Knots with finite interal Menger curvature

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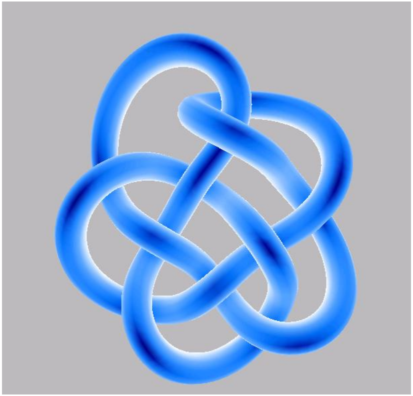
A *knot* *is a curve in* $\mathbb{R}^3$ *homeomorphic to a circle.*



A knot can be considered as its parametrization $\Gamma : S_L \to \mathbb{R}^3$ or as a set $\gamma(S_L)$.

Type of knot

Definition

We say that two knots κ_1 and κ_2 are equivalent if they are isotopic. The equivalence of knots we will denote by $\kappa_1 \approx \kappa_2$.

Let $f_0, f_1 : S_L \to \mathbb{R}^3$ be such that $f_i : S_L \to f_i(S_L)$ is a homeomorphism. We say that f_0 and f_1 are isotopic iff there exists a homeomorphism

$$F : \mathbb{R}^3 \times I \to \mathbb{R}^3 \times I$$

such that

- $F(y, 0) = (y, 0)$ *for* $y \in \mathbb{R}^3$
- $F(f_0(x), 1) = (f_1(x), 1)$ *for* $x \in S_L$.

Equivalent knots - remark

In literature one can find another definition of equivalent knots

Two knots γ_1 *and* γ_2 *are equivalent* $\Leftrightarrow$ *there exists homeomorphism* $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, *such that* $h(\gamma_1) = h(\gamma_2)$.

Those definitions are not equivalent!

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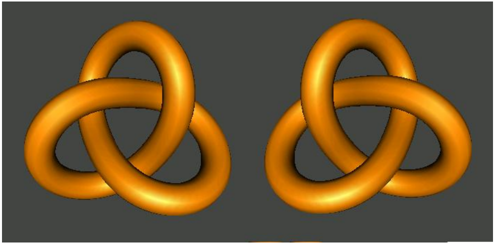
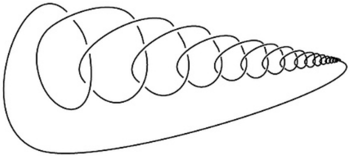
The knot is called chiral when it is not isotopic with his mirror image.



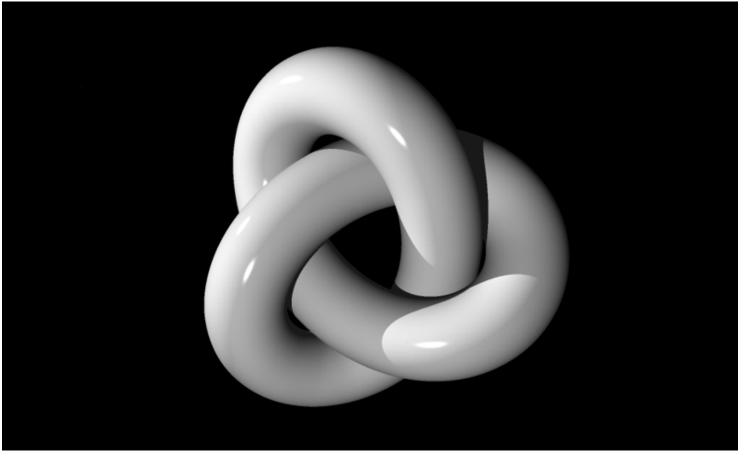
Wild and tame knots

A knot is called tame, if it is ambient isotopic to regular polygon curve.

Example of a wild knot:



Thickness of a curve

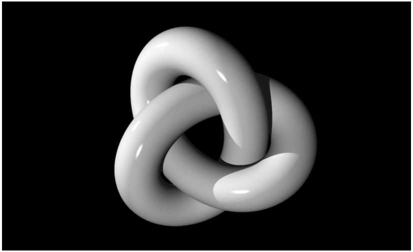


Global Menger curvature

Radius of Menger curvature of three points x, y and z, denoted by $R(x, y, z)$, is a radius of the smallest circle passing through x, y and z.

Radius of Global Menger Curvature: introduced by Gonzalez and Maddocks '99

$$\Delta(\gamma) = \inf\{R(\Gamma(x), \Gamma(y), \Gamma(z)) \quad | \quad x, y, z \in S_L, \; x \neq y \neq z \neq x\},$$



If $\Delta(\gamma) > \theta$, than γ is a core of a curve of a rope with thickness θ.

$\Delta(\gamma) > \theta$ - consequences

Gonzalez, Maddocks, Schuricht and von der Mosel proved [2002] that curves with $\Delta^{-1}(\gamma) < \infty$ then

- the curve is injective
- the derivative of arc-length parametrization Γ' is satisfies

$$|\Gamma'(x) - \Gamma'(y)| < \theta|x - y|$$

- in a family of curves with given length, in each knot type there is a curve $\tilde{\gamma}$ which minimizes Δ^{-1}

Knot energies - ropelength

A knot energy is a scaling invariant functional defined on a family of closed curves in $\mathbb{R}^3$

Example:

$$\text{Ropelength} = \frac{\mathit{length}(\gamma)}{\Delta(\gamma)}$$

Cantarella, Cusner and Sullivan [2002] proved that

- in each link type there is a ropelength minimizer
- the minimizer does not need to be unique
- the minimizer is of class $C^{1,1}$, but does not need to have a higher regularity

O'Hara knot energy

$$\mathcal{E}(\gamma) := \int_0^l \int_0^l \frac{1}{|\Gamma(x) - \Gamma(y)|^2} - \frac{1}{|x-y|^2} dxdy,$$

Different approach - repulsive energies

O'Hara knot energy

$$\mathcal{E}(\gamma) := \int_0^l \int_0^l \frac{1}{|\Gamma(x) - \Gamma(y)|^2} - \frac{1}{|x-y|^2} dxdy,$$

Freedman, He, Wang [1998] proved

- $\mathcal{E}(\gamma) < \infty$, to Γ is bi-lipschitz (i.e. there exists $L > 0$ such that $L^{-1}|x-y| < |\Gamma(x) - \Gamma(y)| < L|x-y|$)
- For each $a > 0$, there are only finitely many knot types that can be represented by curves with $\mathcal{E}(\gamma) < \infty < a$
- A circle is a minimizer among all closed curves
- Each local minimizer of the energy is $C^{1,1}$

He [2000] - minimizers of $\mathcal{E}$ in each knot type are C^∞ curves.

Integral Menger curvature

Integral Menger Curvature of a curve (introduced by Banavar et. al.)

$$\mathcal{M}_p(\gamma) := \int_\gamma \int_\gamma \int_\gamma R^{-p}(x,y,z) d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z)$$

When Γ is an arc-length parametrization of γ we have

$$\mathcal{M}_p(\gamma) := \int_{S_L} \int_{S_L} \int_{S_L} R^{-p}(\Gamma(x), \Gamma(y), \Gamma(z)) dxdydz$$

It is not scaling invariant, but

$$\tilde{\mathcal{M}}_p(\gamma) := L^{p-3} \int_{S_L} \int_{S_L} \int_{S_L} R^{-p}(\Gamma(x), \Gamma(y), \Gamma(z)) dxdydz$$

is a knot energy.

Theorem (Strzelecki, von der Mosel, M.Sz. 2010)

Let $\mathcal{M}_p(\gamma) < \infty$ for some $p > 3$. If γ is a closed curve then γ homeomorphic with a circle.

Theorem (Strzelecki, von der Mosel, M.Sz)

Assume that $p > 3$, $\mathcal{M}_p(\gamma) < \infty$ and Γ is an injective arc-length parametrization then for all $x, y \in [a, b]$ (S_L if Γ is closed)

$$|\Gamma'(x) - \Gamma'(y)| \leq M|x - y|^{1-3/p}(E_{[x,y]})^{1/p},$$

where $M = M(p, \mathcal{M}_p, \mathcal{H}^1(\gamma))$ and

$$E_{[x,y]} = \int_{[x,y]}\int_{[x,y]}\int_{[x,y]} c^p(\Gamma(s), \Gamma(t), \Gamma(u))ds\,dt\,du.$$

$\mathcal{M}_p(\gamma) < \infty$ - consequences - regularity

Theorem (Kolasinski, M.Sz. 2011)

If Γ is $C^{1,\alpha}$ for $\alpha > 1 - \frac{2}{p}$ (for p>2) then $\mathcal{M}_p(\gamma) < \infty$. Moreover there exists a $C^{1,1-2/p}$ curve such that $\mathcal{M}_p(\gamma) = \infty$.

Recently Blatt proved "if and only if" charcterization:

Theorem (Blatt 2011)

$\mathcal{M}_p(\gamma) < \infty$ iff $\Gamma \in W^{2,2-2/p}(S_L)$

Crossing number

Average crossing number of a rectifiable curve over itself was defined by Freedman and He as

$$c(\gamma) = \frac{1}{4\pi} \int_{S_L} \int_{S_L} \frac{|\Gamma'(x), \Gamma'(y), \Gamma(x) - \Gamma(y)|}{|\Gamma(x) - \Gamma(y)|^3} dx\, dy$$

where $\Gamma : X \to \mathbb{R}^3$ denotes the arc-length parametrization of the curve.

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where $\Gamma : X \to \mathbb{R}^3$ denotes the arc-length parametrization of the curve.
Equivalently - for simple rectifiable curve $\gamma \subset \mathbb{R}^3$

$$c(\gamma) = \frac{1}{4\pi} \int \int_{\theta \in S^2} n(\gamma, \theta) dS$$

Crossing number is the number of self-itersections of θ - projection of γ averaged over the sphere.

Estimation of crossing number for curves with $\mathcal{M}_p(\gamma) < \infty$

Proposition

Let γ be a simple closed rectifiable curve with a length L. If $\mathcal{M}_p(\gamma) < \infty$ for some $p > 6$ then there exist constants $c_1(p)$ and $c_2(p)$ such that

$$c(\gamma) < c_1(p)\mathcal{M}^{\frac{1}{p-3}} L + c_2(p) L^2 \mathcal{M}^{\frac{4}{p-3}}.$$

The estimation is scaling invariant and for

$$\mathcal{E}_p := L^{3-p} \int_\gamma \int_\gamma \int_\gamma R^{-p}(x,y,z) d\mathcal{H}^1(x) d\mathcal{H}^1(y) d\mathcal{H}^1(z)$$

$$c(\gamma) < c_1(p)\mathcal{E}^{\frac{1}{p-3}} + c_2(p)\mathcal{E}^{\frac{4}{p-3}}.$$

Proposition

Let $\mathcal{M}_p(\gamma) < \infty$ *for some* $p > 3$*, than* γ *is isotopic with a regular polygon, whose sides are segments with endpoints* $\Gamma(x_i), \Gamma(x_{i+1})$*, where* $0 = x_1 < x_2 < \ldots < x_n < L$*,* $x_{n+1} = x_1$ *and*

$$|\Gamma(x_i) - \Gamma(x_{i+1})| < D(p)(\mathcal{M}_p(\gamma))^{\frac{1}{3-p}}.$$

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Proposition

Assume that γ_1 *and* γ_2 *are closed curves, and for some* $p > 3$

$$\max\{\mathcal{M}_p(\gamma_1), \mathcal{M}_p(\gamma_2)\} < M.$$

Then there exists $\epsilon = \epsilon(p)$ *such that if*

$$d_H(\gamma_1, \gamma_2) < \epsilon M^{\frac{1}{3-p}},$$

then γ_1 *and* γ_2 *are isotopic.*

Sketch of the proof - an arc

$$|\Gamma(x) - \Gamma(y)| < D(p)(\mathcal{M}_p(\gamma))^{\frac{1}{3-p}}.$$

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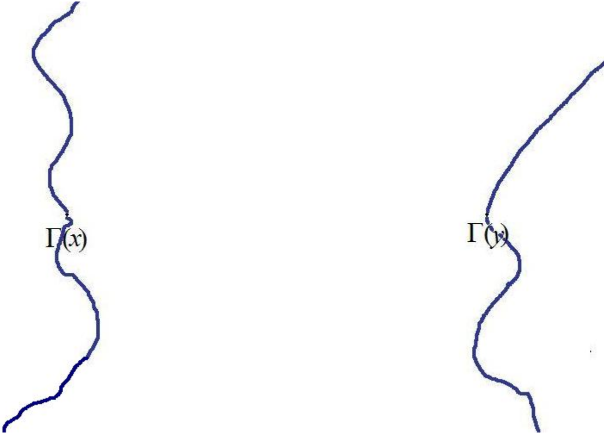

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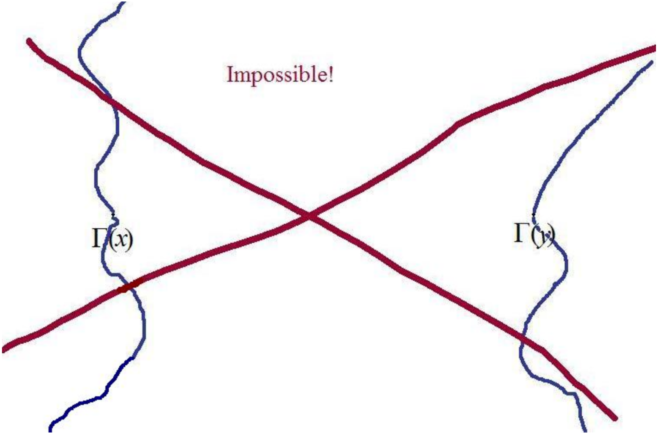

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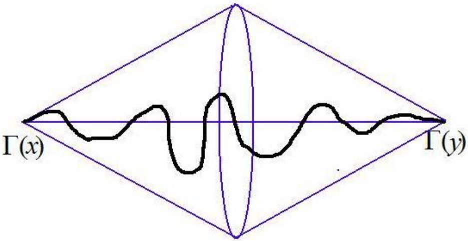

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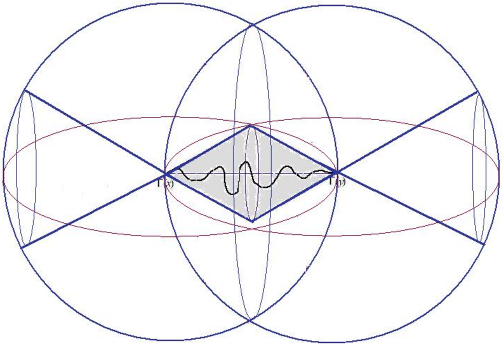

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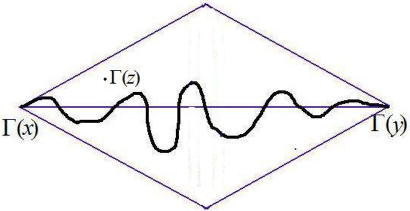

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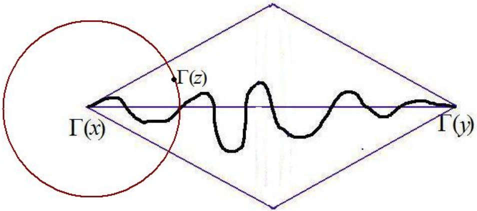

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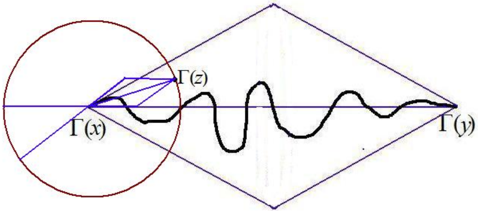

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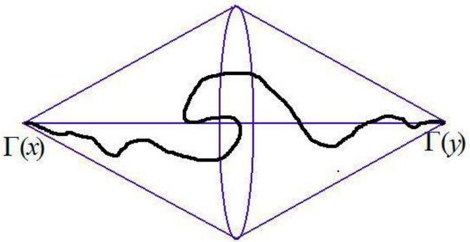

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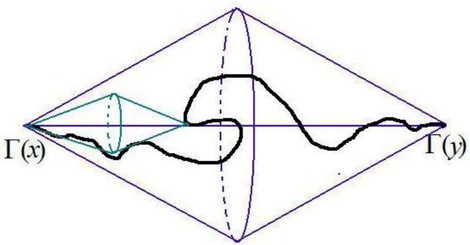

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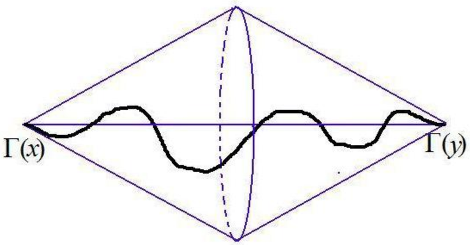
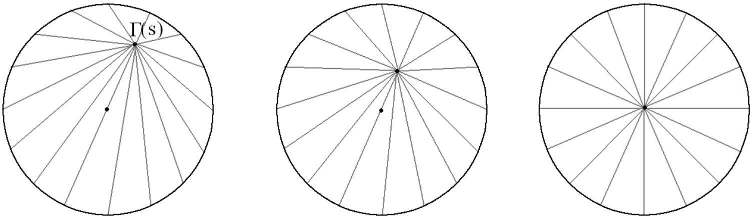
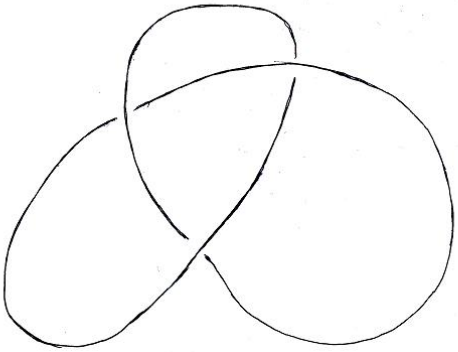


Sketch of the proof - isotopy



$H(D_i \cap \pi_{i,\Gamma(s)}, 0)$ $H(D_i \cap \pi_{i,\Gamma(s)}, t)$ $H(D_i \cap \pi_{i,\Gamma(s)}, 1)$

Sketch of the proof - isotopy



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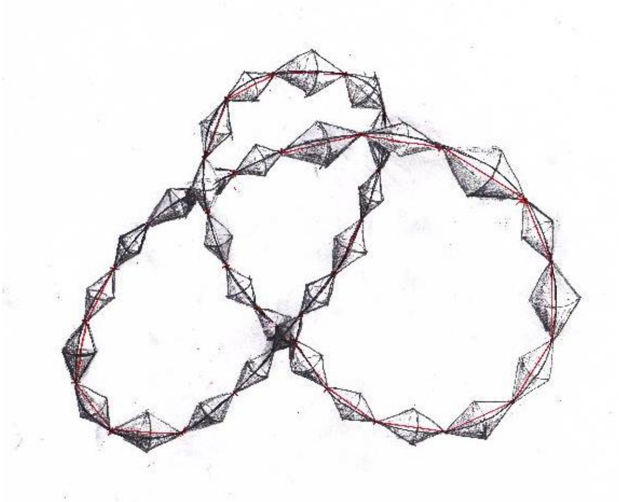
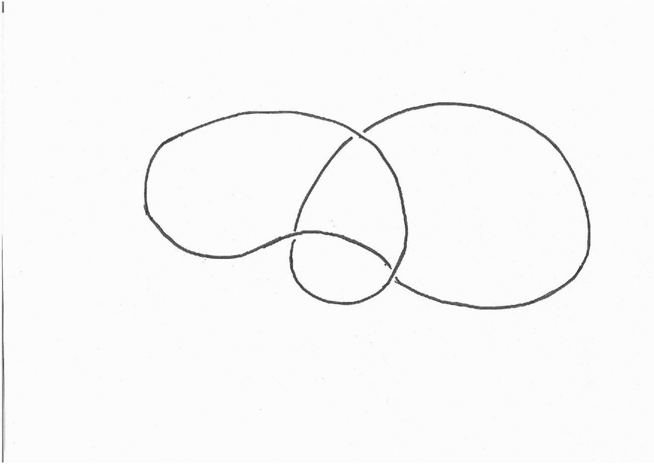
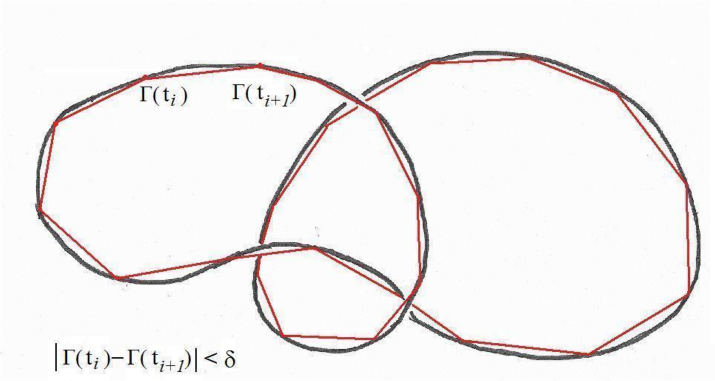


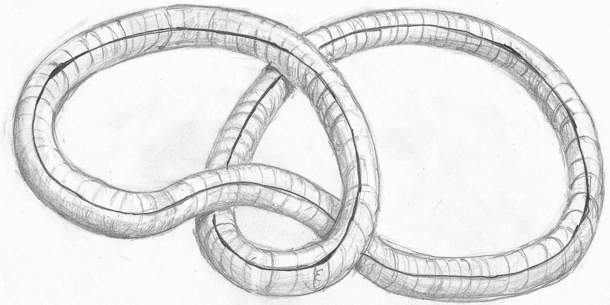
Sketch of the proof - a curve



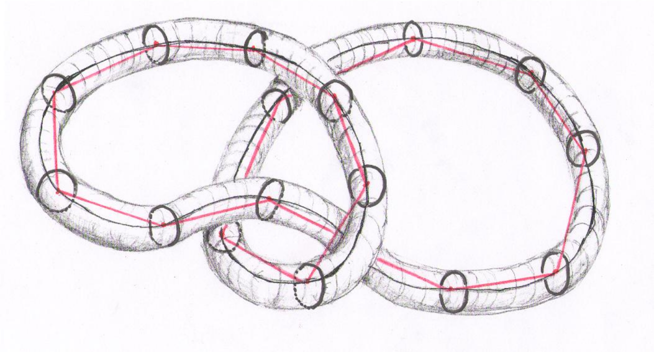
Sketch of the proof - a curve and isotopic polygon curve

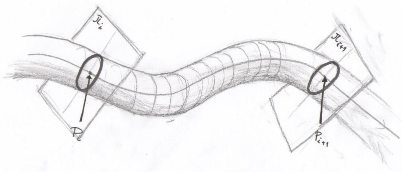


Sketch of the proof - ε- neighbourhood

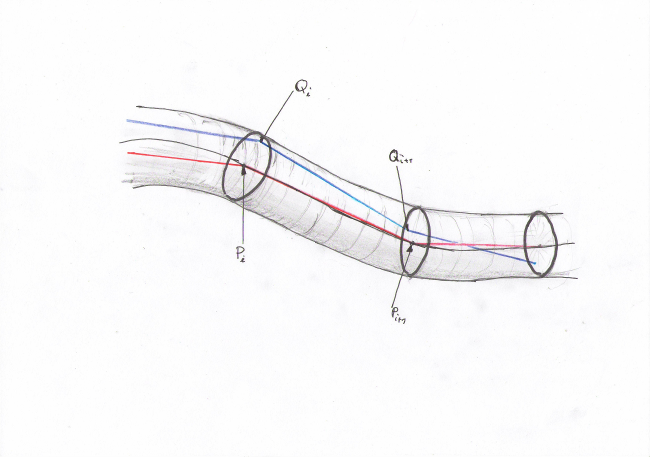


Sketch of the proof - sections





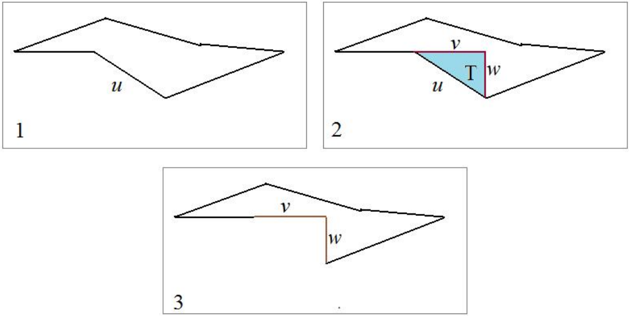
Sketch of the proof



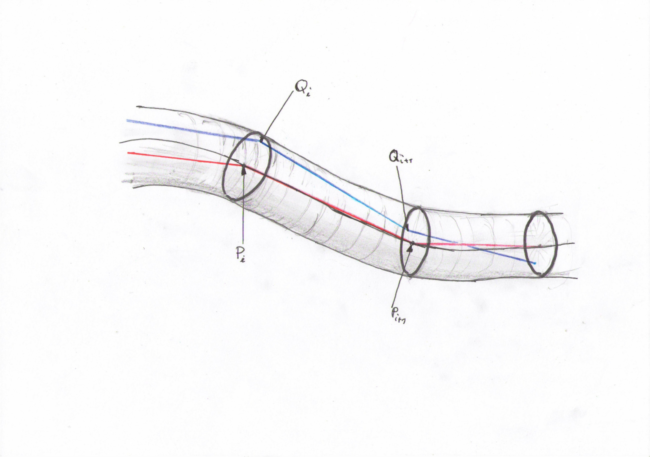
Δ-moves

Definition (Δ-moves)

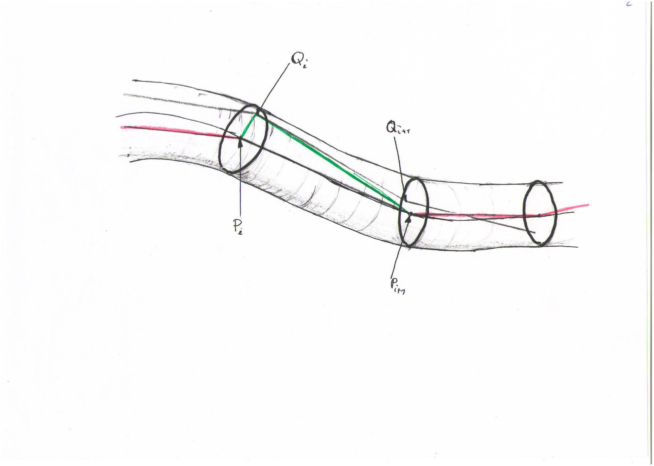
Let u be a segment of a regular polygon closed curve κ in $\mathbb{R}^3$, and let T be a triangle bounded by segments u, v, w.
If $T \cap \kappa = u$, then $\kappa' = (\kappa \setminus u) \cup v \cup w$ is a polygon curve obtained from κ by Δ-move. Inverse action is denoted by Δ^{-1}.



Sketch of the proof - Δ-moves



Sketch of the proof - continuation



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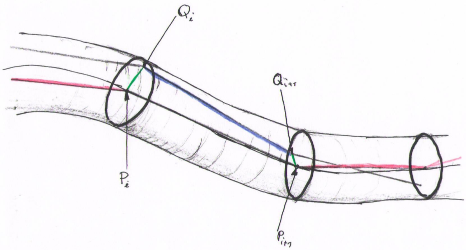

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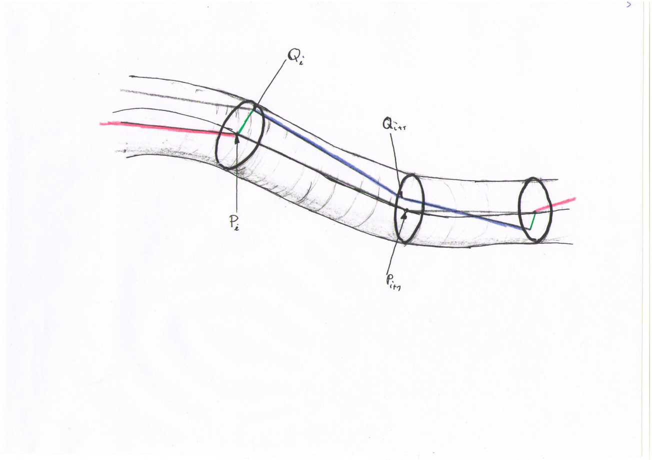

Thank you