Boundedness and Topology

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Definitions

- A bornology $\mathcal{B}$ on a set X is an ideal in $\mathcal{P}(X)$ that contains all singletons.
- A bornological universe is a pair $(X, \mathcal{B})$ where X is a topological space and $\mathcal{B}$ is a bornology.

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Examples

- A topological space and its relatively compact subsets,
- a metric space and the subsets with finite diameter,
- a uniform space and its totally bounded subsets.

Motivation

In a bornological universe both the abstract ideas of *infinitely small* and *infinitely large* are captured.

Definition

A bornological universe $(X, \mathcal{B})$ will be called metrizable iff the topology on X is defined by a metric d and $\mathcal{B}$ is the set of all subsets of X that have finite diameter for the metric d.

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Theorem

A bornological universe $(X, \mathcal{B})$ is metrizable iff the following conditions are satisfied:

- X is metrizable,
- $\mathcal{B}$ has a countable base,
- for each $B_1 \in \mathcal{B}$ there is a $B_2 \in \mathcal{B}$ such that $\overline{B_1} \subseteq \mathring{B_2}$

Reference

- S. Hu, *Boundedness in a topological space*, 1949.

Definition

A locally convex topological vector space is called *bornological* iff each absorbing, absolutely convex set is a neighbourhood of 0.

Proposition

- Each normed vector space is bornological.
- Each bounded seminorm on a bornological space is continuous.
- Each bounded linear map on a bornological space is continuous.
- Each sequentially continuous, linear map on a bornological space is continuous.

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Questions

- How should a *uniformizable* bornological universe be defined?
- How are these bornological universes characterized?

Definition

A subset B of a uniform space $(X, \mathcal{U})$ is called Bourbaki bounded iff B has a finite diameter for each uniformly continuous pseudometric.

Proposition

The following are equivalent:

- B is Bourbaki bounded,
- for each $U \in \mathcal{U}$ there is a finite set K and an $n \in \mathbb{N}$ such that $B \subseteq U^n(K)$,
- each uniformly continuous, real-valued map is bounded on B.

Definition

A bornological universe $(X, \mathcal{B})$ will be called uniformizable iff the topology on X is defined by uniformity $\mathcal{U}$ and $\mathcal{B}$ is the set of all subsets of X that are Bourbaki bounded for $\mathcal{U}$.

Definitions

- $\mathcal{C}_{\mathcal{B}}(X)$ denotes the set of all bounded, continuous maps into the real line,
- a sequence $(G_n)_n$ of open sets is called bounding iff each bounded set is contained in a certain G_n,
- a bornology is called saturated iff it contains each B that is contained in some G_n whenever $(G_n)_n$ is a bounding sequence of open sets that satisfies $\overline{G_n} \subseteq G_{n+1}$.

Theorem

The following are equivalent:

- $(X, \mathcal{B})$ *is uniformizable,*
- $\mathcal{C}_{\mathcal{B}}(X)$ *is an initial source,*
- X *is completely regular and* $\mathcal{B}$ *is saturated.*

If X *is Hausdorff, then* $(X, \mathcal{B})$ *is uniformizable iff it is isomorphic to a subspace of a product of real lines.*

Definition

A bornological universe $(X, \mathcal{B})$ will be called *tb*-uniformizable iff the topology on X is is defined by a uniformity $\mathcal{U}$ and $\mathcal{B}$ is the set of all subsets of X that are totally bounded for $\mathcal{U}$.

Definition

A bornological universe $(X, \mathcal{B})$ will be called *tb*-uniformizable iff the topology on X is is defined by a uniformity $\mathcal{U}$ and $\mathcal{B}$ is the set of all subsets of X that are totally bounded for $\mathcal{U}$.

Theorem

The following are equivalent:

- $(X, \mathcal{B})$ *is uniformizable,*
- $(X, \mathcal{B})$ *is tb-uniformizable.*

Definition

A bornological universe $(X, \mathcal{B})$ for which X is Tychonoff will be called realcompact iff it is complete for the uniformity induced by the source $\mathcal{C}_{\mathcal{B}}(X)$.

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Propositions

- *$(X, \mathcal{B})$ is realcompact iff X is realcompact (as a topological space) and all $B \in \mathcal{B}$ are relatively compact.*

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A bornological universe $(X, \mathcal{B})$ for which X is Tychonoff will be called realcompact iff it is complete for the uniformity induced by the source $\mathcal{C}_{\mathcal{B}}(X)$.

Propositions

- *$(X, \mathcal{B})$ is realcompact iff X is realcompact (as a topological space) and all $B \in \mathcal{B}$ are relatively compact.*
- *$(X, \mathcal{B})$ is realcompact iff X is equal to*

$$\upsilon_{\mathcal{B}}(X) = \{x \in \beta(X) | \forall f \in \mathcal{C}_{\mathcal{B}}(X) : f^{\beta}(x) \neq \infty\},$$

- *$(X, \mathcal{B})$ is realcompact iff X is equal to*

$$\upsilon_{\mathcal{B}}(X) = \upsilon\left(\bigcup_{B \in \mathcal{B}} \overline{B}\right)$$

- *$\upsilon_{\mathcal{B}}(X)$ is the largest realcompact extension of X in $\beta(X)$ to which each bounded, continuous real-valued map can be continuously extended.*

Definition A sequence $(Z_n)_n$ of zero-sets of X will be called unbounded iff for each $B \in \mathcal{B}$ there is an $n \in \mathbb{N}$ such that $Z_n \cap \overline{B} = \phi$.

Proposition

- $(X, \mathcal{B})$ *is realcompact iff each z-ultrafilter that does not contain a decreasing, unbounded sequence of zero-sets, converges.*
- $(X, \mathcal{B})$ *is uniformizable and realcompact iff it is isomorphic to a closed subspace of a product of real lines.*

Proposition *Let x be an element of $\beta(X)$. The following statements are equivalent:*

- $x \in \upsilon_{\mathcal{B}}(X)$,
- *for each sequence $(K_n)_{n\in\mathbb{N}}$ of closed subsets of $\beta(X)$ that satisfies*
$$\forall B \in \mathcal{B} \exists n \in \mathbb{N} : B \subseteq K_n$$
there is an $m \in \mathbb{N}$ such that $x \in K_m$,
- *there is no sequence $(V_n)_n$ of neighbourhoods of x that satisfies*
$$\forall B \in \mathcal{B} \exists n \in \mathbb{N} : B \cap V_n = \phi.$$

Proposition

A realcompactification $Y \subseteq \beta(X)$ of X is defined by a bornology on X iff it is defined by the bornology of subsets of X that are relatively compact in Y.

Theorem

A realcompactification $Y \subseteq \beta(X)$ of X is defined by a bornology on X iff for each sequence of neighbourhoods $(V_n)_n$ of an element $y \in Y$ there is a subset B of X that is relatively compact in Y such that $B \cap V_n \neq \phi$ for all $n \in \mathbb{N}$.

Corollaries

- *If a realcompactification Y of X satisfies the first axiom of countability, then it is defined by a bornology iff each element in Y is in the closure of a subset of X that is relatively compact in Y.*
- *If a realcompactification Y of X is hemicompact, then it is defined by a bornology iff each element in Y is in the closure of a subset of X that is relatively compact in Y.*

Definition

A subset X of a Tychonoff space Y will be called CB-embedded iff each real-valued, continuous map f on X that is bounded on all subsets that are relatively pseudocompact in Y, can be extended to Y.

Proposition

A realcompactification Y of X is bornological iff X is CB-embedded in Y.

Proposition

A realcompactification Y of X is bornological iff each continuous map $f : X \to Z$ into a realcompact space Z that sends subsets of X that are relatively pseudocompact in Y to relatively pseudocompact subsets of Z, can be continuously extended to Y.

Definition

A zero-set $Z \subseteq Y$ is called far from X iff there exists an increasing sequence $(Z_n)_n$ of zero-sets that are disjoint from Z such that each subset of X that is relatively pseudocompact in Y is contained in a certain Z_n.

Definition

A zero-set $Z \subseteq Y$ is called far from X iff there exists an increasing sequence $(Z_n)_n$ of zero-sets that are disjoint from Z such that each subset of X that is relatively pseudocompact in Y is contained in a certain Z_n.

Proposition

A realcompactification Y of X is bornological iff X is C^*-embedded in Y and completely separated from each zero-set that is far from it.

Definition

A filter $\mathcal{F}$ on a topological space is called completely regular if for each $A \in \mathcal{F}$ there is a $B \in \mathcal{F}$ such that B and $X \setminus A$ are completely separated.

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Proposition

A realcompactification Y of X is bornological iff each maximal completely regular filter on X is the trace of a maximal completely regular filter on Y that does not meet any zero-sets that are far from X.