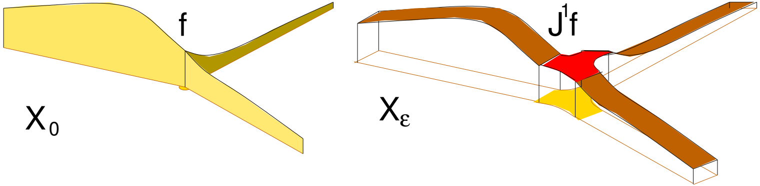
Theorem (CdV:86, Rubinstein-Schatzman:01, Kuchment-Zeng:01, Exner-P:05)

X_ε compact, $H_\varepsilon = \Delta_\varepsilon \geq 0$, (Neumann if $\partial X_\varepsilon \neq \emptyset$), then

$$\lambda_k(H_\varepsilon) - \lambda_k(H_0) = \mathcal{O}(\varepsilon^{1/2}) \qquad \text{as } \varepsilon \to 0.$$

Graph-like manifold converging to metric graph II

- uses variational characterisation of eigenvalues (Min-max principle) and identification operators for quadratic form domains $J^1 \colon \mathsf{H}^1(X_0) \longrightarrow \mathsf{H}^1(X_\varepsilon)$, $J'^1 \colon \mathsf{H}^1(X_\varepsilon) \longrightarrow \mathsf{H}^1(X_0)$



- compare Rayleigh quotients $\dfrac{\|f'\|^2_{\mathsf{L}_2(X_0)}}{\|f\|^2_{\mathsf{L}_2(X_0)}}$ and $\dfrac{\|\mathrm{d}u\|^2_{\mathsf{L}_2(X_\varepsilon)}}{\|u\|^2_{\mathsf{L}_2(X_\varepsilon)}}$

Graph-like manifold converging to metric graph III

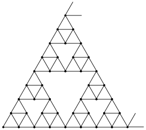
One can show more generally "$H_\varepsilon \to H_0$". Problems:

- operators act in different Hilbert spaces $L_2(X_\varepsilon)$ $\rightsquigarrow$ use identification operators
- operators are unbounded $\rightsquigarrow$ use resolvents

Consequences:

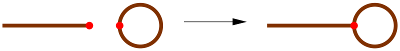
- convergence of heat operators "$e^{-tH_\varepsilon} \to e^{-tH_0}$" and other op. fcts.
- convergence of discrete spectrum (if H_ε has purely discrete spectrum) eigenvalues in gaps; convergence of eigenfcts $\|J\varphi_0 - \varphi_\varepsilon\| \to 0$
- convergence of essential spectrum (uniform in $[0, \Lambda]$): H_ε has spectral gap if H_0 has (provided $\varepsilon > 0$ small enough)

Example Sierpiński graph: non-compact metric graph with fractal spectrum [Teplyaev:98]  $\Rightarrow$ $N_\varepsilon := \#\{\text{components of } \sigma(H_\varepsilon) \cap I\} \to \infty$ for any compact interval $I \subset [0, \infty)$

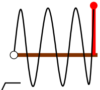


Coupling of spaces: A simple example

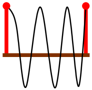
A very simple example:
$-f'' = zf$ on each 1-dim. space
fix boundary value, e.g. $\varphi = 1$



Interval: f_1 on $[0,1]$:
$f_1(1) = \varphi, \quad (f_1(0) = 0)$
$\Lambda_1(z) := f_1'(1) = \sqrt{z}\cot\sqrt{z}$



Loop: f_2 on $[0,1]/\{0,1\}$:
$f_2(0) = f_2(1) = \varphi$
$$\Lambda_2(z) := f_2'(1) - f_2'(0) = \frac{2\sqrt{z}(\cos\sqrt{z} - 1)}{\sin\sqrt{z}}$$



The coupled problem:

- $Hf = -f''$ on each part;
- coupling condition: $f_1(1) \stackrel{!}{=} f_2(0) = f_2(1)$ (and cond. on f_i')
- spectrum of coupled operator: ($\Lambda(z)$ Dirichlet-to-Neumann (DtN) map)
  $$z \in \sigma(H) \text{ iff } \Lambda(z) = \Lambda_1(z) + \Lambda_2(z) = \frac{\sqrt{z}(3\cos\sqrt{z} - 2)}{\sin\sqrt{z}} \stackrel{!}{=} 0$$

Coupling of spaces via a graph

- Branched space $X = \bigcup X_\alpha$: constructed from building blocks X_α according to a graph
- Relate local and global properties via the so-called "Krein's resolvent formula"
  $(\Delta_X - z)^{-1} = \bigoplus_\alpha (\Delta^{\mathrm{D}}_{X_\alpha} - z)^{-1} + S(z)\Lambda(z)^{-1}S(\overline{z})^*$
- Spectral properties, e.g.
  $z \in \sigma(\Delta_X)$ iff $\ker \Lambda(z) \neq 0$
  (global DtN map $\Lambda(z)$ expressed in terms of local DtN maps $\Lambda_\alpha(z)$)
- More abstractly: "Fibred" spaces over metric graphs

Open problems

- Metric graphs: use them as “toy” models: nodal lines, quantum chaos
- Graph-like manifolds: Spectrum of non-compact manifolds mostly unknown:
  - absolutely continuous spectrum? (transport) (e.g. extended states conjecture)
  - spectral gaps? (no transport)
  - infinitely many gaps on a manifold? (Bethe-Sommerfeld conjecture)
  - point spectrum (eigenvalues) (localisation)
- Why graph-like spaces useful for such problems?
  - decompose complicated space into simpler building blocks (analysis)
  - construct spaces with certain (spectral) properties (synthetis)

Conclusion

- Graph-like spaces (spaces having a natural graph structure) provide an interesting class of almost solvable models: "construction kit"
- relation between the Laplacians on such spaces and their spectra (allows calculation of spectra for complicated structures)
- methods for (discrete) graphs can often be used also for graph-like spaces (and vice versa)

Thank you for your attention!