Multiplicity of positive large solutions in a superlinear indefinite problem

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Joint work with Julián López-Gómez and Fabio Zanolin

Introduction to the problem

$$\begin{cases} -u'' = \lambda u + a(t)u^p & \text{in } (0,1) \\ u(0) = M = u(1) & \text{with } M > 0 \text{ large} \end{cases} \tag{1}$$

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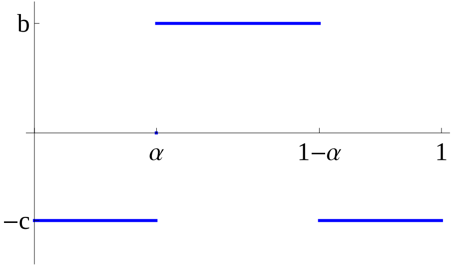

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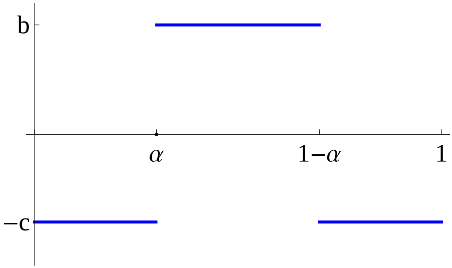
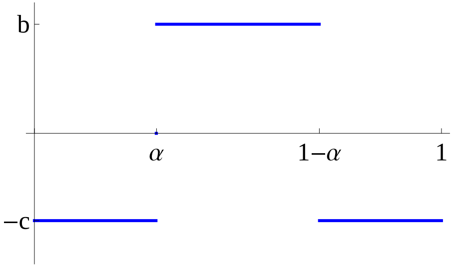


The weight $a(t)$

Positive solutions

Why superlinear indefinite?

$$-u'' = \lambda u + a(t) u^p \quad \text{in } (0,1)$$



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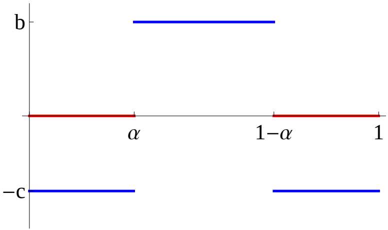
$$-u'' = \lambda u + \underbrace{a(t)}_{<0} u^p \quad \text{in } (0,\alpha)\cup(1-\alpha,1)$$



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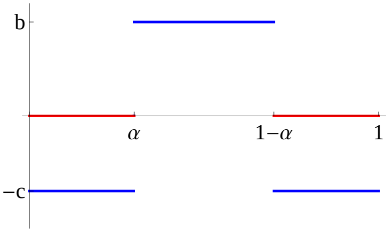

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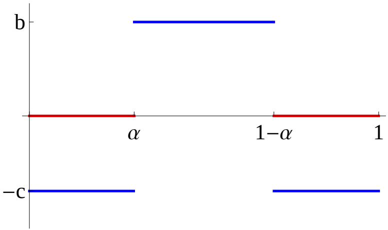

sublinear

Why superlinear indefinite?

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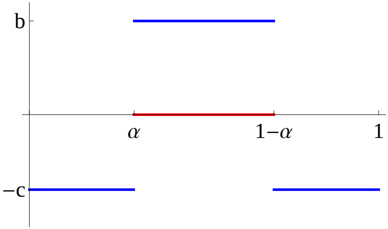

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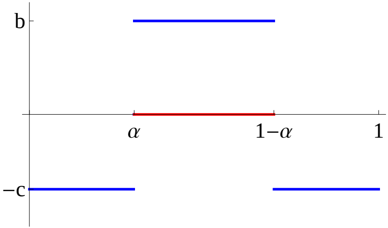

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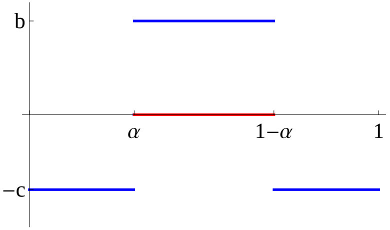

superlinear

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$$\lim_{t \downarrow 0} u(t) = \lim_{t \uparrow 1} u(t) = +\infty$$

Equation in $[0, \alpha]$

$$\begin{cases} -u''(t) = \lambda u(t) - c u^p(t) & \text{in } (0, \alpha) \\ u(0) = M, \quad u(\alpha) = x > 0 \end{cases} \tag{2}$$

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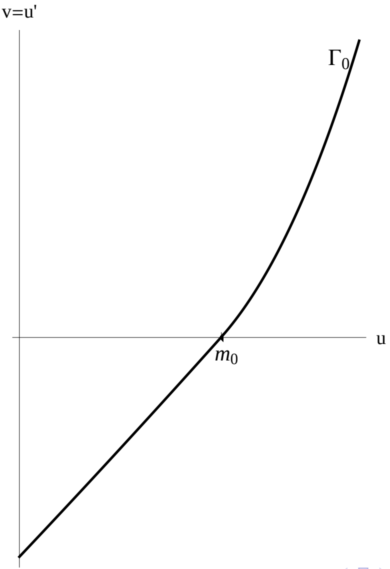
$$\Gamma_0 := \{(x, y_M(x)), x \in (0, \infty)\}$$

The curve Γ_0



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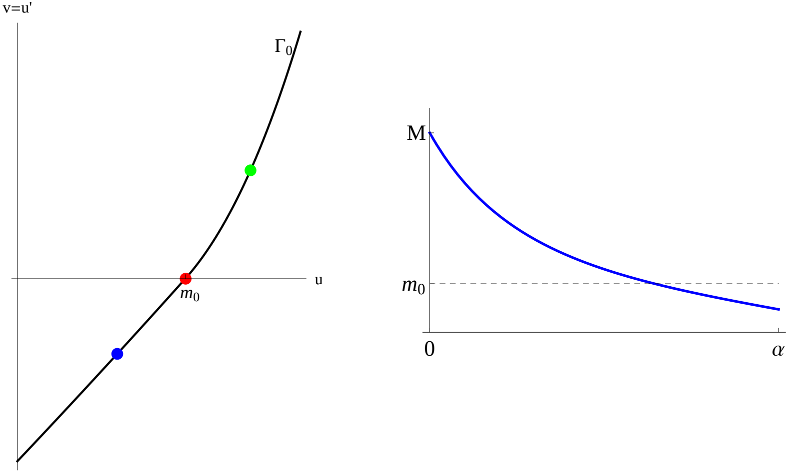

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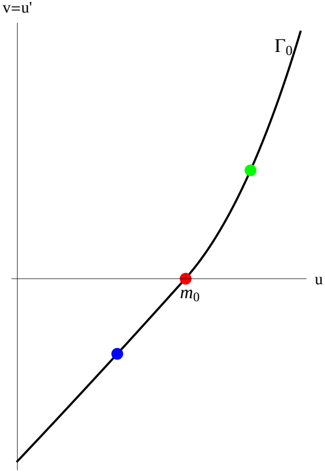

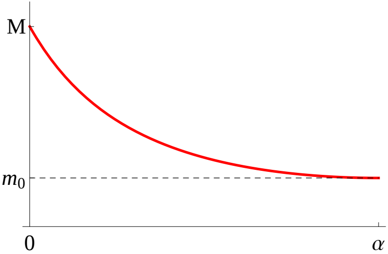

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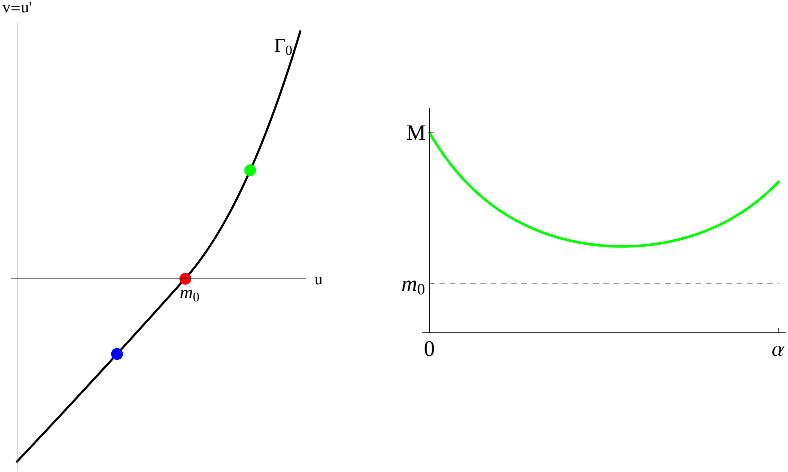

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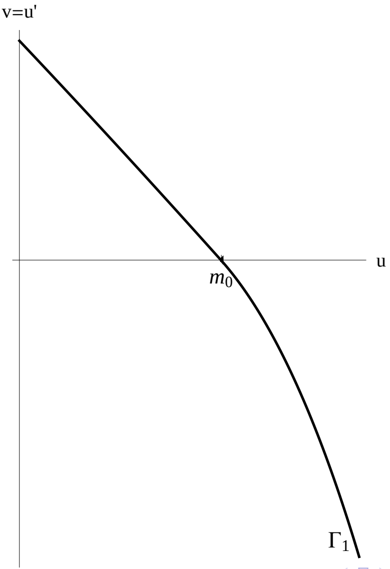

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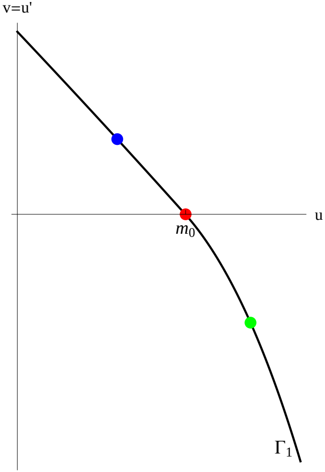

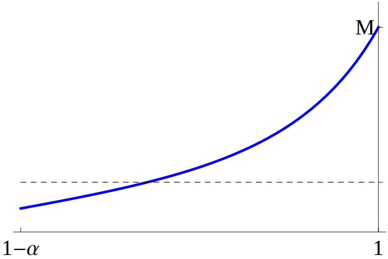

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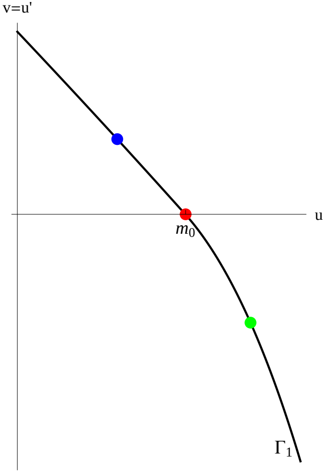

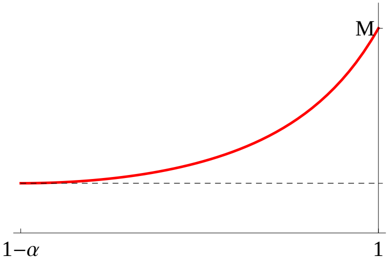

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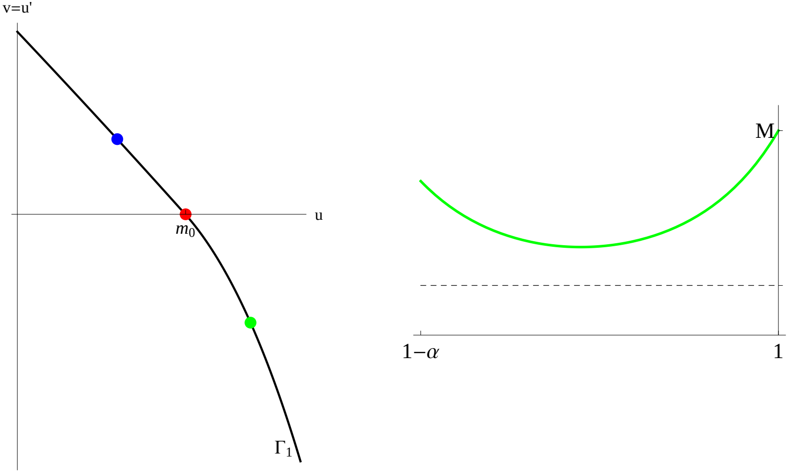

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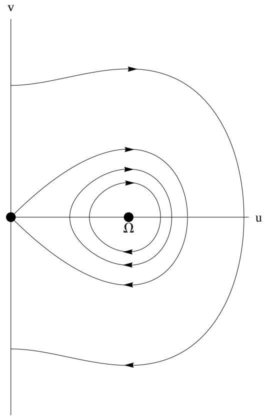

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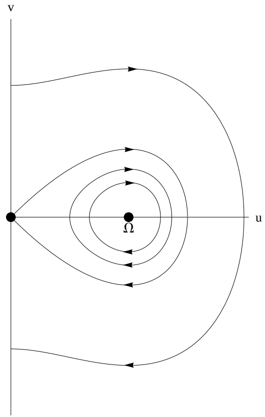

$$\Omega = \left(\frac{-\lambda}{b}\right)^{\frac{1}{p-1}}$$

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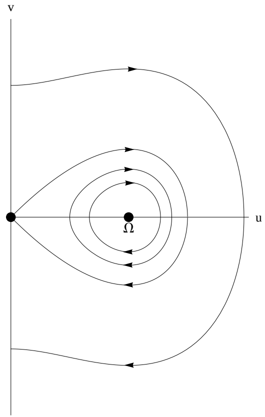
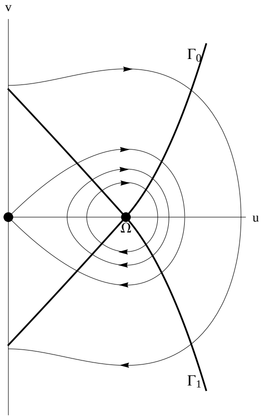


$$\Omega = \left(\frac{-\lambda}{b}\right)^{\frac{1}{p-1}} \Longrightarrow \text{choose } b^* \text{ s.t. } \Omega = m_0$$

Equation in $[\alpha, 1-\alpha]$



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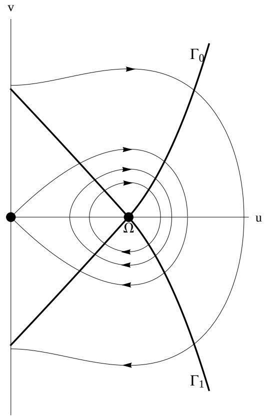
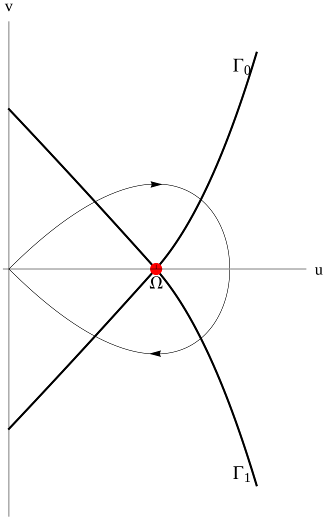
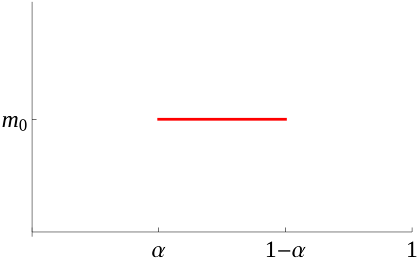


To construct solutions on $[0, 1]$ we have to *connect* Γ_0 to Γ_1 in time $1 - 2\alpha$

Connections between Γ_0 and Γ_1





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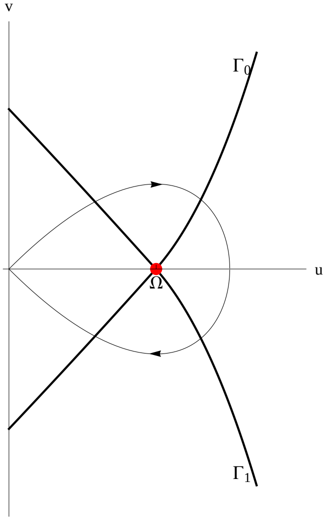

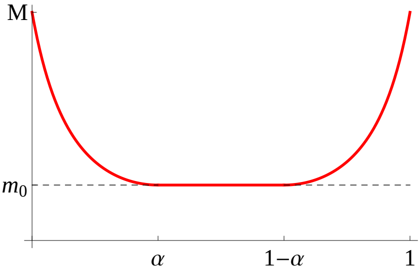

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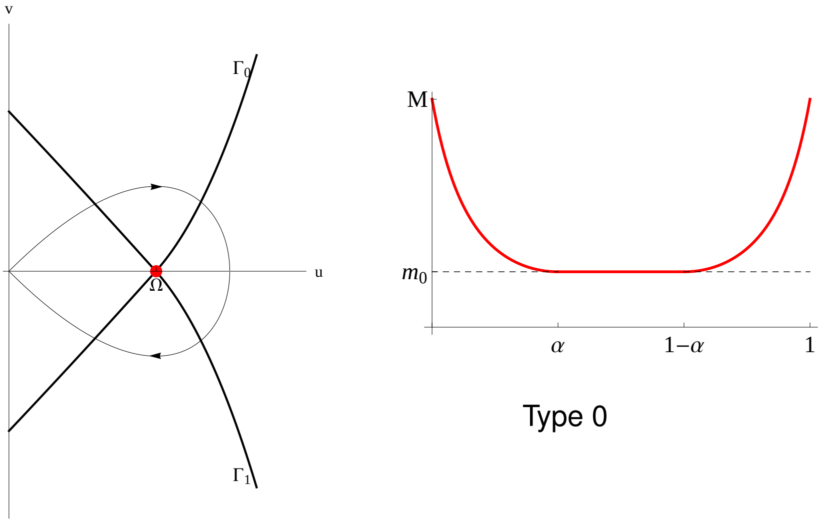

Type 0

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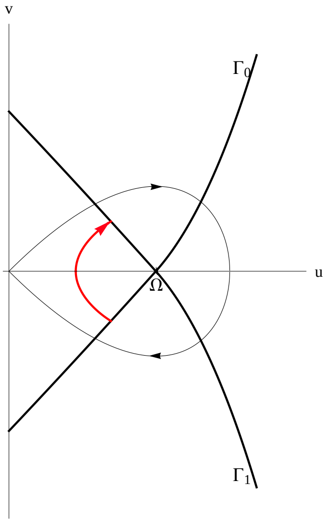

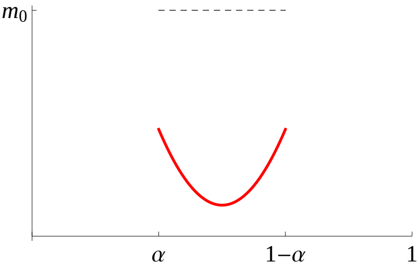

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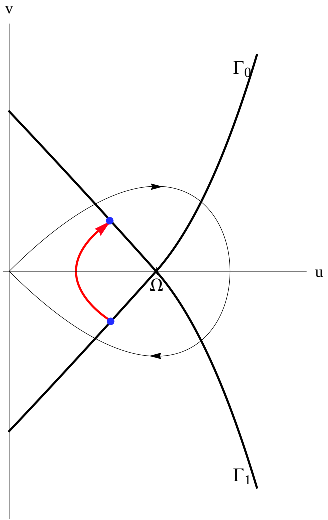

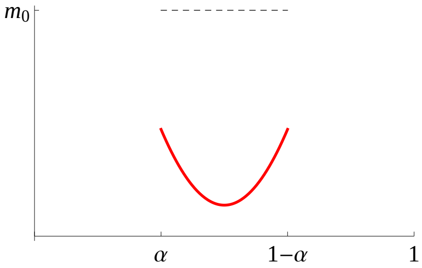

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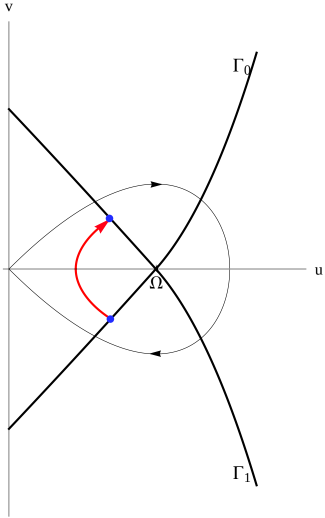
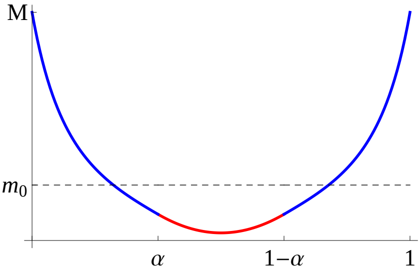
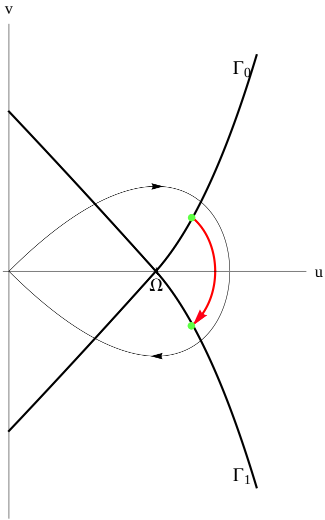
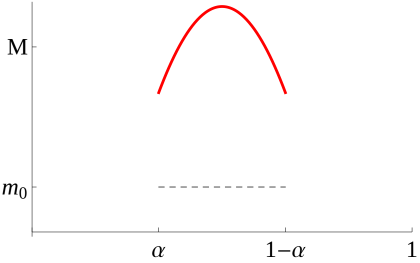




Type 1

Connections between Γ_0 and Γ_1





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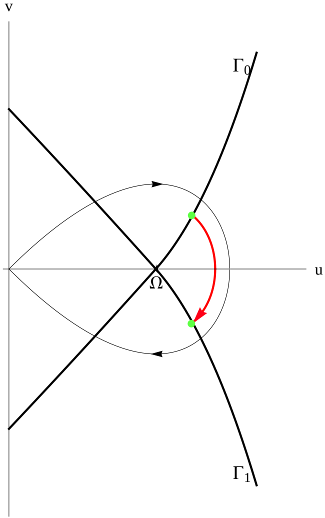

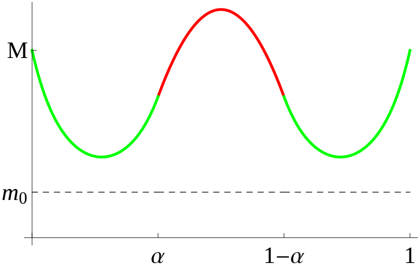

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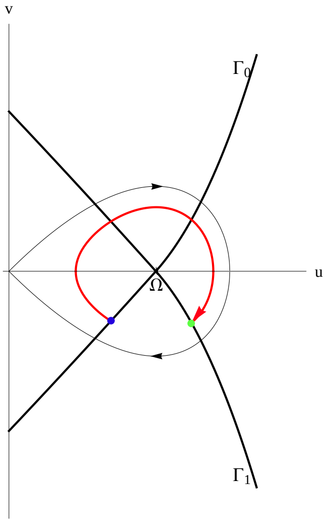

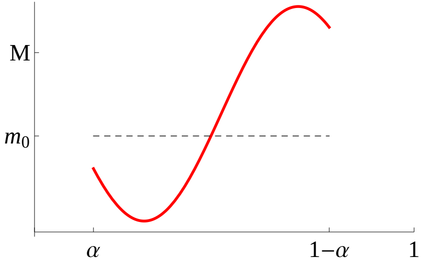

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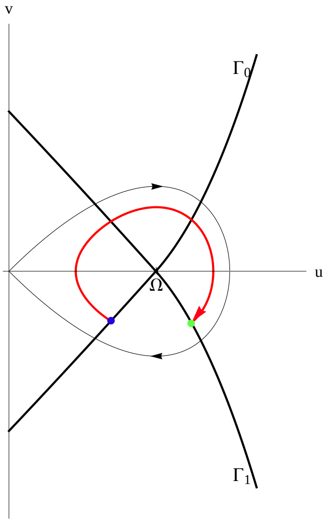
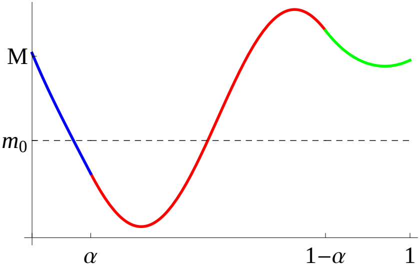
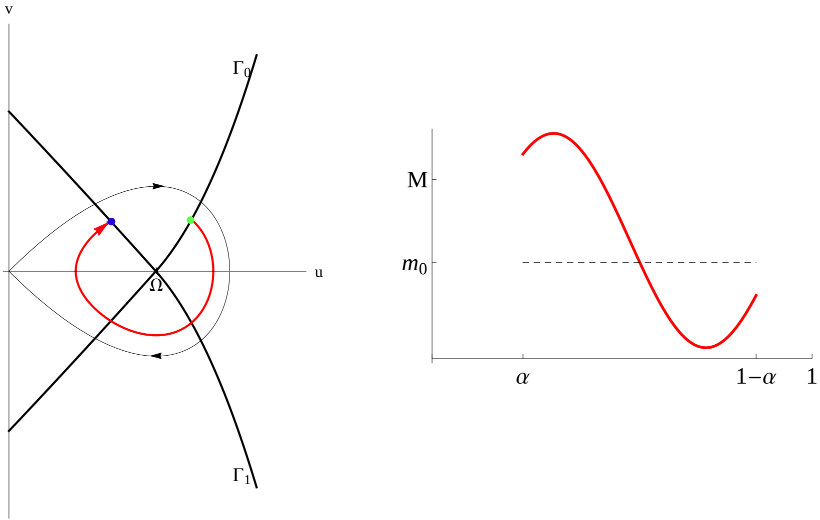




Type 2

Connections between Γ_0 and Γ_1



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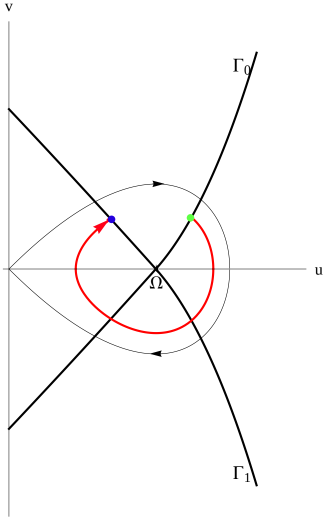

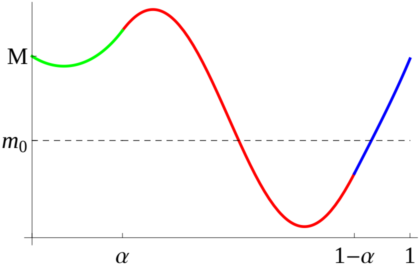

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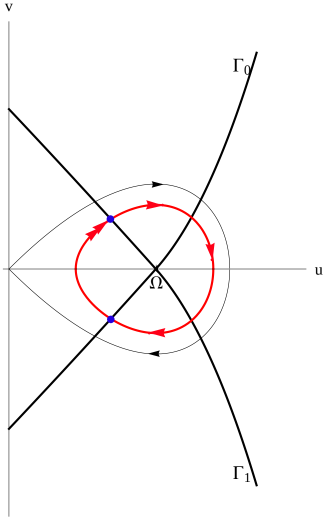

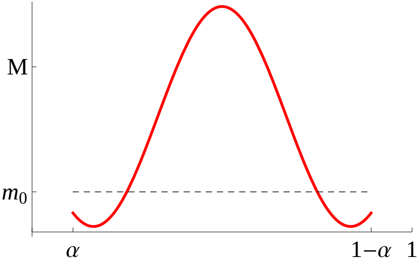

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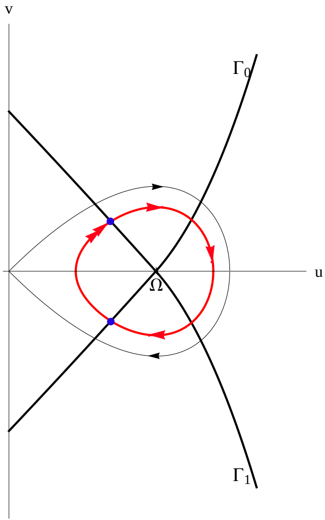
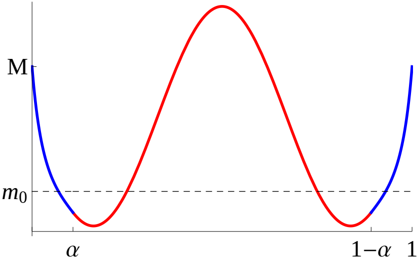
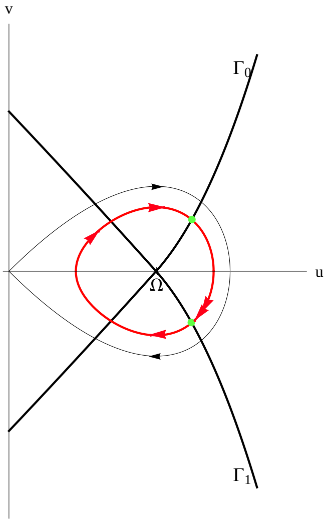
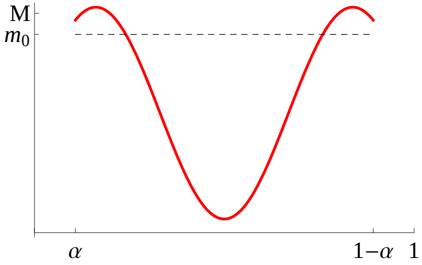




Type 3

Connections between Γ_0 and Γ_1





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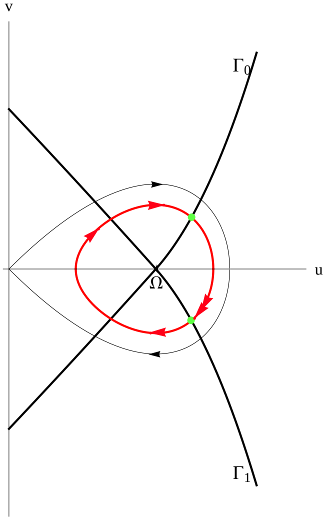
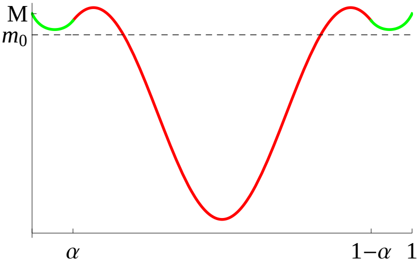
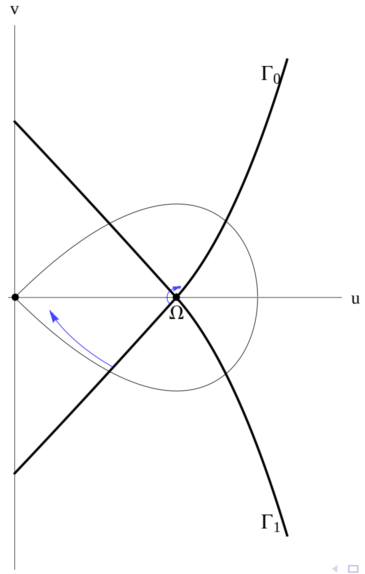




Type 3

Conditions to establish the connections



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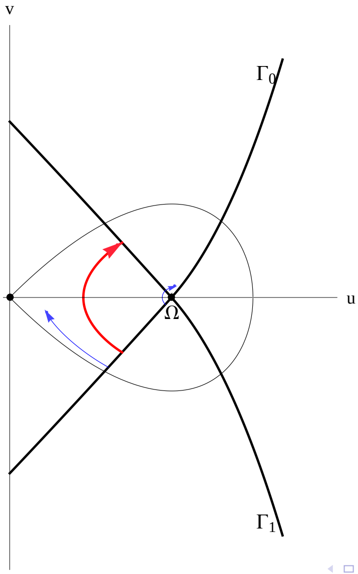

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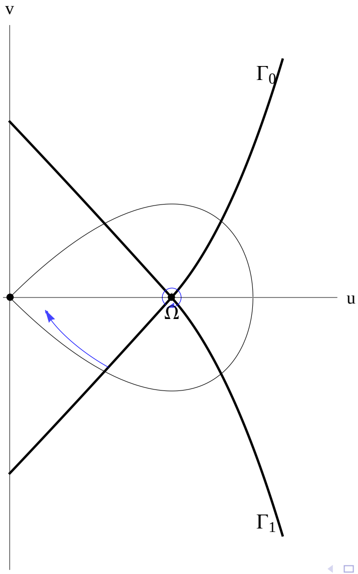

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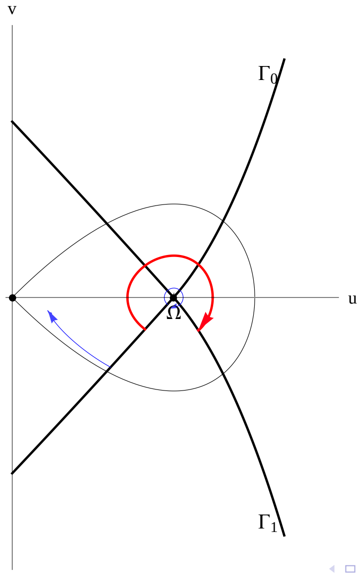

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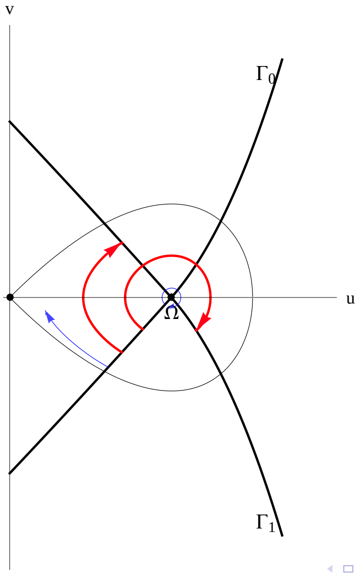

Conditions to establish the connections

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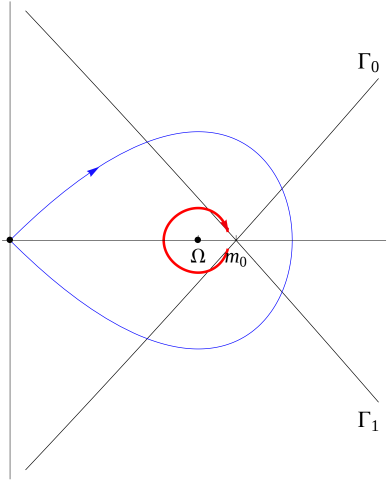
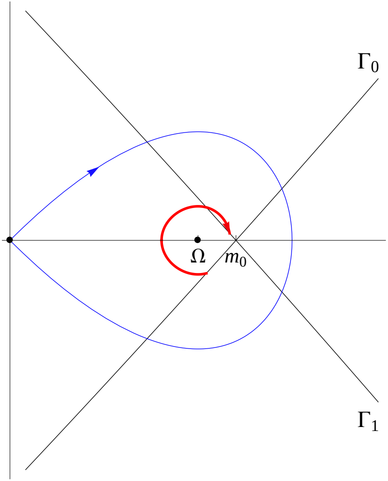
Conditions on the choice of λ can be sharpened!

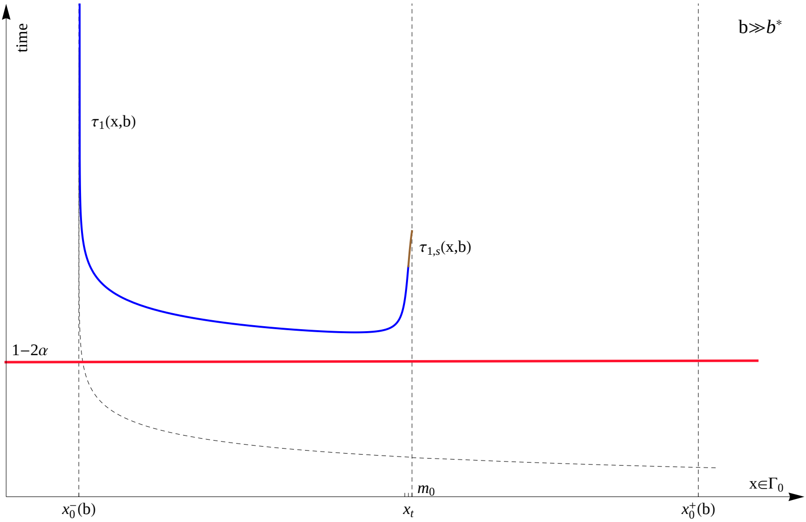
What happens when $b \neq b^*$?

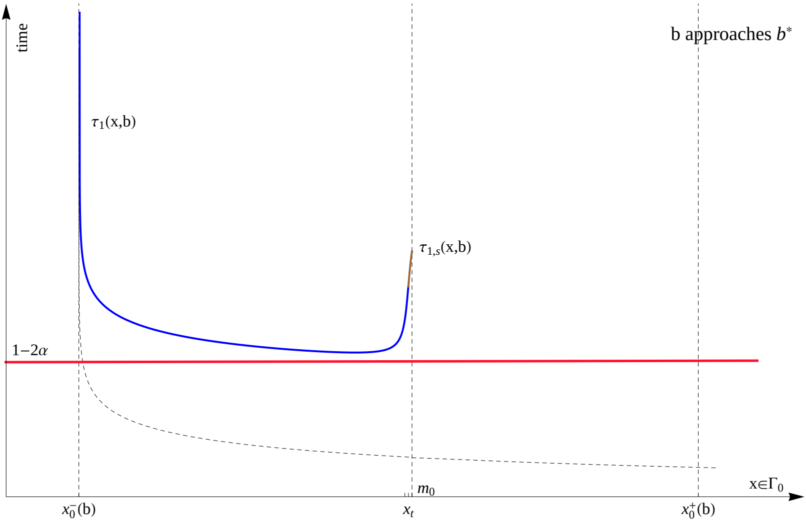


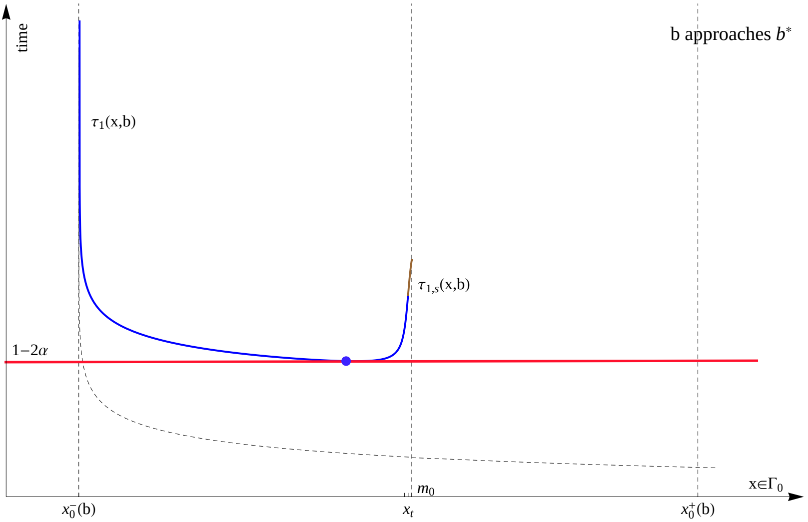
Type 1, *s*



Type 1, *a*


time
b≫b^*
τ_1(x,b)
$\tau_{1,s}$(x,b)
1−2α
m_0
x∈Γ_0
x_0^-(b)
x_t
x_0^+(b)


time
b approaches b*
τ1(x,b)
τ1,s(x,b)
1−2α
m0
x∈Γ0
x0−(b)
xt
x0+(b)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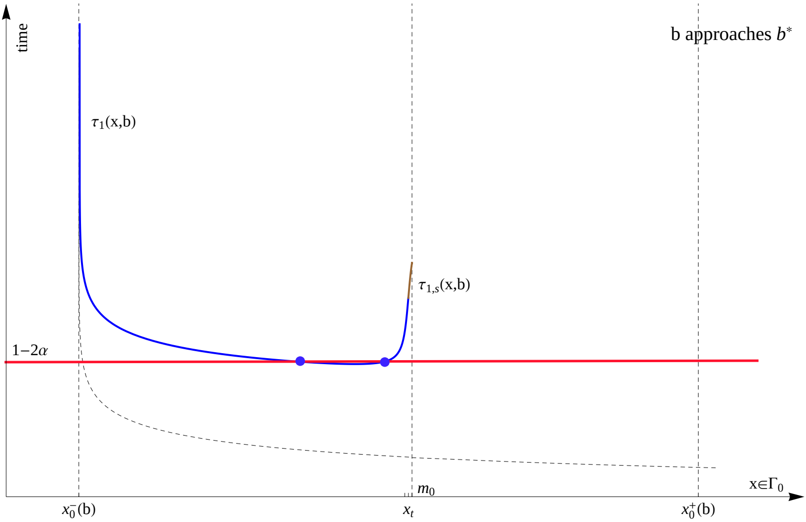
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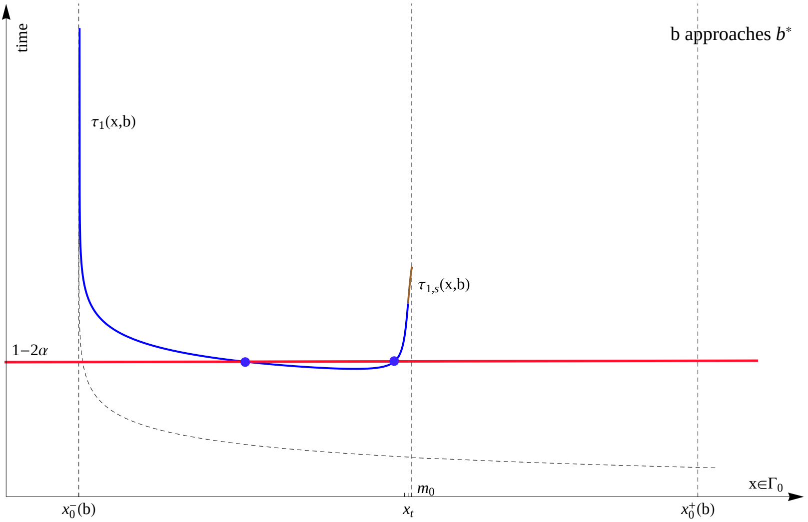
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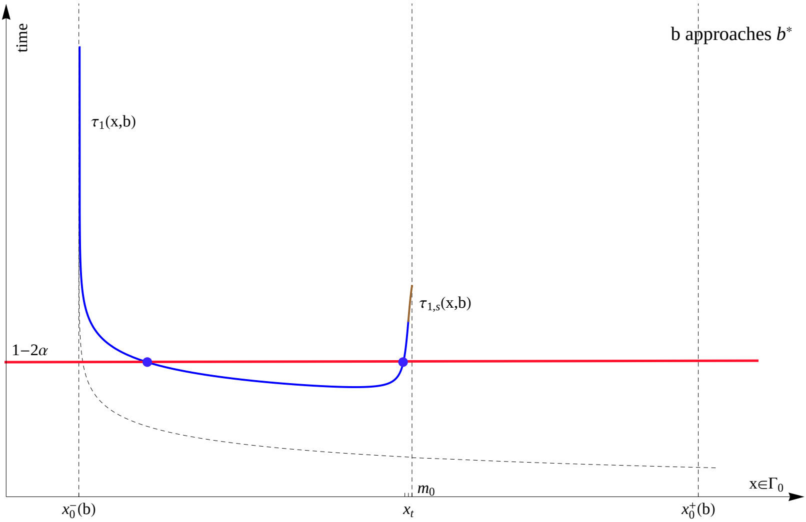
$\tau_{1,s}$(x,b)
$1-2\alpha$
m_0
$x\in\Gamma_0$
x_0^-(b)
x_t
x_0^+(b)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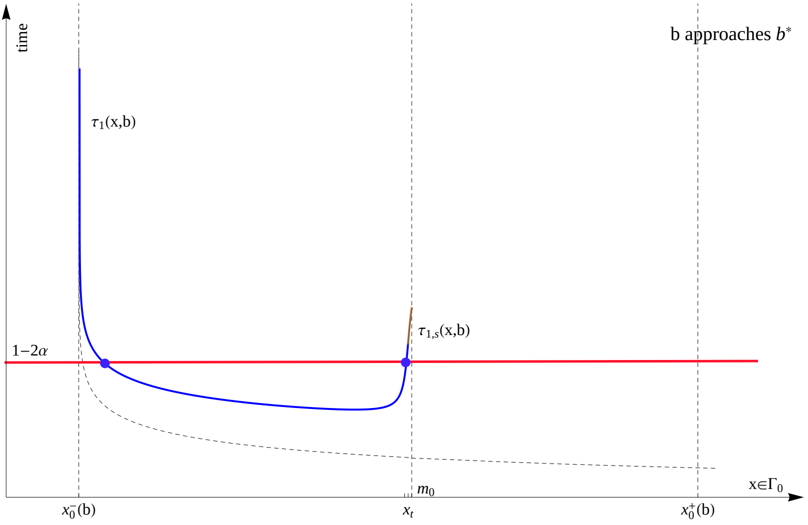

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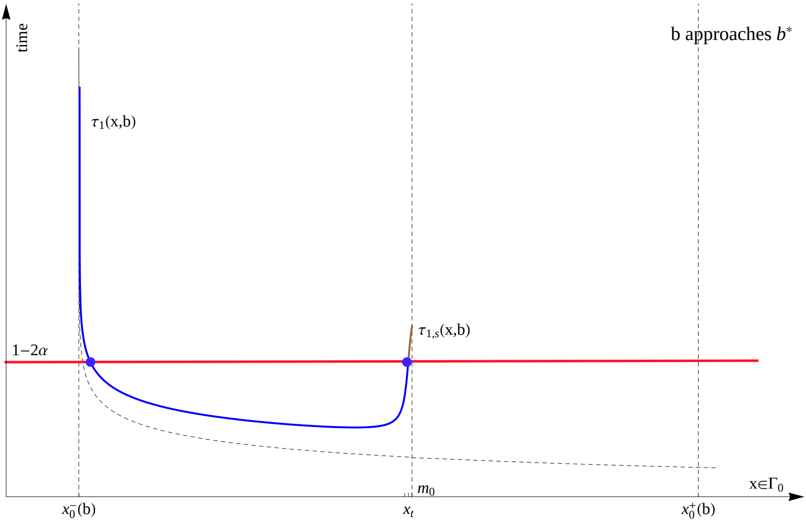

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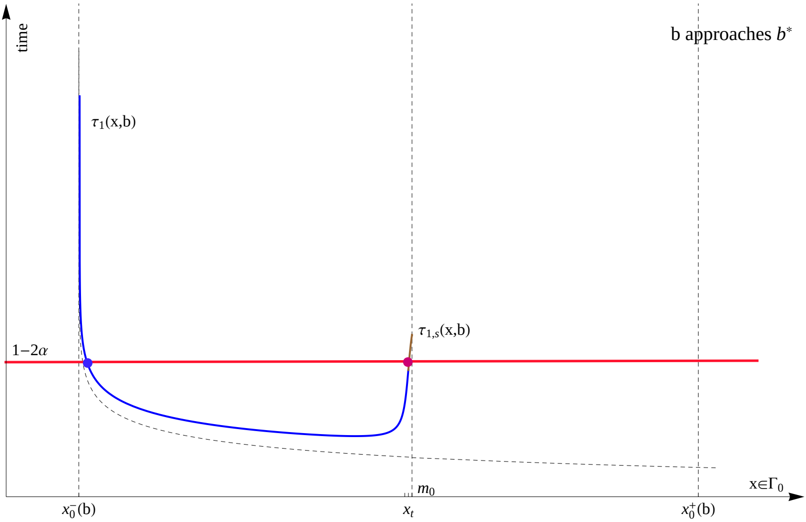

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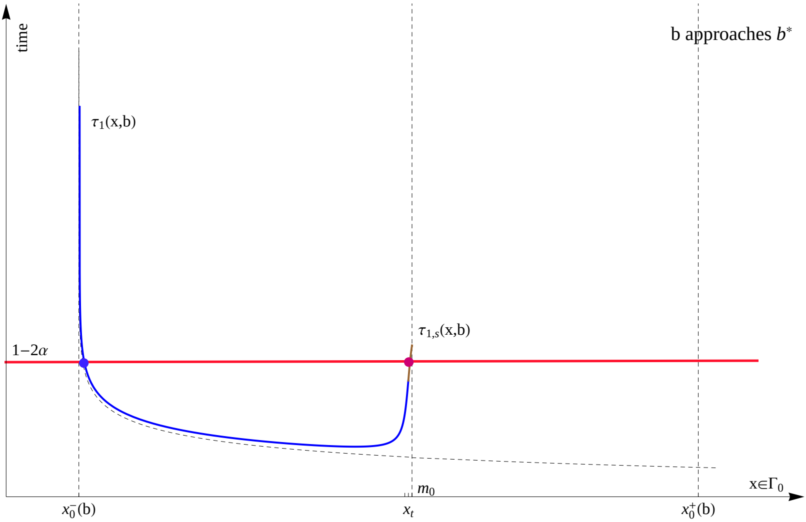
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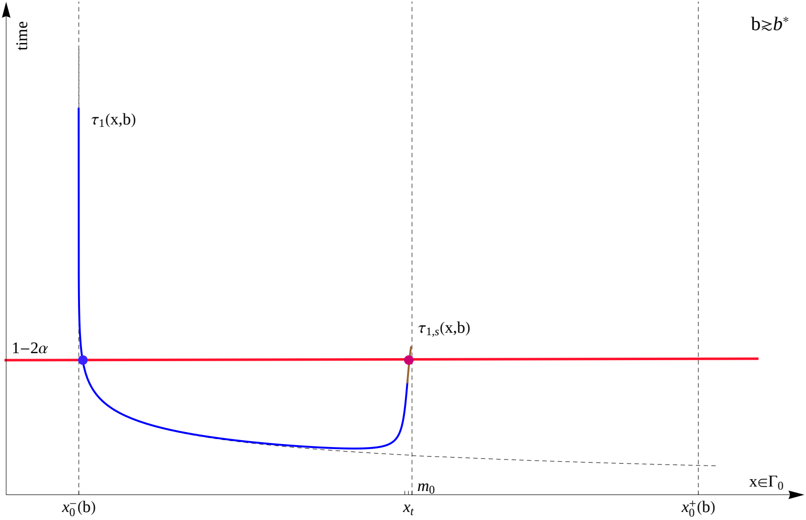
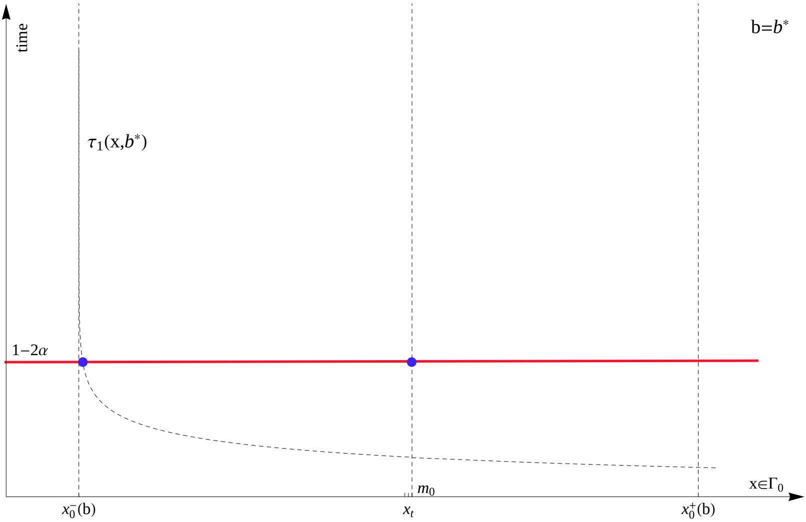


b≳b*
time
τ1(x,b)
τ1,s(x,b)
1−2α
m0
x∈Γ0
x0−(b)
xt
x0+(b)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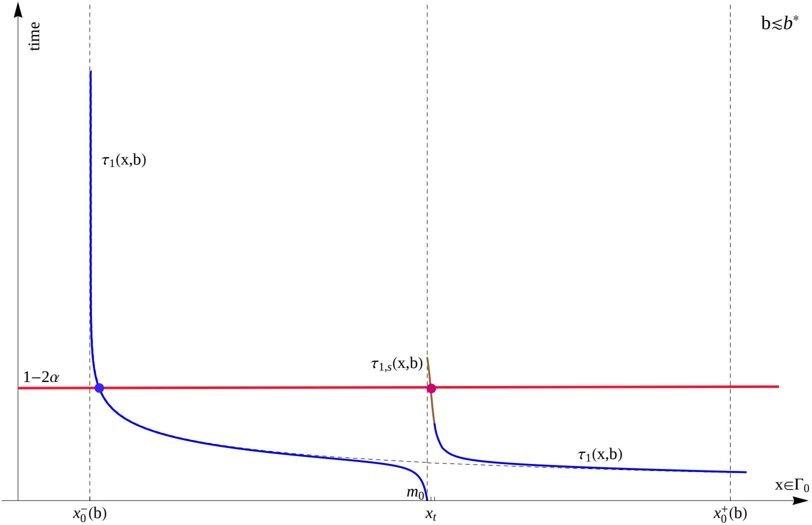

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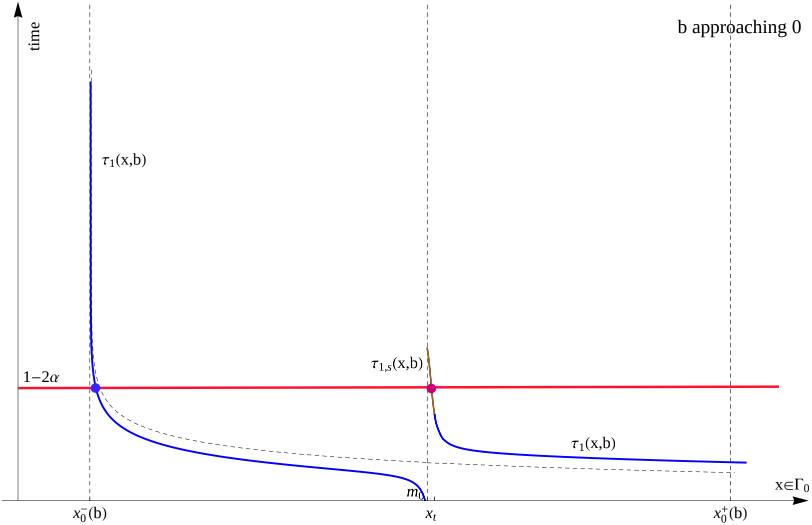

b approaching 0
time
τ_1(x,b)
$1-2\alpha$
$\tau_{1,s}$(x,b)
τ_1(x,b)
m_0
x∈Γ_0
x_0^-(b)
x_t
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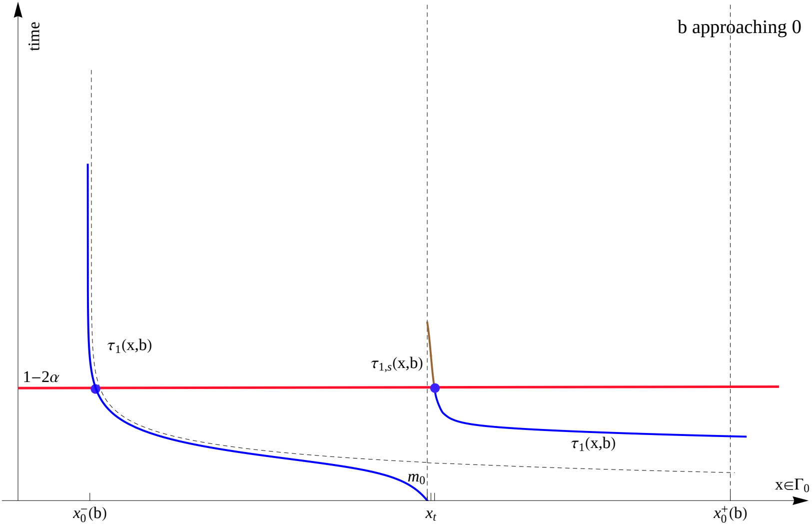
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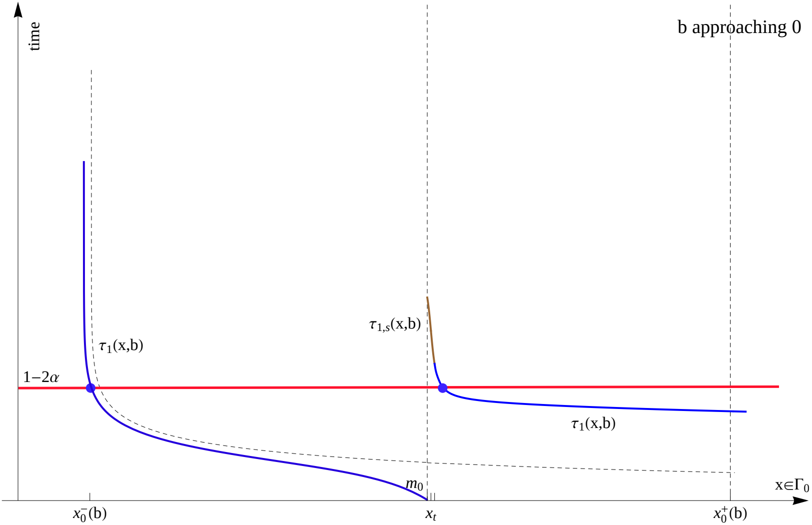
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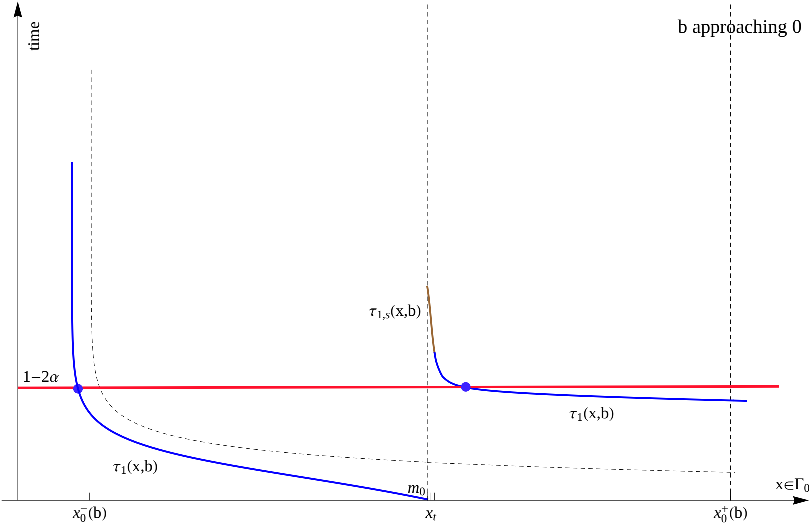
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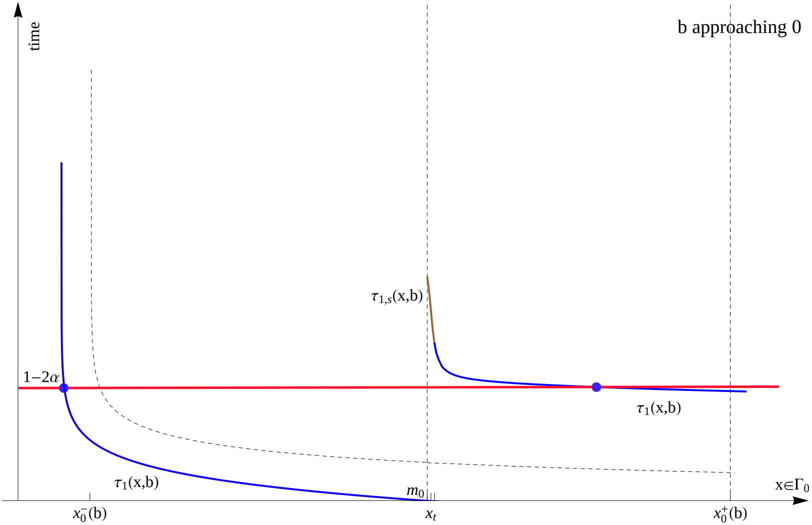

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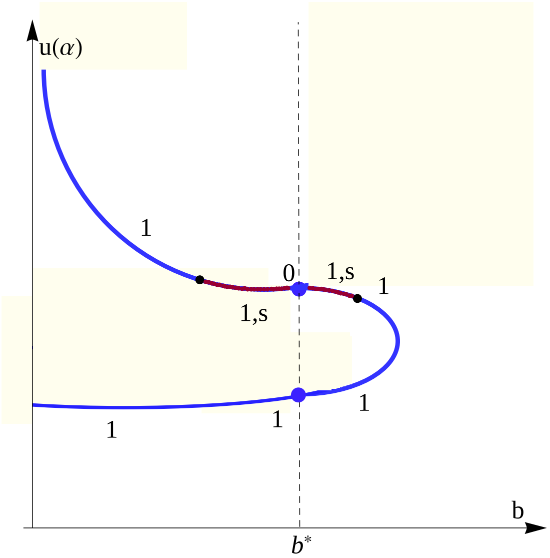
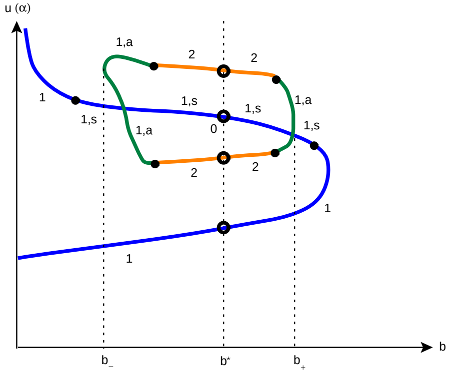
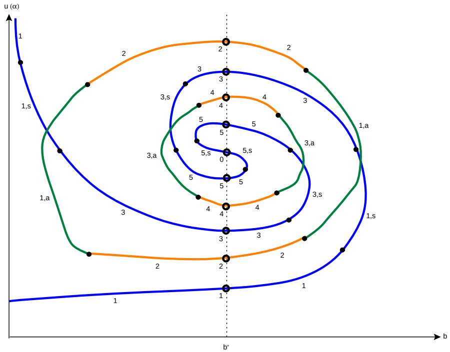
m0
x∈Γ0
x0−(b)
xt
x0+(b)


u(α)
1
0
1,s
1
1,s
1
1
1
b
b*


u (α)
1,a
2
2
1
1,s
1,s
1,a
1,s
1,a
0
1,s
2
2
1
1
b
b_
b*
b+


u (α)
b
b*
1
1,s
1,a
2
3
3,s
3,a
4
5
5,s
0

Large solutions

- Construct two curves $\Gamma_{0,\infty}$ and $\Gamma_{1,\infty}$ as limit of the curves $\Gamma_{0,M}$ and $\Gamma_{1,M}$ as $M \uparrow \infty$

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The same multiplicity result holds and the global behaviour is the same both in case $M \in \mathbb{R}$ large and $M = \infty$