The fractional Laplacian in the Signorini problem: a semigroup approach to the problem

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The fractional Laplacian

$f:\mathbb{R}^n\to\mathbb{R}$, $n\geq 1$,

$$(-\Delta)f(x)=-\sum_{i=1}^n\frac{\partial^2}{\partial x_i^2}f(x)\quad\longleftrightarrow\quad\widehat{(-\Delta)f}(\xi)=|\xi|^2\,\widehat{f}(\xi).$$

$\sigma>0$, **obvious Fourier transform definition:**

$$\widehat{(-\Delta)^\sigma f}(\xi)=|\xi|^{2\sigma}\,\widehat{f}(\xi).$$

$0<\sigma<1$, by inverse Fourier transform, **pointwise formula:**

$$(-\Delta)^\sigma f(x)=c_{n,\sigma}\,\mathrm{P.V.}\int_{\mathbb{R}^n}\frac{f(x)-f(z)}{|x-z|^{n+2\sigma}}\,dz.$$

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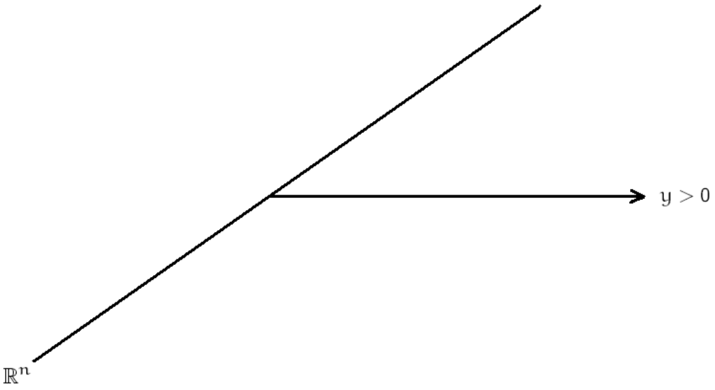
Fourier series: If $f:\mathbb{T}\to\mathbb{R}$ then we have Fourier series $f(x)=\sum_{n\in\mathbb{Z}}c_ne^{inx}$:

$$\left(-\frac{d^2}{dx^2}\right)f(x)=\sum_{n\in\mathbb{Z}}n^2c_ne^{inx}\quad\text{and}\quad\left(-\frac{d^2}{dx^2}\right)^\sigma f(x)=\sum_{n\in\mathbb{Z}}n^{2\sigma}c_ne^{inx}.$$

The Signorini problem



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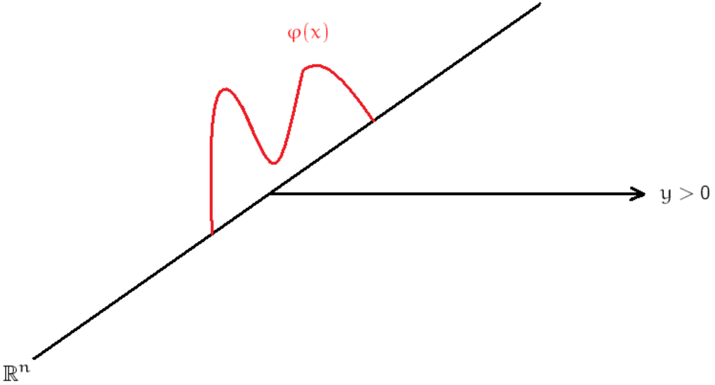

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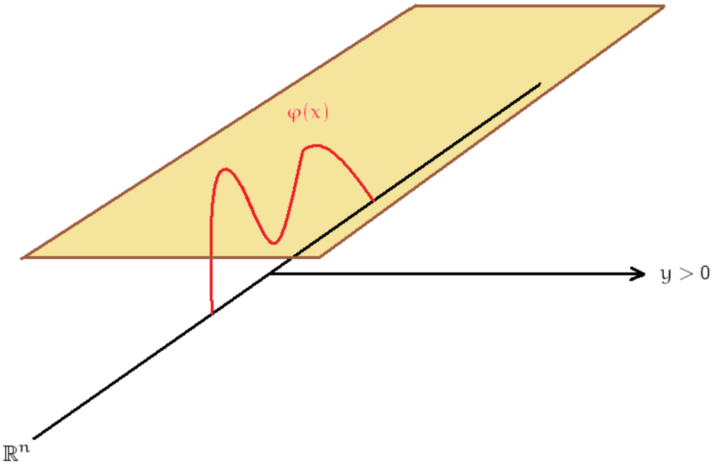

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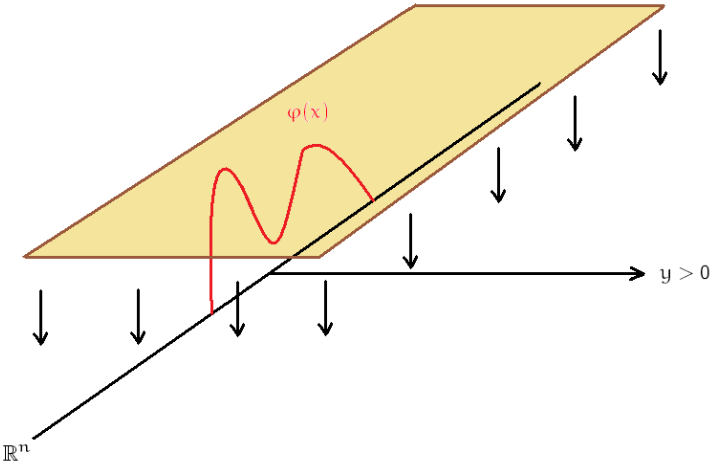

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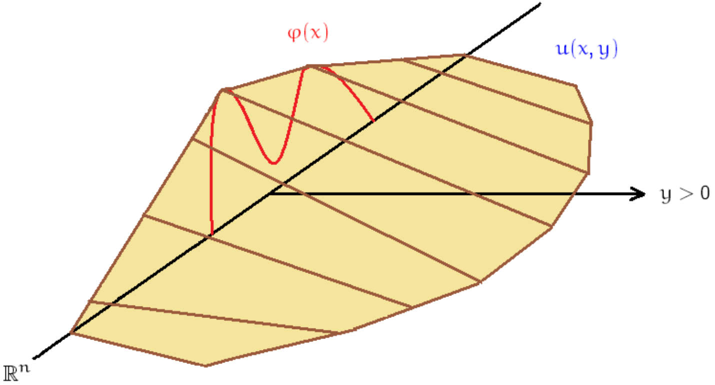

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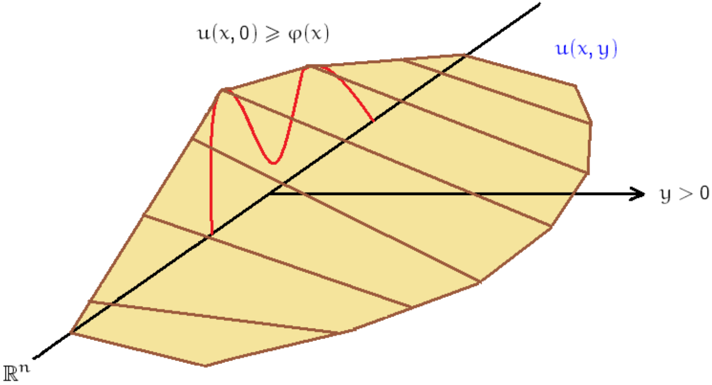

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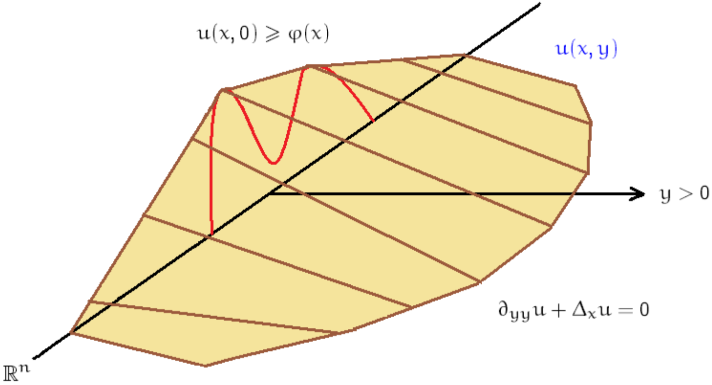

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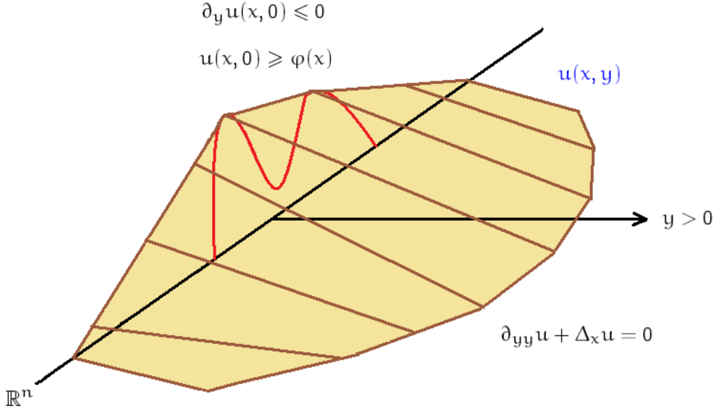

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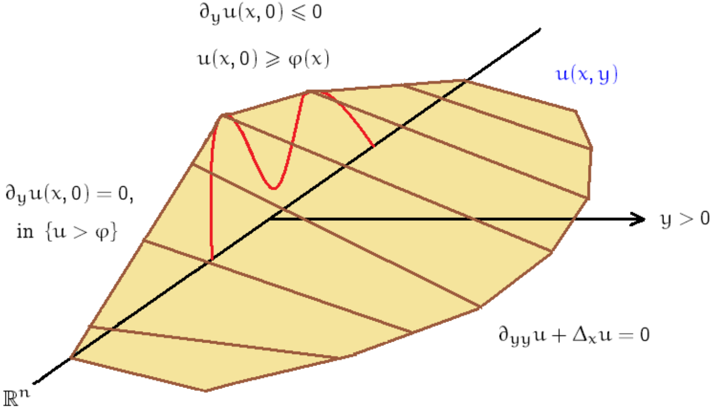

The Signorini problem, semigroups and $(-\Delta)^{1/2}$

Given $\varphi \in C_c(\mathbb{R}^n)$, find $u(x,y) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ such that

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- Fractional powers of L = second order differential operator.

Obstacle problem for the fractional Laplacian

Fix $0 < \sigma < 1$, find $f : \mathbb{R}^n \to \mathbb{R}$ such that

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- Applications. Mathematical Finance, modeling with Lévy process instead of usual Wiener process.
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3. Harnack's inequality for $(-\Delta)^\sigma$.
4. Is there some way to *localize* the operator $(-\Delta)^\sigma$?

First tool: explicit pointwise formula

Our tool: **heat semigroup e^{-tL} generated by L**

$$L^\sigma f(x) = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL}f(x) - f(x)\right) \, \frac{dt}{t^{1+\sigma}}.$$

Here $v(x,t) = e^{-tL}f(x)$ solves the diffusion equation $\partial_t v + Lv = 0$ with $v(x,0) = f(x)$.
Compare with Gamma function formula: $\lambda^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\lambda} - 1\right) \, \frac{dt}{t^{1+\sigma}}$.

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We check it with $L = -\Delta$:

$$\frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f(x) - f(x) \right) \frac{dt}{t^{1+\sigma}} =$$

$$= \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left[\int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{4t}}}{(4\pi t)^{n/2}} (f(z) - f(x))\, dz \right] \frac{dt}{t^{1+\sigma}} \leftarrow \quad (e^{t\Delta} 1 = 1)$$

$$= \int_{\mathbb{R}^n} (f(z) - f(x)) \left[\frac{1}{\Gamma(-\sigma)} \int_0^\infty \frac{e^{-\frac{|x-z|^2}{4t}}}{(4\pi t)^{n/2}} \frac{dt}{t^{1+\sigma}} \right] dz \leftarrow \quad \text{(cancellation)}$$

$$= c_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x - z|^{n+2\sigma}}\, dz, \quad \text{with explicit constant } c_{n,\sigma} > 0.$$

Second tool: extension problem for the fractional Laplacian

- L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian", *Comm. Partial Differential Equations* **32** (2007), 1245–1260.

Let $u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ be a solution to the **extension problem**

$$\begin{cases} u(x,0) = f(x), & \text{on } \mathbb{R}^n, \\ \Delta_x u + \frac{1-2\sigma}{y} u_y + u_{yy} = 0, & \text{in } \mathbb{R}^n \times (0,\infty). \end{cases}$$

Then

$$-\lim_{y\to 0^+} y^{1-2\sigma} u_y(x,y) = c_\sigma (-\Delta)^\sigma f(x), \qquad x \in \mathbb{R}^n.$$

Dirichlet-to-Neumann operator

$$(-\Delta)^\sigma : f(x) \mapsto -y^{1-2\sigma} u_y(x,y)\Big|_{y=0}.$$

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To get an extension of f we **add a new variable** y:

$$\begin{aligned} u(x,y) &= \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-z|^2+y^2}{4t}} g(z)\, dz\, \frac{dt}{t^{1-\sigma}} \\ &= \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-z|^2}{4t}} g(z)\, dz\, e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1-\sigma}} \\ &= \int_0^\infty e^{t\Delta} g(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1-\sigma}} = \int_0^\infty e^{t\Delta} (-\Delta)^\sigma f(x) e^{-\frac{y^2}{4t}}\, \frac{dt}{t^{1-\sigma}}. \end{aligned}$$

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$$f(x) = \int_0^\infty e^{t\Delta}g(x)\,\frac{dt}{t^{1-\sigma}} = \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-z|^2}{4t}} g(z)\,dz\,\frac{dt}{t^{1-\sigma}}.$$

To get an extension of f we **add a new variable** y:

$$\begin{aligned} u(x,y) &= \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-z|^2+y^2}{4t}} g(z)\,dz\,\frac{dt}{t^{1-\sigma}} \\ &= \int_0^\infty \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-z|^2}{4t}} g(z)\,dz\, e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} \\ &= \int_0^\infty e^{t\Delta}g(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}} = \int_0^\infty e^{t\Delta}(-\Delta)^\sigma f(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1-\sigma}}. \end{aligned}$$

It can be checked that this is a solution to the extension problem.

- We can also replace $-\Delta$ by L in the formula above!

Extension problem for L^σ

L: second order differential operator, positive and self-adjoint in some $L^2(\Omega, d\eta)$.

Theorem

Let $f \in \mathrm{Dom}(L^\sigma)$. *A solution of the* **extension problem**

$$\begin{cases} u(x,0) = f(x), & \text{on } \Omega, \\ -L_x u + \frac{1-2\sigma}{y} u_y + u_{yy} = 0, & \text{in } \Omega \times (0,\infty). \end{cases}$$

is given by

$$u(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL}(L^\sigma f)(x) e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1-\sigma}} = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1+\sigma}},$$

and

$$\lim_{y \to 0^+} y^{1-2\sigma} u_y(x,y) = c_\sigma L^\sigma f(x).$$

▶ P. R. Stinga and J. L. Torrea, "Extension problem and Harnack's inequality for some fractional operators", *Comm. Partial Differential Equations* (2010).

Proof of the extension problem

Fundamental tools: Spectral Theory and Functional Calculus.

$$u(x,y) = \frac{y^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty e^{-tL} f(x) e^{-\frac{y^2}{4t}} \, \frac{dt}{t^{1+\sigma}}$$

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$$u(x,y)=\frac{y^{2\sigma}}{4^\sigma\Gamma(\sigma)}\int_0^\infty e^{-tL}f(x)e^{-\frac{y^2}{4t}}\,\frac{dt}{t^{1+\sigma}}$$

Neumann condition:

$$\begin{aligned}
y^{1-2\sigma}u_y(x,y) &= \int_0^\infty e^{-tL}f(x)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}\\
&= \int_0^\infty \left(e^{-tL}f(x)-f(x)\right)e^{-\frac{y^2}{4t}}\left(2\sigma-\frac{y^2}{2t}\right)\,\frac{dt}{t^{1+\sigma}}\\
&\underset{y\to 0}{\longrightarrow} \int_0^\infty \left(e^{-tL}f(x)-f(x)\right)\,\frac{dt}{t^{1+\sigma}}=L^\sigma f(x).
\end{aligned}$$

More about the extension problem

1. Maximum principle and L^p estimates.
2. Computation of the fundamental solution.
3. Adapted Cauchy-Riemann equations and conjugate Poisson integrals.
4. Applications to: divergence form elliptic operators; Schrödinger operators; classical orthogonal expansions such as Ornstein-Uhlenbeck, Laguerre, Jacobi, Laplacian on bounded domains…
5. Using the extension: Harnack's inequalities.

Thank you for your attention!