

WORKSHOP OF YOUNG RESEARCHERS IN MATHEMATICS 2011

Entropy on LCA groups

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Setting and notation:

- G will always denote an LCA group (additively written);
- $\phi : G \rightarrow G$ will always denote a continuous endomorphism and $\text{End}(-)$, $\text{Aut}(-)$, \dots ;
- $\mathcal{C}(G) = \{\text{compact neighborhoods of } 0\}$;
- μ a Haar measure on G .

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(Partial) History of the definition:

- Adler, Konheim, McAndrew (1965) gave the first definition of topological entropy (compact spaces);
- Bowen (1971) modified the definition for metric spaces;
- Hood (1974) extended the definition to uniform spaces;
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History of the definition:

- Adler, Konheim, McAndrew (1965);
 - Weiss (1974/75);
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Comments on the definitions

- given another Haar measure μ' on G , there exists a constant $k \in \mathbb{R}_+$ such that $\mu = k \cdot \mu'$ and so our definitions do not depend on the choice of μ ;
- Peters's entropy only makes sense for automorphisms;
- for $\alpha \in \text{Aut}(G)$, $h_P(\alpha) = h_A(\alpha^{-1})$;
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The modulus is a group morphism

$$\text{mod}_G(-) : \text{Aut}(G) \rightarrow \mathbb{R}_+$$

such that $\mu(\alpha(E)) = \text{mod}_G(\alpha) \cdot \mu(E)$.

Example

- if G is discrete or compact then $\text{mod}_G \equiv 1$;
- if $\alpha \in \text{Aut}(\mathbb{R}^n)$ then $\text{mod}_{\mathbb{R}^n}(\alpha) = |\det_{\mathbb{R}}(\alpha)|$.

Let $\alpha \in \text{Aut}(G)$ then

$$h_A(\alpha) = h_A(\alpha^{-1}) + \log \text{mod}_G(\alpha);$$

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Theorem (SV, 2011)

Let $\phi \in \text{End}(\mathbb{Z}^n)$ and $\{\lambda_i : i = 1, \dots, n\} \subseteq \mathbb{C}$ be the family of the eigenvalues of ϕ .

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Endomorphisms of \mathbb{Q}_p^n

Let $\mathbb{P} = \{\text{primes}\} \cup \{\infty\}$. For every $p \in \mathbb{P}$, let \mathbb{Q}_p be the p -adic numbers, $|\cdot|_p$ be the p -adic norm. In particular, we set $\mathbb{Q}_\infty = \mathbb{R}$ and $|\cdot|_\infty$ be the usual absolute value.

Let also $\phi \in \text{End}(\mathbb{Q}_p^n)$ and consider the eigenvalues $\{\lambda_i : i = 1, \dots, n\}$ of ϕ in some finite extension \mathbb{K}_p of \mathbb{Q}_p .

We denote by $|\cdot|_p$ the extension of the p -adic norm to \mathbb{K}_p .

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Mahler measure

Let $P(X) = a_n X^n + \dots + a_1 X + a_0 \in \mathbb{Z}[X]$ be a primitive polynomial and denote by $\{\lambda_i : i = 1, \dots, n\}$ its roots.

The Mahler measure of $P(X)$ is

$$m(P(X)) := \log |a_n| + \sum_{|\lambda_i| > 1} \log |\lambda_i|$$

For a monic polynomial $Q(X) \in \mathbb{Q}[X]$ there exists a minimal positive integer s such that $s \cdot Q(X) \in \mathbb{Z}[X]$. Then

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The Algebraic Yuzvinski Formula

Theorem (A. Giordano Bruno and SV, 2011)

Let $\phi : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$. Then $h_A(\phi) = m(\chi_\phi(X))$.

Some steps of the proof:

Step 1. Let $\mathbb{P} = \{\text{primes}\} \cup \{\infty\}$, for every $p \in \mathbb{P}$ define

$$\phi_p = \phi \otimes_{\mathbb{Q}} \text{id}_{\mathbb{Q}_p} : \mathbb{Q}_p^n \rightarrow \mathbb{Q}_p^n \quad (\mathbb{Q}_\infty = \mathbb{R})$$

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Bridge Theorem

Theorem (A. Giordano Bruno and D. Dikranjan, 2010)

Let G be a discrete Abelian group and $\phi : G \rightarrow G$. Then

$$h_A(\phi) = h_B(\hat{\phi})$$

where $\hat{\phi} : \hat{G} \rightarrow \hat{G}$ is the Pontryagin–Van Kampen dual.

The torsion discrete/profinite case of this theorem was proved by Weiss. For the general case one needs the Algebraic Yuzvinski Formula and its topological counterpart.

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Question:

Is it possible to extend the Bridge Theorem between h_A and h_B to endomorphisms of LCA groups?

BT fails for Peters's entropy

Let $r \in \mathbb{R} \setminus \{0\}$ and consider the morphism $\varphi_r : \mathbb{R} \rightarrow \mathbb{R}$.
 $x \mapsto x \cdot r$

- $h_A(\varphi_r) = \begin{cases} \log |r| & \text{if } |r| > 1; \\ 0 & \text{if } |r| \leq 1. \end{cases}$
- $h_P(\varphi_r) = h_A(\varphi_r) - \log(\text{mod}_{\mathbb{R}}(\varphi_r))$

- [Bowen] $h_B(\varphi_r) = \begin{cases} \log |r| & \text{if } |r| > 1; \\ 0 & \text{if } |r| \leq 1. \end{cases}$

Applying duality $\widehat{\varphi}_r = \varphi_r : \widehat{\mathbb{R}} = \mathbb{R} \rightarrow \widehat{\mathbb{R}} = \mathbb{R}$, and so

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