Extreme Interpolation Methods

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Workshop of Young Researchers in Mathematics
September 23, 2011

Interpolation Theory

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- P. L. Butzer and H. Berens, Springer, 1967.
- J. Bergh and J. Löfström, Springer, 1976.
- H. Triebel, North-Holland, 1978.
- C. Bennett and R. Sharpley, Academic Press, 1988.

General framework

The Marcinkiewicz interpolation theorem

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, with $q_0 \neq q_1$, and let $(\mathcal{U}, \mu)$ and $(\mathcal{V}, \nu)$ be measure spaces. Let T be a linear operator that is continuous from $L_{p_0}(\mathcal{U}, \mu)$ to $L_{q_0,\infty}(\mathcal{V}, \nu)$ and at the same time from $L_{p_1}(\mathcal{U}, \mu)$ to $L_{q_1,\infty}(\mathcal{V}, \nu)$ with norms M_0 and M_1 respectively. Let $0 < \theta < 1$ and let

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then, if $p \leq q$, the operator

$$T : L_p(\mathcal{U}, \mu) \to L_q(\mathcal{V}, \nu)$$

is also bounded and its norm is $M \leq C M_0^{1-\theta} M_1^{\theta}$, where C does not depend on T.

General framework

- Let A_0 and A_1 be two normed spaces. A_0 and A_1 are *compatible* (or (A_0, A_1) is a *compatible pair*) if there is a Hausdorff topological vector space $\mathcal{A}$ such that $A_0, A_1 \hookrightarrow \mathcal{A}$.

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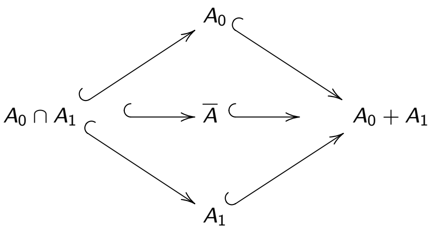
- Intermediate spaces $\overline{A}$



Definitions

The K-Method

- Peetre's K-functional, $a \in A_0 + A_1$

$$K\left(t, a; \overline{A}\right) = \inf\left\{\|a_0\|_{A_0} + t\,\|a_1\|_{A_1} : a = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\right\}$$

for any fixed $t > 0$.

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  for any fixed $t > 0$.

- Let $-\infty < \theta < \infty$, $1 \leq q \leq \infty$ and let (A_0, A_1) be a compatible pair. We define $(A_0, A_1)_{\theta,q;K}$ to be the space of all elements $a \in A_0 + A_1$ such that $\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t} < \infty$, equipped with the norm $\|a\|_{\theta,q;K} = \left(\int_0^\infty \left(t^{-\theta}K(t,a)\right)^q \frac{dt}{t}\right)^{1/q}$.

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- Meaningful whenever $0 < \theta < 1$, $1 \leq q < \infty$ and/or $0 \leq \theta \leq 1$, $q = \infty$. In these cases, they are intermediate spaces.

Definitions
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- Peetre's J-functional, $a \in A_0 \cap A_1$

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$$\|a\|_{\theta,q;J} = \inf\left\{ \left(\int_0^\infty \left(t^{-\theta} J(t,u(t))\right)^q \frac{dt}{t} \right)^{1/q} : a = \int_0^\infty u(t)\frac{dt}{t} \right\}.$$

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Relevant Theorems of Real Interpolation Theory

The Equivalence Theorem

The Equivalence Theorem

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, one has that $(A_0, A_1)_{\theta,q;J} = (A_0, A_1)_{\theta,q;K}$.

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For $0 < \theta < 1$ and $1 \leq q \leq \infty$, one has that $(A_0, A_1)_{\theta,q;J} = (A_0, A_1)_{\theta,q;K}$.

In these cases, we shall write $(A_0, A_1)_{\theta,q}$ instead of $(A_0, A_1)_{\theta,q;K}$ or $(A_0, A_1)_{\theta,q;J}$.

Relevant Theorems of Real Interpolation Theory

The Interpolation Theorem

The Interpolation Theorem

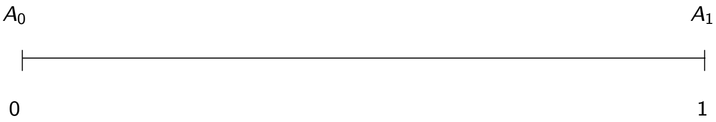
Let $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ be two compatible pairs, let $T \in \mathcal{L}\left(\overline{A}, \overline{B}\right)$. Then, if $0 < \theta < 1$, $1 \leq q \leq \infty$, the restriction of T to $(A_0, A_1)_{\theta,q}$ is a bounded operator,

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q},$$

and its norm is $M \leq M_0^{1-\theta} M_1^{\theta}$.

Relevant Theorems of Real Interpolation Theory

The Reiteration Theorem



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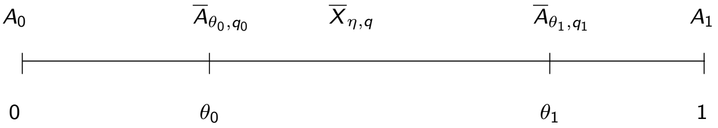


Relevant Theorems of Real Interpolation Theory

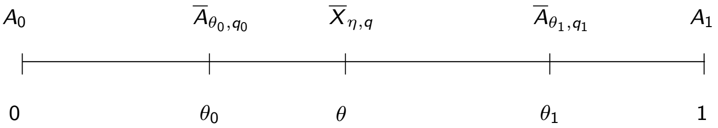
The Reiteration Theorem

The Reiteration Theorem

Let (A_0, A_1) be a compatible pair of Banach spaces, let $0 < \theta_j < 1$, $1 \leq q_j \leq \infty$, $j = 0, 1$, $\theta_0 \neq \theta_1$. Take $0 < \eta < 1$ and $1 \leq q \leq \infty$, and let $\theta = (1 - \eta)\theta_0 + \eta\theta_1$. Then the following holds:

$$\left(\overline{A}_{\theta_0, q_0}, \overline{A}_{\theta_1, q_1}\right)_{\eta, q} = (A_0, A_1)_{\theta, q},$$

with equivalent norms.



Relevant Theorems of Real Interpolation Theory

The Duality Theorem

The Duality Theorem

Let (A_0, A_1) be a regular pair of Banach spaces, let $1 \leq q < \infty$ and $0 < \theta < 1$. Then, $\left((A_0, A_1)_{\theta,q}\right)' = \left(A_0', A_1'\right)_{\theta,q'}$.

Extreme Classes of Real Interpolation Spaces

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- Ordered couples, $A_0 \hookrightarrow A_1$.
- $$\|a\|_{\theta,q;J} = \inf \left\{ \left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^q \frac{dt}{t} \right)^{1/q} : a = \int_0^\infty u(t) \frac{dt}{t} \right\}$$
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- $\|a\|_{\theta,q;K} = \left(\int_0^\infty \left(t^{-\theta} K(t, a)\right)^q \frac{dt}{t} \right)^{1/q} \sim \left(\int_1^\infty \left(t^{-\theta} K(t, a)\right)^q \frac{dt}{t} \right)^{1/q}$
- $\theta = 0 \rightsquigarrow$ close to A_0,
  $\theta = 1 \rightsquigarrow$ close to A_1.

The $(0, q; J)$-Method

Definition

Let A_0, A_1 be Banach spaces such that $A_0 \hookrightarrow A_1$ and let $1 \leq q \leq \infty$. We define $(A_0, A_1)_{0,q;J}$ as the space of all elements $a \in A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0 such that

$$a = \int_1^\infty u(t) \frac{dt}{t} \qquad (\text{in } A_1) \tag{1}$$

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and

$$\left(\int_1^\infty J\left(t, u(t)\right)^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{2}$$

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and

$$\left(\int_1^\infty J(t, u(t))^q \, \frac{dt}{t} \right)^{1/q} < \infty. \tag{2}$$

We set the following norm.

$$\|a\|_{0,q;J} = \inf \left\{ \left(\int_1^\infty J(t, u(t))^q \, \frac{dt}{t} \right)^{1/q} : u \text{ satisfies 1 and 2} \right\}.$$

The $(0, q; J)$-Method

Interpolation Theorem

Interpolation Theorem for the $(0, q; J)$-method [CFKU]

Let $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$ be Banach spaces and let $T \in \mathcal{L}\left(\overline{A}, \overline{B}\right)$. Then, $T : (A_0, A_1)_{0,q;J} \to (B_0, B_1)_{0,q;J}$ and

$$\|T\|_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}} \leq \|T\|_{A_0,B_0} \left[1 + \left(\log \frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}\right)_+\right],$$

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whereas before

$$\|T\|_{(A_0,A_1)_{\theta,q},(B_0,B_1)_{\theta,q}} \leq \|T\|_{A_0,B_0}^{1-\theta} \|T\|_{A_1,B_1}^{\theta}.$$

The $(0, q; J)$-Method
Reiteration Formulas

Reiteration Formulas for the $(0, q; J)$-method I [CFKU]

Let A_0, A_1 be Banach spaces such that $A_0 \hookrightarrow A_1$. Then, if $0 < \theta_0 < \theta_1 < 1$, $1 \leq q \leq \infty$ and q' is the conjugate exponent of q, we have that

$$\left((A_0, A_1)_{\theta_0,q}, (A_0, A_1)_{\theta_1,q}\right)_{0,q;J} =$$

$$\left\{a \in A_1 : \|a\| = \left(\sum_{n=1}^{\infty}\left(n^{-1/q'} 2^{-\theta_0 n} K\left(2^n, a\right)\right)^q\right)^{1/q} < \infty\right\}.$$

The $(0, q; J)$-Method

Reiteration Formulas

Reiteration Formulas for the $(0, q; J)$-method II [CFKU]

If $1 < q \leq \infty$, $0 < \theta < 1$ and q' is the conjugate exponent of q, up to equivalence of norms, the following hold:

1. $\left((A_0, A_1)_{\theta,q}, A_1\right)_{0,q;J} =$
$$\left\{a \in A_1 : \left(\int_1^\infty \left(t^{-\theta}\,(1 + \log t)^{-1/q'}\, K(t, a; A_0, A_1)\right)^q \frac{dt}{t}\right)^{1/q} < \infty\right\}.$$
2. $\left(A_0, (A_0, A_1)_{\theta,q}\right)_{0,q;J} = (A_0, A_1)_{0,q;J}$.
3. $\left(A_0, (A_0, A_1)_{0,q;J}\right)_{0,q;J} =$
$$\left\{a \in A_1 : \left(\int_1^\infty \left(\frac{K(t,a;A_0,A_1)}{(1+\log t)^{1/q}(1+|\log\log t|)}\right)^q \frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

The $(0, q; J)$-Method

An example

Let $(\mathcal{U}, \mu)$ be a finite measure space, let $1 < q \leq \infty$ and let q' be the conjugate exponent of q. Choose two weights on $\mathcal{U}$, ω_0 and ω_1, such that $\omega_0(x) \geq \omega_1(x)$ μ-a.e. and put

$$\omega(x) = \omega_0(x) \left(1 + \log \frac{\omega_0(x)}{\omega_1(x)}\right)^{-1/q'} .$$

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$$\omega(x) = \omega_0(x) \left(1 + \log \frac{\omega_0(x)}{\omega_1(x)} \right)^{-1/q'} .$$

Then, up to equivalence of norms, we have

$$\left(L_q \left(\omega_0 \right), L_q \left(\omega_1 \right) \right)_{0,q;J} = L_q \left(\omega \right).$$

The $(1, q; K)$-Method

Definition

Let $A_0 \hookrightarrow A_1$ be two Banach spaces and let $1 \leq q < \infty$. We define $(A_0, A_1)_{1,q;K}$ to be the space of all $a \in A_1$ for which the following norm is finite

$$\|a\|_{1,q;K} = \left(\int_1^\infty \left(t^{-1} K(t,a) \right)^q \frac{dt}{t} \right)^{1/q}.$$

The $(1, q; K)$-Method

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Let $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$ be Banach spaces and let $T \in \mathcal{L}\left(\overline{A}, \overline{B}\right)$. Then, $T : (A_0, A_1)_{1,q;K} \to (B_0, B_1)_{1,q;K}$ and

$$\|T\|_{(A_0,A_1)_{1,q;K},(B_0,B_1)_{1,q;K}} \leq \|T\|_{A_1,B_1} \left[1 + \left(\log \frac{\|T\|_{A_0,B_0}}{\|T\|_{A_1,B_1}}\right)_+\right],$$

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whereas the theorem for the $(0, q; J)$-method stated that

$$\|T\|_{(A_0,A_1)_{0,q;J},(B_0,B_1)_{0,q;J}} \leq \|T\|_{A_0,B_0} \left[1 + \left(\log \frac{\|T\|_{A_1,B_1}}{\|T\|_{A_0,B_0}}\right)_+\right].$$

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Reiteration Formulas

Reiteration Formula for the $(1, q; K)$-method [CFKU]

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The $(1, q; K)$-Method

Reiteration Formulas

Reiteration Formula for the $(1, q; K)$-method [GM]

Let $A_0 \hookrightarrow A_1$ be two Banach spaces and let $1 \leq q < \infty$. Then,

$$\Big((A_0, A_1)_{1,q;K}\,, A_1\Big)_{1,q;K} = \left\{ a \in A_1 : \left(\int_1^\infty \left(\frac{K(t, a; A_0, A_1)}{t\,(1+\log t)^{1/q}} \right)^q \frac{dt}{t} \right)^{1/q} < \infty \right\}$$

The $(1, q; K)$-Method

An example

Let $(\mathcal{U}, \mu)$ be a finite measure space, let $1 \leq q < \infty$. Choose two weights on $\mathcal{U}$, ω_0 and ω_1, such that $\omega_0(x) \geq \omega_1(x)$ μ-a.e. and put

$$\omega(x) = \omega_1(x) \left(1 + \log \frac{\omega_0(x)}{\omega_1(x)}\right)^{1/q}.$$

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Then, up to equivalence of norms, we have

$$\left(L_q\left(\omega_0\right), L_q\left(\omega_1\right)\right)_{1,q;K} = L_q\left(\omega\right).$$

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Before, we put

$$\omega(x) = \omega_0(x) \left(1 + \log \frac{\omega_0(x)}{\omega_1(x)}\right)^{-1/q'}.$$

Duality

The Duality Theorem [CFKU]

Let $A_0 \hookrightarrow A_1$ be Banach spaces such that A_0 is dense in A_1, let $1 < q < \infty$ and let q' be the conjugate exponent of q. Then, the following hold

$$\begin{aligned} (A_0, A_1)'_{0,q;J} &= \big(A'_0, A'_1\big)_{1,q';K}, \\ (A_0, A_1)'_{1,q;K} &= \big(A'_1, A'_0\big)_{0,q';J} \end{aligned}$$

with equality of norms.

Multidimensional Interpolation

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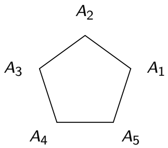
- Convex polygon $\Pi = \overline{P_1, P_2, \ldots, P_N}$ (in $\mathbb{R}^2$).



Multidimensional Interpolation

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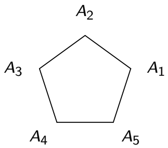

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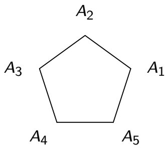

- $P_j = \left(x_j, y_j\right)$.
- $K\left(t, s; a\right) = \inf\left\{\sum_{j=1}^{N} t^{x_j} s^{y_j} \left\|a_j\right\|_{A_j} : a = \sum_{j=1}^{N} a_j,\ a_j \in A_j\right\}, t, s > 0, a \in \Sigma\left(\overline{A}\right),$

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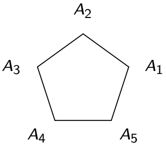

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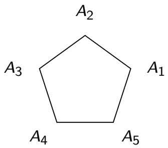

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- (α, β) in the interior of Π, $1 \le q \le \infty$.

Multidimensional Interpolation

Definition of $\overline{A}_{(\alpha,\beta),q;K}$

We define the space $\overline{A}_{(\alpha,\beta),q;K}$ as the set of all $a \in \Sigma\left(\overline{A}\right)$ for which the following norm is finite

$$\|a\|_{(\alpha,\beta),q;K} = \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha} s^{-\beta} K\left(t,s;a\right)\right)^q \frac{dt}{t}\frac{ds}{s}\right)^{1/q}.$$

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Definition of $\overline{A}_{(\alpha,\beta),q;J}$

We define the space $\overline{A}_{(\alpha,\beta),q;J}$ as the one consisting of all elements $a \in \Sigma\left(\overline{A}\right)$ which can be represented as

$$a = \int_0^\infty \int_0^\infty u(t,s) \frac{dt}{t} \frac{ds}{s} \qquad \text{(convergence in } \Sigma\left(\overline{A}\right)\text{)},$$

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We define the norm in $\overline{A}_{(\alpha,\beta),q;J}$ as follows:

$$\|a\|_{(\alpha,\beta),q;J} = \inf \left\{ \left(\int_0^\infty \int_0^\infty \left(t^{-\alpha} s^{-\beta} J\left(t,s;u(t,s)\right)\right)^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} \right\}$$

where the infimum is taken over all representations $u(t,s)$ of a as above.

Multidimensional Interpolation

Interpolation over the Unit Square

4-tuple $\{A_0, A_1, A_1, A_0\}$:

Multidimensional Interpolation

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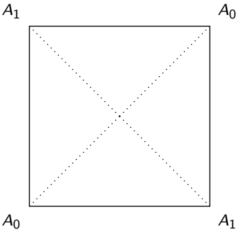

Multidimensional Interpolation

Interpolation over the Unit Square

Theorem [CFKU]

Let $A_0 \hookrightarrow A_1$ be two Banach spaces, let $0 < \alpha < 1$ and $1 \leq q \leq \infty$. Then, up to equivalence of norms, the following hold:

$$(A_0, A_1, A_1, A_0)_{(\alpha,1-\alpha),q;J} = \begin{cases} (A_0, A_1)_{1-2\alpha,q} & \text{if } 0 < \alpha < 1/2, \\ (A_0, A_1)_{2\alpha-1,q} & \text{if } 1/2 < \alpha < 1, \\ (A_0, A_1)_{0,q;J} & \text{if } \alpha = 1/2, \end{cases}$$

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$$(A_0, A_1, A_1, A_0)_{(\alpha,\alpha),q;J} = (A_0, A_1)_{0,q;J} \qquad \text{for any } 0 < \alpha < 1.$$

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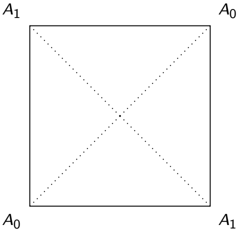

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Interpolation over the Unit Square

Theorem [CFM]

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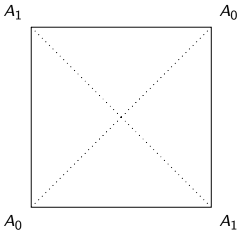

Thank you!