Universidad Complutense de Madrid
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Analytical Techniques on Multilinear Problems

(Técnicas Analíticas en Problemas Multilineales)

Memoria para optar al grado de doctor
presentada por

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O esforço é grande e o homem é pequeno.
Eu, Diogo Cão, navegador, deixei
Este padrão ao pé do areal moreno
E para adiante naveguei.

A alma é divina e a obra é imperfeita.
Este padrão assinala ao vento e aos céus
Que, da obra ousada, é minha a parte feita:
O por-fazer é só com Deus.

E ao imenso e possível oceano
Ensinam estas Quinas, que aqui vês,
Que o mar com fim será grego ou romano:
O mar sem fim é português.

E a Cruz ao alto diz que o que me há na alma
E faz a febre em mim de navegar
Só encontrará de Deus na eterna calma
O porto sempre por achar.

Fernando Pessoa – Padrão

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Sobre esta Tesis

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Fruto del trabajo de estos cuatro años han sido los artículos:

1. G. Botelho, D. Cariello, V.V. Fávaro, D. Pellegrino, J.B. Seoane-Sepúlveda. On very non-linear subsets of continuous functions. *The Quarterly Journal of Mathematics*. 65 (2014), 841–850.

2. D. Cariello, J.B. Seoane-Sepúlveda. Basic sequences and spaceability in ℓ_p spaces. *Journal of Functional Analysis*. 266 (2014), 3797–3814.

3. D. Cariello. Separability for weakly irreducible matrices. *Quantum Inf. Comp.* 14 (2014), 1308–1337.

4. D. Cariello. Does symmetry imply PPT property? *Quantum Inf. Comp.* 15 (2015), 812–824.

5. D. Cariello. Completely Reducible Maps in Quantum Information Theory. *IEEE Transactions on Information Theory* 62 (2016) 1721–1732.

Contents

Resumen

Esta tesis doctoral se centra principalmente en tres problemas multilineales y su objetivo es describir las técnicas analíticas y topológicas útiles para "atacar" estos problemas. El primer problema tiene su origen en la Teoría de Información Cuántica, es el llamado problema de la separabilidad de los estados cuánticos, y los otros dos fueron propuestos por Vladimir I. Gurariy. A grandes rasgos, en nuestro primer problema se emplea la teoría de Perron-Frobenius, que está relacionada con las aplicaciones positivas que actúan sobre C^*- álgebras, con el fin de obtener una reducción del problema de la separabilidad a un caso particular y algunas otras aplicaciones a la teoría de información cuántica. Para el segundo problema, se utilizó el teorema de Borsuk-Ulam para demostrar que la dimensión de cierto espacio vectorial debe estar comprendida en cierto intervalo. Para el tercer problema, hemos construido sucesiones básicas con propiedades especiales a fin de obtener una solución completa del mismo.

Denotemos por M_k al conjunto de las matrices complejas de orden k y P_k será el conjunto de matrices Hermíticas semidefinidas positivas de M_k. El problema de la separabilidad de los estados cuánticos es un problema famoso y bien establecido en el campo de la teoría de información cuántica debido a su importancia y, sobre todo, a su gran dificultad.

El objetivo de este problema es encontrar un criterio determinístico para distinguir los estados separables de los estados entrelazados. Aqui sólo trabajamos con el caso bipartito de dimensión finita, luego los estados son los elementos del producto tensorial $M_k \otimes M_m$, que pueden ser interpretados como matrices en M_{km} a través del producto de Kronecker.

Decimos que $B \in M_k \otimes M_m$ es separable si $B = \sum_{i=1}^{n} C_i \otimes D_i$, donde $C_i \in P_k$ y $D_i \in P_m$, para cada i. Si B no es separable entonces B está entrelazada.

Este problema fue resuelto por completo por Horodecki en el espacio $M_k \otimes M_m$ donde $km \leq 6$, por el llamado criterio PPT (ver [29]). Este criterio establece que una matriz $A = \sum_{i=1}^{k} A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$, $km \leq 6$, es separable si y sólo si A permanece positiva bajo transposición parcial (PPT), es decir, A y $A^{t_2} = \sum_{i=1}^{k} A_i \otimes B_i^t$ son matrices Hermíticas semidefinidas positivas (definición 3.1).

El caso general, incluso para el caso de dimensión finita, sigue siendo un gran desafío. Se han desarrollado algoritmos con el fin de resolver el problema de la separabilidad, pero se sabe que este problema es NP-hard (véase [28]). Por lo tanto, cualquier restricción del problema a un conjunto más pequeño de matrices es, sin duda, muy importante. Por ejemplo, Peres en [39] fue el primero en darse cuenta de la importancia de la propiedad PPT que más tarde se demostró ser necesaria y suficiente para la separabilidad en $M_k \otimes M_m$ de $km \leq 6$, en [29] .

Otra reducción ha sido obtenida para el caso positivo definido en $M_k \otimes M_m$. Con el fin de encontrar las matrices Hermíticas positivas definidas y separables sólo tenemos que distinguir las matrices separables entre las matrices positivas definidas del tipo siguiente:

$$Id \otimes Id + \sum_{i=1}^{l} a_i E_i \otimes F_i$$

donde $tr(E_i) = tr(F_i) = 0$, $\{E_1, ..., E_l\}$, $\{F_1, ..., F_l\}$ son conjuntos de matrices ortonormales hermíticas con respecto al producto interno de la traza y $a_i \in \mathbb{R}$. Este resultado se obtiene por medio de una forma normal (véase la subsección 3.3.2 y [23, 34, 46]).

Los autores de [34] también obteneran una notable reducción del problema de separabilidad en $M_2 \otimes M_2$ para el caso general, no sólo para el caso positivo definido. Ellos mostraron que, para resolver el problema en $M_2 \otimes M_2$, es suficiente descubrir qué matrices de la siguiente familia son separables:

$$Id \otimes Id + d_2\gamma_2 \otimes \gamma_2 + d_3\gamma_3 \otimes \gamma_3 + d_4\gamma_4 \otimes \gamma_4,$$

donde $d_2, d_3, d_4 \in \mathbb{R}$ y $\gamma_2, \gamma_3, \gamma_4$ son las matrices de Pauli diferentes de Id. Ellos demostraron que una matriz de esta familia es separable si y sólo si es PPT, y si y sólo si $|d_2| + |d_3| + |d_4| \leq 1$. Esta es una segunda demonstración del criterio PPT en $M_2 \otimes M_2$.

El lector interesado puede encontrar más información en relación con el problema de separabilidad en [25].

A continuación, vamos a describir cómo utilizamos la teoría Perron-Frobenius con el fin de reducir el problema de separabilidad a un cierto subconjunto de matrices PPT y para obtener algunas otras aplicaciones.

Denotemos por VM_kW el conjunto $\{VXW, X \in M_k\}$, donde $V, W \in M_k$ son proyecciones ortogonales. Si $V = W$ entonces el conjunto VM_kV es una C^*- subálgebra hereditaria de M_k. Se dice que una transformación lineal $T : VM_kV \to WM_mW$ es una aplicación positiva, si $T(P_k \cap VM_kV) \subset P_m \cap WM_mW$. Se dice que una aplicación no nula positiva $T : VM_kV \to VM_kV$ es irreducible si $V'M_kV' \subset VM_kV$ es tal que $T(V'M_kV') \subset V'M_kV'$ entonces $V' = V$ o $V' = 0$.

Por la teoría de Perron-Frobenius, sabemos que si $T : VM_kV \to VM_kV$ es una aplicación positiva, entonces su radio espectral, λ, es un valor propio y hay $0 \neq \gamma \in P_k \cap VM_kV$ de tal manera que $T(\gamma) = \lambda\gamma$. Por otra parte, si $T : VM_kV \to VM_kV$ es irreducible, la multiplicidad del radio espectral es 1 y las imágenes de γ y V son iguales (ver proposiciones 2.3 y 2.5 en [21]).

Para ciertos tipos de aplicaciones positivas vale el recíproco del último teorema. Por ejemplo, si $T : VM_kV \to VM_kV$ es una aplicación positiva autoadjunta con respecto al producto interno de la traza ($\langle X, Y\rangle = tr(XY^*)$), si su radio espectral tiene multiplicidad 1 y $\Im(\gamma) = \Im(V)$ entonces $T : VM_kV \to VM_kV$ es irreducible (ver lema 2.11). Otro ejemplo es una aplicación completamente positiva (véase la definición en [44]).

Una extensión natural del concepto de aplicación irreducible positiva es una suma directa de aplicaciones irreducibles positivas. Digamos que $T : VM_kV \to VM_kV$ es una aplicación completamente reducible, si es positiva y si hay proyecciones ortogonales $V_1, \ldots, V_s \in M_k$ tales que $V_iV_j = 0$ $(i \neq j)$, $V_iV = V_i$ $(1 \leq i \leq s)$, $VM_kV = V_1M_kV_1 \oplus \ldots \oplus V_sM_kV_s \oplus R$, $R \perp V_1M_kV_1 \oplus \ldots \oplus V_sM_kV_s$ y que satisfacen: $T(V_iM_kV_i) \subset V_iM_kV_i$ $(1 \leq i \leq s)$, $T|_{V_iM_kV_i}$ es irreducible $(1 \leq i \leq s)$, $T|_R \equiv 0$. Observe que cualquier aplicación irreducible es completamente reducible. Este concepto está relacionado con el de la matriz completamente reducible (ver [42]).

La única restricción fuerte en la definición de aplicación completamente reducible es $T|_R \equiv 0$. Por ejemplo, la existencia de las subálgebras $V_iM_kV_i$ que cumplen las condiciones requeridas está garantizada para cualquier aplicación positiva autoadjunta, sin embargo la condición $T|_R \equiv 0$ es (en general) falsa. La aplicación positiva autoadjunta más simple que no es completamente reducible es la identidad $Id : M_k \to M_k$, $k > 1$. Nuevamente, como ocurre con las aplicaciones irreducibles, para aplicaciones autoadjuntas hay una propiedad simple equivalente a la propiedad de ser completamente reducible (proposición 2.13). Llamamos a esta propiedad de propiedad de descomposición (definición 2.10).

Ahora vamos a centrarnos en determinados tipos de aplicaciones autoadjuntas positivas. Sea $A = \sum_{i=1}^{n} A_i \otimes B_i \in M_k \otimes M_m$ e identifique $M_k \otimes M_m \simeq M_{km}$, a través del producto de Kronecker. Defina $G_A : M_k \to M_m$, $G_A(X) = \sum_{i=1}^{n} tr(A_iX)B_i$ y $F_A : M_m \to M_k$, $F_A(X) = \sum_{i=1}^{n} tr(B_iX)A_i$. Si $A \in M_k \otimes M_m \simeq M_{km}$ es Hermítica entonces F_A y G_A son adjuntas con respecto al producto interno de la traza. Por otra parte, si $A \in P_{km}$ entonces F_A y G_A son positivas y $F_A \circ G_A : M_k \to M_k$ es una aplicación positiva autoadjunta.

Sea S_4 el grupo de permutaciones de $\{1, 2, 3, 4\}$ y considere la notación de ciclos. Sea $\sigma \in S_4$ y defina $L_\sigma : M_k \otimes M_k \to M_k \otimes M_k$ como la transformación lineal que satisface $L_\sigma(v_1v_2^t \otimes v_3v_4^t) = v_{\sigma(1)}v_{\sigma(2)}^t \otimes v_{\sigma(3)}v_{\sigma(4)}^t$, para todos $v_1, v_2, v_3, v_4 \in \mathbb{C}^k$. Defina $P_\sigma = \{A \in M_k \otimes M_k,\ A \in P_{k^2} \text{ and } L_\sigma(A) \in P_{k^2}\}$ y $I_\sigma = \{A \in M_k \otimes M_k,\ A \in P_{k^2} \text{ and } L_\sigma(A) = A\}$. Entre estes tipos de matrices estamos interesados especialmente en 3:

(1) $P_{(34)}$, que es el conjunto de las matrices PPT (definición 3.1)

(2) $P_{(243)}$, que es el conjunto de las matrices SPC (definiton 3.6)

(3) $I_{(23)}$, que es el conjunto de las matrices invariantes por realineamiento (definición 3.8).

Finalmente, podemos describir nuestros principales resultados. Si $A \in M_k \otimes M_m$ es positiva bajo transposición parcial (PPT) o simétrica con coeficientes positivos (SPC) o invariante bajo realineamiento luego $F_A \circ G_A : M_k \to M_k$ es completamente reducible (teoremas 3.2, 3.12 y 3.13). Vamos a aplicar nuestros principales resultados a la teoría de información cuántica.

La aplicación $F_A \circ G_A : M_k \to M_k$ es responsable por la descomposición de Schmidt de la matriz Hermítica $A \in M_k \otimes M_m$. Nuestros principales teoremas dicen que bajo una de estas tres hipótesis la aplicación $F_A \circ G_A : M_k \to M_k$ se descompone como una suma de aplicaciones irreducibles. Por lo tanto, A también se descompone como una suma de matrices débilmente irreducibles (definición 3.15 y la proposición 3.18).

Una condición necesaria para la separabilidad de $A \in M_k \otimes M_m$ es ser PPT. Podemos utilizar la descomposición de una matriz PPT como una suma de matrices débilmente irreducibles para reducir el problema de separabilidad para el caso PPT débilmente irreducible (corolario 3.20). También proporcionamos una descripción completa de las matrices PPT débilmente irreducibles (proposición 3.17).

Una herramienta importante para estudiar la separabilidad de las matrices Hermitícas definidas positivas en $M_k \otimes M_m$ es una forma normal denominada forma nomal de filtro (véase la sección IV.D de [23] y la subsección 3.3.2). La única prueba conocida de esta forma normal depende de A ser positiva definida. En realidad, la descomposición de una matriz PPT como una suma de matrices débilmente irreducibles proporciona otro caso en el que la forma normal de filtro puede ser utilizada (véase la subsección 3.3.2). Esto plantea una pregunta importante: Podemos demostrar la forma normal de filtro para las matrices PPT débilmente irreducibles? Si la respuesta fuera sí, seríamos capazes de usar la forma normal de filtro para cualquier matriz PPT.

Todavía podemos obtener algunas desigualdades que impliquen separabilidad para matrices PPT débilmente irreducibles, incluso sin la forma normal de filtro. Estas desigualdades se basan en el hecho de que toda matriz Hermítica positiva semidefinida con rango tensorial 2 es separable (ver teorema 3.44). Queremos enfatizar que la forma normal de filtro también sería útil para mejorar estas desigualdades (ver ejemplo 3.38).

Otra de las aplicaciones de nuestros resultados principales es la siguiente: Si $F_A \circ G_A : M_k \to M_k$ es completamente reducible con los únicos valores propios 1 o 0 entonces A es separable. El uso de este teorema para una matriz invariante bajo realineamiento, proporciona una prueba diferente del siguiente resultado publicado recientemente en [47]: Si hay k bases mutuamente imparciales en $\mathbb{C}^k$ entonces existe otra base ortonormal que es mutuamente imparcial con estas k bases. Por lo tanto, si $\mathbb{C}^k$ contiene k bases mutuamente imparciales entonces $\mathbb{C}^k$ contiene $k+1$. El caso real sigue de manera análoga: Si $\mathbb{R}^{2k}$ contiene k bases mutuamente imparciales entonces $\mathbb{R}^{2k}$ contiene $k+1$.

Este resultado es bastante sorprendente, ya que algunos conjuntos de bases mutuamente imparciales se demostraron inextensibles (véase, por ejemplo, [36]). En la teoría de información cuántica, el concepto de bases mutuamente imparciales (definición 3.23) se ha demostrado útil. Tiene aplicaciones en tomografía y criptografía (vea [17, 31, 48, 49]) . Se sabe que $k+1$ es un límite superior para el número de bases mutuamente imparciales de $\mathbb{C}^k$ y la existencia de este número de bases es un problema abierto, cuando k no es una potencia de números primos. Cuando k es una potencia de cierto número primo, se han utilizados métodos constructivos para obtener estos $k+1$ bases (ver [4, 31, 49]).

Además de la información que nuestros principales teoremas proporcionan, también nos proporcionan una intuición: Los tres tipos de matrices que aparecen en nuestros teoremas están conectados. Por lo tanto, podemos preguntarnos si cada matriz SPC es PPT o si cada matriz invariante bajo realineamiento es PPT. Se demuestra que las matrices SPC y las matrices invariantes bajo realineamiento son PPT en $M_2 \otimes M_2$, sin embargo, en $M_k \otimes M_k$, $k > 2$, hay contraejemplos.

Observemos que la propiedad de ser completamente reducible es muy fuerte. Es una sorpresa que $F_A \circ G_A : M_k \to M_k$ sea completamente reducible, cuando A es PPT o SPC o invariante bajo realineamiento. Una matriz PPT es un tipo muy común de estado en teoría de información cuántica. Por otra parte, se sabe que un estado invariante bajo la multiplicación por el operador "Flip" es PPT si y sólo si es SPC (ver [45] y la proposición 3.33), por lo tanto, las matrices SPC son relativamente conocidas. Matrices invariante bajo realineamiento no son muy comunes, pero el realineamiento es bien conocido debido a su uso con el fin de detectar entrelazamiento. Muy a menudo la teoría de información cuántica se beneficia de las ideas y de los teoremas de la teoría de las aplicaciones positivas. Por ejemplo, la solución completa del problema de la separabilidad en $M_k \otimes M_m$, $km \leq 6$, se obtuvo mediante la clasificación completa de las aplicaciones positivas, $T : M_k \to M_m$, $km \leq 6$. Como la teoría de información cuántica está proporcionando sus tipos especiales de estados como la hipótesis de nuestros principales resultados, podemos interpretar estos resultados como la retroalimentación de esta teoría a la teoría de aplicaciones positivas.

Nuestro segundo problema fue propuesto originalmente por Vladimir I. Gurariy y, más tarde, estudiado por Gurariy y Quarta en [27]. Sea K un espacio topológico. Consideremos $C(K)$ el espacio vectorial de las funciones reales continuas con dominio K. Denotemos por $\widehat{C}(K)$ el subconjunto de $C(K)$ formado por aquellas funciones que alcanzan su máximo en un solo punto de K. El conjunto $\widehat{C}(K)$ no es un espacio vectorial por muchas razones, por ejemplo, la función de cero no es un elemento de este conjunto.

Gurariy y Quarta se hicieron la siguiente pregunta: ¿Podemos encontrar un subespacio V de $C(K)$ dentro de $\widehat{C}(K) \cup \{0\}$? En caso positivo, ¿cómo de grande puede llegar a ser la dimensión de V ?

Los principales resultados obtenidos por Gurariy y Quarta en este sentido son los siguientes:

(A) Existe un subespacio de dimensión 2 de $C[a,b)$ contenido en $\widehat{C}[a,b) \cup \{0\}$.

(B) Hay un subespacio de dimensión 2 de $C(\mathbb{R})$ contenido en $\widehat{C}(\mathbb{R}) \cup \{0\}$.

(C) No existe ningún subespacio de dimensión 2 de $C\,[a,b]$ contenido en $\widehat{C}\,[a,b] \cup \{0\}$.

Nuestro principal resultado es una generalización de (C). Hemos demostrado que si K es un subconjunto compacto de $\mathbb{R}^n$ y si V es un subespacio de $C(K)$ dentro $\widehat{C}(K) \cup \{0\}$ entonces $\dim(V) \leq n$. Mientras que Gurariy y Quarta [27] emplearon técnicas analíticas clásicas, nuestra generalización requiere de un teorema topológico: el famoso teorema de Borsuk-Ulam.

La razón por la cual el teorema de Borsuk-Ulam es útil en este contexto es la siguiente: Supongamos que $f_1, \ldots, f_k$ es una base de un subespacio V de $C(K)$ dentro $\widehat{C}(K) \cup \{0\}$. Sea S^{k-1} la esfera Euclídea dentro de $\mathbb{R}^k$ y definamos la función $g : S^{k-1} \to K$ como $g(a_1, \ldots, a_k) =$ el único punto de máximo en K de $\sum_{i=1}^k a_i f_i$. Demostramos que esta función es continua si K es un subconjunto compacto de $\mathbb{R}^n$. Por el teorema de Borsuk-Ulam, si la dimensión k del subespacio V es mayor que n, entonces hay un par de puntos antipodales en S^{k-1} con la misma imagen. Por lo tanto, tenemos que hay $f, -f$ dentro de este subespacio con el mismo punto de máximo. Entonces, f es constante y no alcanza su máximo en un solo punto, lo cual es absurdo.

En general, la función g no es continua. Por ejemplo, si $K = [0, 2\pi)$ entonces el subespacio de $C([0,2\pi))$ generado $\{\cos(t), \sin(t)\}$ es un subconjunto de $\widehat{C}([0,2\pi)) \cup \{0\}$. La función $g : S^1 \to [0, 2\pi)$, $g(a_1, a_2) =$ el único punto de máximo en $[0, 2\pi)$ de $a_1 \cos(t) + a_2 \sin(t)$, no es continua en $(1, 0)$. La continuidad de g bajo la hipótesis de compacidad de K es una sorpresa.

Nuestro tercer y último problema fue propuesto por Richard M. Aron Y Vladimir I. Gurariy.

> ¿Es posible obtener un subespacio cerrado de dimensión infinita de ℓ_∞ tal que cada sucesión de este espacio tiene solamente un número finito de coordenadas nulas?

Esta cuestión ha aparecido en varios trabajos recientes (véase, por ejemplo, [9, 20, 22, 38]) y, durante la última década, ha habido varios intentos de responder parcialmente, aunque no hay nada concluyente en relación con el original problema que se ha obtenido hasta ahora.

Si X denota un espacio de sucesiones, designaremos por $Z(X)$ el subconjunto de X formado por sucesiones que tienen sólo un número finito de coordenadas cero. A continuación, mostraremos (entre otros resultados) la respuesta definitiva a esta pregunta. Es decir, si X representa c_0 o ℓ_p, con $p \in [1, \infty]$, se prueba lo siguiente:

(i) No hay subespacio cerrado de dimensión infinita de X dentro $Z(X) \cup \{0\}$ (corolarios 5.7 y 5.16).

(ii) Hay un subespacio cerrado de dimensión infinita de X dentro $V \setminus Z(V)$, para cualquier subespacio cerrado V (con dimensión infinita) de X (teorema 5.18).

Para obtener estos resultados, se construyen sucesiones básicas dentro de cualquier subespacio cerrado de dimensión infinita de $X = c_0$ o ℓ_p, $p \in [1, +\infty]$, y verificando propiedades "especiales". Una de estas propiedades es que cada elemento de esta sucesión básica tiene un número infinito de coordenadas nulas.

Observemos que hay un ejemplo muy simple de un subespacio de dimensión infinita de X dentro $Z(X) \cup \{0\}$, que por supuesto, no es cerrado. Sea V el subespacio generado por $\{(\lambda^n)_{n\in\mathbb{N}} \mid 0 < \lambda < 1\}$. Tenga en cuenta que $V \subset X$, si $X = c_0$ o ℓ_p, $p \in [1, +\infty]$, y cualquier combinación lineal no trivial de distintos $(\lambda_1^n)_{n\in\mathbb{N}}, \ldots, (\lambda_k^n)_{n\in\mathbb{N}}$ está dominado por $(\lambda_i^n)_{n\in\mathbb{N}}$ con el mayor λ_i. Por lo tanto, las coordenadas de esta combinación lineal no son cero después de una cierta coordenada, que depende de la combinación. Por lo tanto, $V \subset Z(X) \cup \{0\}$.

Usando terminología moderna (acuñada originalmente por V.I. Gurariy), un subconjunto M de un espacio vectorial topológico X se llama *lineable* (resp. *espaciable*) en X si existe un subespacio lineal de dimensión infinita (resp. subespacio lineal *cerrado* de dimensión infinita) $S \subset M \cup \{0\}$ (véase [1, 2, 5, 9, 10, 20, 27]). Por lo tanto, hemos demostrado que $Z(X)$ es lineable y no espaciable.

No hay muchos ejemplos de conjuntos (no triviales) que son lineables y no espaciables. Uno de los primeros en este sentido, se debe a Levine y Milman (1940, [35]) que mostraran que el subconjunto de $\mathcal{C}[0, 1]$ de todas las funciones de variación acotada no es espaciable (obviamente lineable, ya que es un espacio lineal de dimensión infinita). Una más reciente se debe a Gurariy (1966, [26]), que probó que el conjunto de funciones diferenciables en todas partes $[0, 1]$ (que es también un espacio lineal de dimensión infinita) no es espaciable en $\mathcal{C}([0, 1])$. Sin embargo, Bernal-González ([8], 2010) mostró que $\mathcal{C}^\infty(]0, 1[)$ es, en realidad, espaciable en $\mathcal{C}(]0, 1[)$.

Chapter 1

Introduction

This Ph.D. dissertation mainly focuses on three multilinear problems and its aim is to describe analytical and topological techniques that we found useful to tackle these problems. The first problem comes from Quantum Information theory, it is the so-called the Separability Problem, and the other two were proposed by Gurariy. In our first problem we used Perron-Frobenius Theory, which is related to positive maps acting on C^*-algebras, in order to obtain a reduction of the Separability Problem to a particular case and some other applications to Quantum Information theory. For the second problem, we used Borsuk-Ulam theorem to show that the dimension of a particular vector space must be within a certain range in order to exist. For the third problem, we constructed basic sequences with special properties in order to obtain a complete solution.

Let M_k denote the set of complex matrices of order k and let P_k be the set of positive semidefinite Hermitian matrices of M_k. The Separability Problem is a well established problem in the field of Quantum Information Theory due to its importance and difficulty. The aim of this problem is to find a deterministic criterion to distinguish the separable states from the entangled states. In this work we shall only deal with the bipartite finite dimensional case, therefore the states are elements in the tensor product space $M_k \otimes M_m$, which can be interpreted as matrices in M_{km} via the Kronecker product. We say that $B \in M_k \otimes M_m$ is separable if $B = \sum_{i=1}^n C_i \otimes D_i$, where $C_i \in P_k$ and $D_i \in P_m$, for every i. If B is not separable then B is entangled.

This problem was completely solved by Horodecki in the space $M_k \otimes M_m$ for $km \leq 6$, by the so-called PPT criterion (see [29]). This criterion states that a matrix $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$, $km \leq 6$, is separable if and only if A is positive under partial transposition (PPT), i.e., A and $A^{t_2} = \sum_{i=1}^n A_i \otimes B_i^t$ are positive semidefinite Hermitian matrices (definition 3.1).

The general case, even for the finite dimensional case, is still a great challenge. Algorithms have been developed in order to solve the separability problem, but it is known that this problem is NP-hard (see [28]). Therefore, any restriction of the problem to a smaller set of matrices is, certainly, important. For example, Peres in [39] was the first to notice the importance of the PPT property which was later proved to be necessary and sufficient for separability in $M_k \otimes M_m$ for $km \leq 6$, in [29].

Another remarkable reduction was obtained for the positive definite case in $M_k \otimes M_m$. In order to find the separable positive definite Hermitian matrices we only need to distinguish the separable matrices among the positive definite matrices of the following type:

$$Id \otimes Id + \sum_{i=1}^{l} a_i E_i \otimes F_i$$

where $tr(E_i) = tr(F_i) = 0$, $\{E_1, ..., E_l\}$, $\{F_1, ..., F_l\}$ are orthonomal sets of Hermitian matrices with respect to the trace inner product and $a_i \in \mathbb{R}$. This result is obtained via the filter normal form (see subsection 3.3.2 and [23, 34, 46]).

The authors of [34] also obtained a remarkable reduction of the separability problem in $M_2 \otimes M_2$ for the general case, not only for the positive definite case. They showed that, in order to solved it, it suffices to discover which matrices from the following family of matrices are separable:

$$Id \otimes Id + d_2\gamma_2 \otimes \gamma_2 + d_3\gamma_3 \otimes \gamma_3 + d_4\gamma_4 \otimes \gamma_4,$$

where $d_2, d_3, d_4 \in \mathbb{R}$ and $\gamma_2, \gamma_3, \gamma_4$ are the matrices of the Pauli's basis of M_2 different from the Id. They proved that a matrix within this family is separable if and only if it is PPT, and if and only if $|d_2| + |d_3| + |d_4| \leq 1$. This is a second proof of the PPT criterion in $M_2 \otimes M_2$.

The interested reader can find more information concerning the Separability Problem in the survey [25].

Next, let us describe how we used the Perron-Frobenius theory in order to reduce the separability problem to a certain subset of PPT matrices and to obtain some other applications.

Denote by VM_kW the set $\{VXW, X \in M_k\}$, where $V, W \in M_k$ are orthogonal projections. If $V = W$ then the set VM_kV is an hereditary C^*−subalgebra of M_k. We say that a linear transformation $T : VM_kV \to WM_mW$ is a positive map, if $T(P_k \cap VM_kV) \subset P_m \cap WM_mW$. We say that a non null positive map $T : VM_kV \to VM_kV$ is irreducible if $V'M_kV' \subset VM_kV$ is such that $T(V'M_kV') \subset V'M_kV'$ then $V' = V$ or $V' = 0$.

By Perron-Frobenius theory, we know that if $T : VM_kV \to VM_kV$ is a positive map then its spectral radius, λ, is an eigenvalue and there is $0 \neq \gamma \in P_k \cap VM_kV$ such that $T(\gamma) = \lambda\gamma$. Moreover, if $T : VM_kV \to VM_kV$ is irreducible then the multiplicity of the spectral radius is 1 and the images of γ and V are equal (see propositions 2.3 and 2.5 in [21]).

There are certain types of positive maps such that the converse of the last theorem is valid. For example, if $T : VM_kV \to VM_kV$ is a self-adjoint positive map with respect to the trace inner product ($\langle X, Y\rangle = tr(XY^*)$), if its spectral radius has multiplicity 1 and $\Im(\gamma) = \Im(V)$ then $T : VM_kV \to VM_kV$ is irreducible (see lemma 2.11). Another example is a completely positive map (see definition 1 in [44]) .

A natural extension of the concept of irreducible positive map is a direct sum of irreducible positive maps. Let us say that $T : VM_kV \to VM_kV$ is a completely reducible map, if it is a positive map and if there are orthogonal projections $V_1, \dots, V_s \in M_k$ such that $V_iV_j = 0$ $(i \neq j)$, $V_iV = V_i$ $(1 \leq i \leq s)$, $VM_kV = V_1M_kV_1 \oplus \dots \oplus V_sM_kV_s \oplus R$, $R \perp V_1M_kV_1 \oplus \dots \oplus V_sM_kV_s$ satisfying: $T(V_iM_kV_i) \subset V_iM_kV_i$ $(1 \leq i \leq s)$, $T|_{V_iM_kV_i}$ is irreducible $(1 \leq i \leq s)$, $T|_R \equiv 0$. Notice that any irreducible map is completely reducible. This concept is related to that of completely reducible matrix (see [42]).

The only strong restriction in the definition of completely reducible map is $T|_R \equiv 0$. For example, the existence of the subalgebras $V_iM_kV_i$ satisfying the required conditions is granted for any self-adjoint positive map, however the condition $T|_R \equiv 0$ is (in general) false. The simplest self-adjoint positive map that is not completely reducible is the identity map $Id : M_k \to M_k$, $k > 1$. As for irreducible maps, if $T : VM_kV \to VM_kV$ is a self-adjoint map then there is a very neat property equivalent to the complete reducibility of T (proposition 2.13). We call this property the decomposition property (definition 2.10).

Now, let us focus on specific types of self-adjoint positive maps. Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m$ and identify $M_k \otimes M_m \simeq M_{km}$, via Kronecker product. Define $G_A : M_k \to M_m$, as $G_A(X) = \sum_{i=1}^n tr(A_iX)B_i$ and $F_A : M_m \to M_k$, as $F_A(X) = \sum_{i=1}^n tr(B_iX)A_i$. If $A \in M_k \otimes M_m \simeq M_{km}$ is Hermitian then F_A and G_A are adjoint with respect to the trace inner product. Moreover, if $A \in P_{km}$ then F_A and G_A are positive maps and $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map.

Next, let S_4 be the group of permutations of $\{1,2,3,4\}$ and consider the cycle notation. Let $\sigma \in S_4$ and define $L_\sigma : M_k \otimes M_k \to M_k \otimes M_k$ as the linear transformation that satisfies $L_\sigma(v_1v_2^t \otimes v_3v_4^t) = v_{\sigma(1)}v_{\sigma(2)}^t \otimes v_{\sigma(3)}v_{\sigma(4)}^t$, for every $v_1, v_2, v_3, v_4 \in \mathbb{C}^k$. Define $P_\sigma = \{A \in M_k \otimes M_k,\ A \in P_{k^2} \text{ and } L_\sigma(A) \in P_{k^2}\}$ and $I_\sigma = \{A \in M_k \otimes M_k,\ A \in P_{k^2} \text{ and } L_\sigma(A) = A\}$. Among these types of matrices we are specially interested in 3 types:

(1) $P_{(34)}$, which is the set of PPT matrices (definition 3.1)

(2) $P_{(243)}$, which is the set of SPC matrices (definiton 3.6)

(3) $I_{(23)}$, which is the set of matrices invariant under realignment (definiton 3.8).

We can finally describe our main results. If $A \in M_k \otimes M_m$ is positive under partial transposition (PPT) or symmetric with positive coefficients (SPC) or invariant under realignment then $F_A \circ G_A : M_k \to M_k$ is completely reducible (theorems 3.2, 3.12 and 3.13). We shall apply our main results to Quantum Information Theory.

The map $F_A \circ G_A : M_k \to M_k$ is responsible for the Schmidt decomposition of the Hermitian matrix A. Our main theorems say that under one of these three hypothesis the map $F_A \circ G_A : M_k \to M_k$ decomposes as a sum of irreducible maps. Hence, A shall also decomposes as a sum of weakly irreducible matrices (definition 3.15 and proposition 3.18).

A necessary condition for the separability of $A \in M_k \otimes M_m$ is to be PPT. We can use the decomposition of a PPT matrix as a sum of weakly irreducible matrices to reduce the Separability Problem to the weakly irreducible PPT case (corollary 3.20). We also provide a complete description of weakly irreducible PPT matrices (proposition 3.17).

An important tool to study separability of positive definite Hermitian matrices in $M_k \otimes M_m$ is the so-called filter normal form (see section IV.D of [23] and subsection 3.3.2). The only known proof of this normal form depends heavily on the positive definiteness of A. Actually, the decomposition of a PPT matrix as a sum of weakly irreducible matrices provides another case where the filter normal form can be used (see subsection 3.3.2). This raises an important question: Can we prove the filter normal form for weakly irreducible PPT matrices? If so, we would be able to use the filter normal form for every PPT matrix.

We can still obtain some inequalities for weakly irreducible PPT matrices that imply separability, even without the filter normal form. These inequalities are based on the fact that every positive semidefinite Hermitian matrix with tensor rank 2 is separable (see theorem 3.44). We want to emphasize that the filter normal form would also be useful to sharpen these inequalities (see example 3.38).

Another application of our main results is the following one: If $F_A \circ G_A : M_k \to M_k$ is completely reducible with the only eigenvalues 1 or 0 then A is separable. Using this theorem for a matrix invariant under realignment, we can provide a different proof of the following result published recently in [47]: If there are k mutually unbiased bases in $\mathbb{C}^k$ then there exists another orthonormal basis which is mutually unbiased with these k bases. Hence, if $\mathbb{C}^k$ contains k mutually unbiased bases then $\mathbb{C}^k$ contains $k+1$. The real case follows analogously: If $\mathbb{R}^{2k}$ contains k mutually unbiased bases then $\mathbb{R}^{2k}$ contains $k+1$.

This result is quite surprising, since some sets of mutually unbiased bases were proved to be unextendible (see, e.g., [36]). In Quantum Information Theory, the concept of mutually unbiased bases (definition 3.23) has been shown to be useful. It has applications in state determination, quantum state tomography, cryptography (see [17, 31, 48, 49]). It is known that $k+1$ is an upper bound for the number of mutually unbiased bases in $\mathbb{C}^k$ and the existence of this number of bases is an open problem, when k is not the power of prime number. When k is a power of certain prime number, some constructive methods were used to obtain these $k+1$ bases (see [4, 31, 49]).

Besides the information that our main theorems provide, they also provide an intuition: The three types of matrices that our main theorems concern are connected. Thus, we can wonder if every SPC matrix is PPT or if every matrix invariant under realignment is PPT. We show that SPC matrices and matrices invariant under realignment are PPT in $M_2 \otimes M_2$, however in $M_k \otimes M_k$, $k > 2$, there are counterexamples.

Notice that the complete reducibility of $F_A \circ G_A : M_k \to M_k$ is a very stong property. It is quite a surprise that $F_A \circ G_A : M_k \to M_k$ is completely reducible, if A is PPT or SPC or invariant under realignment. A PPT matrix is a very common type of state in Quantum Information Theory. Moreover, it is known that a state invariant under multiplication by the flip operator is PPT if and only if is SPC (see [45] and proposition 3.33), thus SPC matrices are relatively known. Matrices invariant under realignment are not very common, but the realignment map is well known due to its use in order to detect entanglement. Very often Quantum Information Theory benefits from the ideas and theorems of the theory of positive maps. For example the complete solution of the Separability Problem in $M_k \otimes M_m$, $km \leq 6$, was obtained by the complete classification of the positive maps, $T : M_k \to M_m$, $km \leq 6$. Since Quantum Information Theory is providing its special types of states as the hypothesis of our main results, we can interpret these results as the feedback of this theory to the theory of positive maps.

Next, our second problem was originally proposed by Gurariy and, later, studied by Gurariy and Quarta in [27]. Let K be a topological space. Consider $C(K)$ the vector space of real-valued continuous functions with domain K. Denote by $\widehat{C}(K)$ the subset of $C(K)$ formed by those functions that attain their maximum at only one point of K. The set $\widehat{C}(K)$ fails to be a vector space for many reasons, for example the zero function is not an element of this set.

Gurariy and Quarta asked the following question: Can we find a subspace V of $C(K)$ inside $\widehat{C}(K) \cup \{0\}$. If so, how big can be the dimension of V?

The main results obtained by Gurariy and Quarta in this direction are the following:

(A) There is a 2-dimensional linear subspace of $C[a,b)$ contained in $\widehat{C}[a,b) \cup \{0\}$.

(B) There is a 2-dimensional linear subspace of $C(\mathbb{R})$ contained in $\widehat{C}(\mathbb{R}) \cup \{0\}$.

(C) There is no 2-dimensional linear subspace of $C[a,b]$ contained in $\widehat{C}[a,b] \cup \{0\}$.

Our main result is a generalization of (C). We proved that if K is a compact subset of $\mathbb{R}^n$ and if V is a subspace of $C(K)$ inside $\widehat{C}(K) \cup \{0\}$ then $\dim(V) \leq n$. While Gurariy and Quarta [27] used typical analytical techniques, our generalization requires a topological theorem: Borsuk-Ulam theorem.

The reason why the Borsuk-Ulam theorem is useful within this context is the following: Assume that $f_1, \ldots, f_k$ is a basis of a subspace V of $C(K)$ inside $\widehat{C}(K) \cup \{0\}$. Let S^{k-1} be the Euclidean sphere inside $\mathbb{R}^k$ and define the function $g : S^{k-1} \to K$ as $g(a_1, \ldots, a_k) =$ the unique point of maximum in K of $\sum_{i=1}^k a_i f_i$. We proved that this function is continuous if K is a compact subset of $\mathbb{R}^n$. By Borsuk-Ulam theorem, if the dimension k of the subspace V is bigger than n then there is a pair of antipodal points in S^{k-1} with the same image. Thus, there are $f, -f$ inside this subspace with the same point of maximum. Hence, f is constant and does not attain its maximum at only one point, which is absurd.

In general, the function g is not continuous. For example, if $K = [0, 2\pi)$ then the subspace $\operatorname{span}\{\cos(t), \sin(t)\}$ of $C([0, 2\pi))$ is a subset of $\widehat{C}([0, 2\pi)) \cup \{0\}$. The function $g : S^1 \to [0, 2\pi)$, $g(a_1, a_2) =$ the unique point of maximum in $[0, 2\pi)$ of $a_1 \cos(t) + a_2 sin(t)$, is not continuous at $(1, 0)$. It was quite a surprise to obtain the continuity of g under the hypothesis of compactness of K.

Our third and final problem was proposed by Gurariy and Aron.

> Is it possible to obtain an infinite dimensional closed subspace of ℓ_∞ such that every sequence of this space has finitely many zero coordinates?

This question has appeared in several recent works (see, e.g., [9, 20, 22, 38]) and, for the last decade, there have been several attempts to partially answer it, although nothing conclusive in relation to the original problem has been obtained so far.

If X denotes a sequence space, we shall denote by $Z(X)$ the subset of X formed by sequences having only a finite number of zero coordinates. Here, we shall provide (among other results) the definitive answer to this question. Namely, if X stands for c_0, or ℓ_p, with $p \in [1, \infty]$, we prove the following:

- (i) There is no infinite dimensional closed subspace of X inside $Z(X) \cup \{0\}$ (Corollaries 5.7 and 5.16).
- (ii) There exists an infinite dimensional closed subspace of X inside $V \setminus Z(V) \cup \{0\}$, for any infinite dimensional closed subspace V of X (Theorem 5.18).

In order to obtain these results, we construct basic sequences within any infinite dimensional closed subspace of $X = c_0$ or ℓ_p, $p \in [1, +\infty]$, satisfying special properties. One of this properties is each element of this basic sequence has infinitely many zero coordinates.

Observe that there is a very simple example of an infinite dimensional subspace of X inside $Z(X) \cup \{0\}$, of course this subspace is not closed. Consider $V = \operatorname{span}\{(\lambda^n)_{n \in \mathbb{N}} \mid 0 < \lambda < 1\}$. Notice that $V \subset X$, for $X = c_0$ or ℓ_p, $p \in [1, +\infty]$, and any non-trivial linear combination of distinct $(\lambda_1^n)_{n\in\mathbb{N}}, \dots, (\lambda_k^n)_{n\in\mathbb{N}}$ is dominated by $(\lambda_i^n)_{n\in\mathbb{N}}$ with the largest λ_i. Hence, the coordinates of this linear combination are not zero after a certain coordinate, which depends on the combination. Hence, $V \subset Z(X) \cup \{0\}$.

Using modern terminology (originally coined by Gurariy himself), a subset M of a topological vector space X is called *lineable* (resp. *spaceable*) in X if there exists an infinite dimensional linear space (resp. an infinite dimensional *closed* linear space) $Y \subset M \cup \{0\}$ (see [1, 2, 5, 9, 10, 20, 27]). Thus, we have proved that $Z(X)$ is lineable and not spaceable.

There are not many examples of (nontrivial) sets that are lineable and not spaceable. One of the first ones in this direction, is due to Levine and Milman (1940, [35]) who showed that the subset of $\mathcal{C}[0,1]$ of all functions of bounded variation is not spaceable (it is obviously lineable, since it is an infinite dimensional linear space itself). A more recent one is due to Gurariy (1966, [26]), who showed that the set of everywhere differentiable functions on $[0,1]$ (which is also an infinite dimensional linear space) is not spaceable in $\mathcal{C}([0,1])$. However, Bernal-González ([8], 2010) showed that $\mathcal{C}^\infty(]0,1[)$ is, actually, spaceable in $\mathcal{C}(]0,1[)$.

Chapter 2

Some Results from the Perron-Frobenius Theory

All definitions and results in this chapter can, also, be found in [14].

Let M_k denote the set of complex matrices of order k. Let us denote by VM_kW the set $\{VXW|\ X \in M_k\}$, where $V, W \in M_k$ are orthogonal projections. If $V = W$ then VM_kV is a hereditary finite dimensional C^*-algebra (see [21]). Let P_k denote the set of positive semidefinite Hermitian matrices in M_k. A linear transformation $L : VM_kV \to WM_mW$ is said to be a positive map if $L(P_k \cap VM_kV) \subset P_m \cap WM_mW$. A non-null positive map $L : VM_kV \to VM_kV$ is called irreducible, if $V'M_kV' \subset VM_kV$ is such that $L(V'M_kV') \subset V'M_kV'$ then $V' = V$ or $V' = 0$.

Within the context of positive maps, sometimes the term *self-adjoint* means $L(A^*) = L(A)^*$ (see, e.g., [21]). Here, we shall use this terminology with its usual meaning. We shall say that $L : VM_kV \to VM_kV$ is self-adjoint if L is equal to its adjoint L^* (i.e., $\langle L(A), B\rangle = \langle A, L(B)\rangle$). We shall consider the usual inner product in M_k, $\langle A, B\rangle = tr(AB^*)$.

In this chapter, we use well known theorems from the Perron-Frobenius Theory to describe some properties of completely reducible maps (see definition 3.16). These are theorems 2.3 and 2.5 in [21]: If $L : VM_kV \to VM_kV$ is a positive map then there exists $\gamma \in P_k \cap VM_kV$ such that $L(\gamma) = \lambda\gamma$, where λ is the spectral radius of L. Moreover, if L is irreducible then this eigenvalue has multiplicity 1. We present elementary proofs of these theorems in Section 2.1 (see theorems 2.7 and 2.8).

In Section 2.2, we prove that if $L : VM_kV \to VM_kV$ is a self-adjoint positive map then L is completely reducible if and only if L has the decomposition property (proposition 2.13). In the next chapter, we provide an equivalent way to prove that L has the decomposition property (lemma 3.14) and we shall give two applications of completely reducible maps to Quantum Information Theory.

2.1 Two theorems from the Perron-Frobenius Theory

For the convenience of the reader and for the completeness of this dissertation, we present here elementary proofs of theorems 2.7 and 2.8. These theorems are well known results from the Perron-Frobenius theory (see propositions 2.3 and 2.5 in [21]). Notice that there are general versions of them in the literature (check, for example, the appendix of [43]). Here, the proofs of theorems 2.7 and 2.8 follow the ideas of [6] and [21].

The first two lemmas are well known and their proofs are omitted. Recall that a positive map $L : VM_kV \to WM_mW$ preserves hermiticity, since every Hermitian matrix is a difference of two positive semidefinite Hermitian matrices.

Lemma 2.1. *If $A \in P_k$ and $B \in M_k$ is Hermitian then $\Im(B) \subset \Im(A)$ if and only if there exists $\epsilon > 0$ such that $A \pm \epsilon B \in P_k$.*

Lemma 2.2. *Let γ_1, γ_2 be Hermitian matrices in M_k, $\gamma_1 \in P_k$ and $\gamma_2 \neq 0$. Suppose that $\Im(\gamma_2) \subset \Im(\gamma_1)$ and γ_2 is not a multiple of γ_1. There exists $0 \neq \lambda \in \mathbb{R}$ such that $\gamma_1 - \lambda\gamma_2 \in P_k$ and $0 \neq v \in \ker(\gamma_1 - \lambda\gamma_2) \cap \Im(\gamma_1)$.*

Lemma 2.3. *Let $L : VM_kV \to WM_mW$ be a positive map. If $\gamma \in P_k \cap VM_kV$ and $L(\gamma) = \delta$ then $L(V_1M_kV_1) \subset W_1M_mW_1$, where V_1 is the orthogonal projection onto $\Im(\gamma)$ and W_1 is the orthogonal projection onto $\Im(\delta)$.*

Proof. Let $\gamma_1 \in V_1M_kV_1$ be a Hermitian matrix. Thus, $\Im(\gamma_1) \subset \Im(V_1) = \Im(\gamma)$. So there is $\epsilon > 0$ such that $\gamma \pm \epsilon\gamma_1 \in P_k$, by lemma 2.1.

Now, since $L(\gamma_1)$ is Hermitian and $L(\gamma) \pm \epsilon L(\gamma_1) = L(\gamma \pm \epsilon\gamma_1) \in P_k$ then $\Im(L(\gamma_1)) \subset \Im(L(\gamma)) = \Im(\delta) = \Im(W_1)$, by lemma 2.1. Therefore, $L(\gamma_1) \in W_1M_mW_1$.

Finally, since every matrix in $V_1M_kV_1$ is a linear combination of Hermitian matrices within $V_1M_kV_1$ then $L(V_1M_kV_1) \subset W_1M_mW_1$. □

Corollary 2.4. *Let $L : VM_kV \to VM_kV$ be a positive map and $\gamma \in P_k \cap VM_kV$ be such that $L(\gamma) = \lambda\gamma$, $\lambda > 0$. Then, $L(V_1M_kV_1) \subset V_1M_kV_1$, where V_1 is the orthogonal projection onto $\Im(\gamma)$.*

Recall that the largest absolute value of all eigenvalues of $L : VM_kV \to VM_kV$ is called the spectral radius of L.

Lemma 2.5. *Let $L : VM_kV \to VM_kV$ be a positive map. If $L(V) = V$ then the spectral radius of L is 1.*

Proof. Let $U \in VM_kV$ be a normal matrix such that $UU^* = U^*U = V$. Thus, $U = \sum_{i=1}^{s} \lambda_i v_i \overline{v_i}^t$, where s is the rank of V, $\{\lambda_1, \ldots, \lambda_s\}$ are complex numbers of norm 1 and $\{v_1, \ldots, v_s\}$ is an orthonormal basis of $\Im(V)$. Recall that $L(U^*) = L(U)^*$, since L preserves hermiticity, and $L(U^*)V = L(U^*)$, $VL(U) = L(U)$.

Now, consider the matrix $B = \sum_{i=1}^{s} \begin{pmatrix} 1 & \lambda_i \\ \overline{\lambda_i} & 1 \end{pmatrix} \otimes L(v_i \overline{v_i}^t) = \begin{pmatrix} L(V) & L(U) \\ L(U^*) & L(V) \end{pmatrix} =$

$$= \begin{pmatrix} V & L(U) \\ L(U)^* & V \end{pmatrix} = \begin{pmatrix} Id & 0 \\ L(U)^* & Id \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & V - L(U)^* L(U) \end{pmatrix} \begin{pmatrix} Id & L(U) \\ 0 & Id \end{pmatrix}.$$

Since $B \in P_{2k}$ then $V - L(U)^* L(U) \in P_k$. So $\|L(U)\|_2 \leq 1$, where $\|L(U)\|_2$ is the spectral norm of $L(U)$.

Thus, for every normal matrix U such that $UU^* = U^*U = V$, we have$\|L(U)\|_2 \leq 1$

Next, let $A \in VM_kV$ be an eigenvector of L associated to some eigenvalue α and $\|A\|_2 = 1$. Let $\sum_{j=1}^{s} a_j m_j n_j^t$ be a SVD decomposition of A, where $0 \leq a_j \leq 1$. Since $a_j = \cos(\theta_j) = \frac{e^{i\theta_j}}{2} + \frac{e^{-i\theta_j}}{2}$ then $A = \frac{1}{2}(\sum_{j=1}^{s} e^{i\theta_j} m_j n_j^t) + \frac{1}{2}(\sum_{j=1}^{s} e^{-i\theta_j} m_j n_j^t) = \frac{1}{2}U_1 + \frac{1}{2}U_2$. Notice that $U_1 U_1^* = U_1^* U_1 = U_2 U_2^* = U_2^* U_2 = V$.

Finally, $|\alpha| = |\alpha| \|A\|_2 = \|L(A)\|_2 = \|L(\frac{1}{2}U_1 + \frac{1}{2}U_2)\|_2 \leq \frac{1}{2}\|L(U_1)\|_2 + \frac{1}{2}\|L(U_2)\|_2 = 1$ and since 1 is an eigenvalue of $L : VM_kV \to VM_kV$, $L(V) = V$, then 1 is the spectral radius of L. □

Lemma 2.6. *Let $L : VM_kV \to VM_kV$ be an irreducible positive map. If $0 \neq X \in P_k \cap VM_kV$ then $\Im((Id + L)^{s-1}(X)) = \Im(V)$, where $s = rank(V)$.*

Proof. If $0 \neq X \in P_k \cap VM_kV$ then $L(X) \in P_k \cap VM_kV$ and $\Im(x) \subset \Im(x + L(x)) \subset \Im(V)$.

Now, if $\Im(X + L(X)) = \Im(X)$ then $\Im(L(X)) \subset \Im(X)$ and, by lemma 2.3, $L(V' M_k V') \subset V' M_k V'$, where V' is the orthogonal projection onto $\Im(X)$. Since L is irreducible then $V = V'$ and $\Im(X) = \Im(V)$. Thus, $\Im(X) \neq \Im(V)$ implies $\text{rank}((Id + L)(X)) > \text{rank}(X)$. Repeating the argument at most $s - 1$ times, we obtain $\Im((Id + L)^{s-1}(X)) = \Im(V)$. □

Theorem 2.7. *Let $L : VM_kV \to VM_kV$ be an irreducible positive map. The spectral radius r of L is an eigenvalue of L associated to some eigenvector $Z \in P_k \cap VM_kV$ such that $\Im(Z) = \Im(V)$. Moreover, the geometric multiplicity of r is 1.*

Proof. Let $Z = \{X \in P_k \cap VM_kV, \Im(X) = \Im(V)\}$. Define $f : Z \to [0, \infty[$ as $f(X) = \sup\{\lambda \in \mathbb{R},\ L(X) - \lambda X \in P_k\}$.

Denote by B^+ the pseudo-inverse of B and by $\|B\|_2$ the spectral norm of B. Notice that if $B \in Z$ then $B^+ B = BB^+ = V$ and $B^+ V = VB^+ = B^+$.

Now, for every $X \in Z$, there is $Y \in Z$ such that $Y^2 = X$. Notice that $f(X)$ is the minimal positive eigenvalue of $Y^+ L(X) Y^+$, which is the minimal positive eigenvalue of $L(X)Y^+Y^+ = L(X)X^+$.

Next, if $(A_n)_{n\in\mathbb{N}} \in Z$ converges to $A \in Z$, then the smallest positive eigenvalue of A_n, $\|A_n^+\|_2^{-1}$, converges to the smallest positive eigenvalue of A, $\|A^+\|_2^{-1}$(see pg 154 in [7]). Thus, there is $N \in \mathbb{N}$ such that if $n > N$ then $\|A_n^+\|_2^{-1} \geq (2\|A^+\|_2)^{-1}$. Hence, for $n > N$, $\|A_n^+\|_2 \leq 2\|A^+\|_2$ and $\|A_n^+ - A^+\|_2 = \|A_n^+(A - A_n)A^+\|_2 \leq \|A_n^+\|_2\|A^+\|_2\|A_n - A\|_2 \leq 2\|A^+\|_2^2\|A_n - A\|_2$. Therefore A_n^+ converges to A^+ and A^+ depends continuously on A, for A varying on Z.

Since the eingevalues of a matrix vary continuously with a matrix (see pg 154 in [7]) then $f : Z \to [0, \infty[$ is a continuous function.

Consider the compact set $Z' = \{X \in P_k \cap VM_kV,\ \|X\|_2 = 1\}$, where $\|X\|_2$ is the spectral norm of X. By lemma 2.6, $(Id+L)^{s-1}(Z')$ is a compact subset of Z. Therefore, $f|_{(Id+L)^{s-1}(Z')}$ attains its maximum r at some point $W \in (Id+L)^{s-1}(Z') \subset Z$. By definiton of $r = f(W)$, we have $L(W) - rW \in P_k$.

Next, if $0 \neq L(W) - rW$ then the range of $(Id + L)^{s-1}(L(W) - rW)$ is the range of V, by lemma 2.6. Thus, $f((Id+L)^{s-1}(\frac{W}{\|W\|_2})) > r = f(W)$, which is a contradiction. So $L(W) = rW$, $W \in P_k \cap VM_kV$ and $\Im(W) = \Im(V)$.

In order to complete this proof, we must show that r is the spectral radius of L with geometric multiplicity 1.

Define $L_1 : VM_kV \to VM_kV$ as $L_1(x) = \frac{1}{r}M^+L(MxM)M^+$, where $M \in Z$ is such that $M^2 = W$. Notice that L_1 is a positive map such that $L_1(V) = V$.

Next, if A is an eigenvector of L associated to some eigenvalue α then M^+AM^+ is an eigenvector of L_1 associated to α/r. By lemma 2.5, $|\frac{\alpha}{r}| \leq 1$. Hence, $|\alpha| \leq r$ and the spectral radius of L is r.

Finally, assume $W_2 \in VM_kV$ is a Hermitian eigenvector of L associated to r. If W_2 and W are linear independent then there is $0 \neq \mu \in \mathbb{R}$ such that $W - \mu W_2 \in P_k$ and $\text{rank}(W - \mu W_2) < \text{rank}(W)$, by lemma 2.2. Thus, $L(V'M_kV') \subset V'M_kV'$, where V' is the orthogonal projection onto $\Im(W - \mu W_2)$, by corollary 2.4. This contradicts the irreducibility of L, since $\Im(W - \mu W_2) \neq \Im(W)$ and $V' \neq V$. Therefore, W_2 and W are linear dependent. Since L preserves Hermiticity and $r > 0$ then every eingevector of L associated to r is a linear combination of Hermitian eigenvectors of L associated to r, thus the geometric multiplicity of r is 1. □

Theorem 2.8. *Let $L : VM_kV \to VM_kV$ be a positive map. The spectral radius r of L is an eigenvalue of L associated to some $Z \in P_k \cap VM_kV$.*

Proof. Consider the sequence of irreducible positive maps $L_n : VM_kV \to VM_kV$ defined by $L_n(x) = L(x) + \frac{1}{n}tr(xV)V$ converging to $L : VM_kV \to VM_kV$.

Let $Z_n \in P_k \cap VM_kV$ be the unique eigenvector of L_n associated to the spectral radius r_n of L_n satisfying $\|Z_n\|_2 = 1$, where $\|\cdot\|_2$ is the spectral norm, by theorem 2.7.

Notice that $\{X \in P_k \cap VM_kV,\ \|X\|_2 = 1\}$ is a compact set, therefore there is a subsequence $(Z_{n_k})_{k\in\mathbb{N}}$ converging to some $Z \in \{X \in P_k \cap VM_kV,\ \|X\|_2 = 1\}$.

Finally, since the spectral radius changes continuosly with a matrix then $\lim_{k\to\infty} r_{n_k} = r$, where r is the spectral radius of L. Thus, $L(Z) = \lim_{k\to\infty} L_{n_k}(Z_{n_k}) = \lim_{k\to\infty} r_{n_k} Z_{n_k} = rZ$. □

2.2 Two related properties: completely reducibility and the decomposition property

The main result of this section is the equivalence of the next two properties for a self-adjoint positive map $L : VM_kV \to VM_kV$ (proposition 2.13).

Definition 2.9. (*Completely Reducible Maps*)*: A positive map* $L : VM_kV \to VM_kV$ *is called completely reducible, if there are orthogonal projections* $V_1, \dots, V_s \in M_k$ *such that* $V_iV_j = 0$ $(i \neq j)$, $V_iV = V_i$, $VM_kV = V_1M_kV_1 \oplus \dots \oplus V_sM_kV_s \oplus R$, $R \perp V_1M_kV_1 \oplus \dots \oplus V_sM_kV_s$ *and*

(1) $L(V_iM_kV_i) \subset V_iM_kV_i$,

(2) $L|_{V_iM_kV_i}$ *is irreducible,*

(3) $L|_R \equiv 0$.

Definition 2.10. *Let* $L : VM_kV \to VM_kV$ *be a self-adjoint positive map. We say that* L *has the decomposition property if for every* $\gamma \in P_k \cap VM_kV$ *such that* $L(\gamma) = \lambda\gamma$, $\lambda > 0$ *and* $V_1 \in M_k$ *is the orthogonal projection onto* $\Im(\gamma)$ *then* $L|_R \equiv 0$, *where* $R = (V - V_1)M_kV_1 \oplus V_1M_k(V - V_1)$. *Notice that* R *is the orthogonal complement of* $V_1M_kV_1 \oplus (V - V_1)M_k(V - V_1)$ *in* VM_kV.

Lemma 2.11. *Let* $L : VM_kV \to VM_kV$ *be a self-adjoint positive map.* L *is irreducible if and only if the largest eigenvalue has multiplicity* 1 *with respect to an eigenvector* $\gamma \in P_k \cap VM_kV$ *such that* $\Im(\gamma) = \Im(V)$.

Proof. Since L is self-adjoint, the eigenvalues of L are real numbers. Since $L : VM_kV \to VM_kV$ is a positive map, by theorem 2.8, the spectral radius λ is an eigenvalue and there exists $\gamma \in P_k \cap VM_kV$ such that $L(\gamma) = \lambda\gamma$. Therefore the spectral radius is the largest eigenvalue of L. Since L is irreducible, the multiplicity of λ is 1, by theorem 2.7. Let $V_1 \in M_k$ be the orthogonal projection onto $\Im(\gamma)$. Notice that $\Im(V_1) \subset \Im(V)$. By the previous corollary $L(V_1M_kV_1) \subset V_1M_kV_1$. Since L is irreducible then $V_1 = V$ and $\Im(\gamma) = \Im(V)$.

For the converse, if $L(V_1M_kV_1) \subset V_1M_kV_1$, $\Im(V_1) \subset \Im(V)$ then the positive map $L : V_1M_kV_1 \to V_1M_kV_1$ has an eigenvector $\gamma' \in P_k \cap V_1M_kV_1$, by theorem 2.8. If $\Im(V_1) \neq \Im(V)$ then $\Im(\gamma') \neq \Im(\gamma)$ and γ' is not a multiple of γ. Since the multiplicity of the largest eigenvalue is 1 then γ' is associated to a different eigenvalue. Thus, γ' is orthogonal to γ, since L is self-adjoint. However, γ' and γ are positive semidefinite Hermitian matrices and $\Im(\gamma') \subset \Im(V_1) \subset \Im(V) = \Im(\gamma)$, thus they can not be orthogonal. Thus, $\Im(V_1) = \Im(V)$ and $V_1 = V$, and L is irreducible. □

Lemma 2.12. *Let $L : VM_kV \to VM_kV$ be a self-adjoint positive map. Let us assume that L has the decomposition property (definition 2.10). Let $V'M_kV' \subset VM_kV$ be such that $L(V'M_kV') \subset V'M_kV'$ then $L|_{V'M_kV'}$ also has the decomposition property.*

Proof. Let $\gamma \in P_k \cap V'M_kV'$ be such that $L(\gamma) = \lambda\gamma, \lambda > 0$. Since $L : VM_kV \to VM_kV$ has the decomposition property (definition 2.10) then $L|_R \equiv 0$, where $R = (V - V_1)M_kV_1 \oplus V_1M_k(V - V_1)$ and $V_1 \in M_k$ is the orthogonal projection such that $\Im(V_1) = \Im(\gamma)$. Notice that $\Im(V_1) = \Im(\gamma) \subset \Im(V') \subset \Im(V)$.

Consider now $R' = (V' - V_1)M_kV_1 \oplus V_1M_k(V' - V_1)$. Since $(V' - V_1)M_kV_1 = (V - V_1)(V' - V_1)M_kV_1 \subset (V - V_1)M_kV_1$ and $V_1M_k(V' - V_1) = V_1M_k(V' - V_1)(V - V_1) \subset V_1M_k(V - V_1)$ then $R' \subset R$ and $L|_{R'} \equiv 0$. Thus, $L : V'M_kV' \to V'M_kV'$ has the decomposition property. □

Proposition 2.13. *If $L : VM_kV \to VM_kV$ is a self-adjoint positive map then L has the decomposition property if and only if L is completely reducible. Moreover, the orthogonal projections $V_1, \dots, V_s$ in definition 2.9 are unique and $s \geq$ the multiplicity of the largest eigenvalue of L.*

Proof. First, suppose that L has the decomposition property and let us prove that L is completely reducible by induction on the rank of V. Notice that if $\text{rank}(V) = 1$ then $\dim(VM_kV) = 1$ and L is irreducible on VM_kV. Thus, L is completely reducible by definition 2.9. Let us assume that $\text{rank}(V) > 1$.

Since L is a positive map then $S = \{\gamma |\ 0 \neq \gamma \in P_k \cap VM_kV,\ L(\gamma) = \lambda\gamma, \lambda > 0\} \neq \varnothing$, theorem 2.8. Let $\gamma \in S$ be such that $\text{rank}(\gamma) = \min\{\text{rank}(\gamma')|\ \gamma' \in S\}$.

By corollary 2.4, $L(V_1M_kV_1) \subset V_1M_kV_1$, where V_1 is the orthogonal projection onto $\Im(\gamma)$. Now, if $L|_{V_1M_kV_1}$ is not irreducible then there exists $V_1'M_kV_1' \subset V_1M_kV_1$ with $\text{rank}(V_1') < \text{rank}(V_1)$ and $L(V_1'M_kV_1') \subset V_1'M_kV_1'$.

By theorem 2.8, there exists $0 \neq \delta \in P_k \cap V_1' M_k V_1'$ such that $L(\delta) = \mu\delta$, $\mu > 0$. However, $\text{rank}(\delta) \leq \text{rank}(V_1') < \text{rank}(V_1) = \text{rank}(\gamma)$. This contradicts the choice of γ. Thus, $L|_{V_1 M_k V_1}$ is irreducible.

Now, if $\text{rank}(V_1) = \text{rank}(V)$ then $V_1 = V$ and $L|_{V M_k V}$ is irreducible. Therefore, $L : V M_k V \to V M_k V$ is completely reducible by definition 2.9.

Next, suppose $\text{rank}(V_1) < \text{rank}(V)$. Since $L(V_1 M_k V_1) \subset V_1 M_k V_1$ and L is self-adjoint then $L((V_1 M_k V_1)^{\perp}) \subset (V_1 M_k V_1)^{\perp}$. Therefore, $tr(L(V - V_1)V_1) = 0$. Since $L(V - V_1)$ and V_1 are positive semidefinite then $\Im(L(V - V_1)) \subset \Im(V - V_1)$. By lemma 2.3, $L((V - V_1)M_k(V - V_1)) \subset (V - V_1)M_k(V - V_1)$.

Notice that $L|_{(V-V_1)M_k(V-V_1)}$ is a self-adjoint positive map with the decomposition property by lemma 2.12. Since $\text{rank}(V - V_1) < \text{rank}(V)$, by induction on the rank, $L|_{(V-V_1)M_k(V-V_1)}$ is completely reducible.

Thus, there are orthogonal projections $V_2, \ldots, V_s \in M_k$ satisfying $V_i V_j = 0$ $(i \neq j)$, $V_i(V - V_1) = V_i$ $(i \geq 2)$, $(V - V_1)M_k(V - V_1) = V_2 M_k V_2 \oplus \ldots \oplus V_s M_k V_s \oplus \widetilde{R}$ with $\widetilde{R} \perp V_2 M_k V_2 \oplus \ldots \oplus V_s M_k V_s$, $L|_{V_i M_k V_i}$ is irreducible for $2 \leq i \leq s$ and $L|_{\widetilde{R}} \equiv 0$.

Since L has the decomposition property then $V M_k V = V_1 M_k V_1 \oplus (V - V_1)M_k(V - V_1) \oplus R$, where $L|_R \equiv 0$ and $R \perp V_1 M_k V_1 \oplus (V - V_1)M_k(V - V_1)$.

Thus, we obtained $V M_k V = V_1 M_k V_1 \oplus V_2 M_k V_2 \oplus \ldots \oplus V_s M_k V_s \oplus \widetilde{R} \oplus R$ such that $L|_{V_i M_k V_i}$ is irreducible for $1 \leq i \leq s$ and $L|_{\widetilde{R} \oplus R} \equiv 0$. Notice that $V_i V_j = 0$, for $2 \leq i \neq j \leq s$ and $V_1 V_i = 0$, for $2 \leq i \leq s$, because $\Im(V_i) \subset \Im(V - V_1)$.

Notice that $\widetilde{R} \perp V_2 M_k V_2 \oplus \ldots \oplus V_s M_k V_s$ and $\widetilde{R} \perp V_1 M_k V_1$, because $\widetilde{R} \subset (V - V_1)M_k(V - V_1)$. Therefore $\widetilde{R} \oplus R \perp V_1 M_k V_1 \oplus V_2 M_k V_2 \oplus \ldots \oplus V_s M_k V_s$ and $L|_{\widetilde{R} \oplus R} \equiv 0$. Thus, L is completely reducible.

For the converse, let us assume that L is completely reducible and let us prove that L has the decomposition property. Thus, $V M_k V = V_1 M_k V_1 \oplus \ldots \oplus V_s M_k V_s \oplus R$, $R \perp V_1 M_k V_1 \oplus \ldots \oplus V_s M_k V_s$, $L(V_i M_k V_i) \subset V_i M_k V_i$, $L|_{V_i M_k V_i}$ is irreducible and $L|_R \equiv 0$.

Assume $L(\gamma') = \lambda\gamma'$, $\lambda > 0$ and $\gamma' \in P_k \cap V M_k V$ and let $V' \in M_k$ be the orthogonal projection onto $\Im(\gamma')$. By corollary 2.4, we have $L(V' M_k V') \subset V' M_k V'$.

Notice that, $\gamma' = \gamma_1' + \ldots + \gamma_s'$, where $\gamma_i' \in V_i M_k V_i$. Now, since $\Im(\gamma_i') \subset \Im(V_i)$ and $\Im(V_i) \perp \Im(V_j)$, for $i \neq j$, then each $\gamma_i' \in P_k$. Since each $V_i M_k V_i$ is an invariant subspace of L then we also conclude that $L(\gamma_i') = \lambda\gamma_i'$. Note that, not for every i, one has $\gamma_i' = 0$. Assume, without loss of generality, that $\gamma' = \gamma_1' + \ldots + \gamma_m'$ and $\gamma_i' \neq 0$, for $1 \leq i \leq m \leq s$.

Now, if for some $1 \leq i \leq m$, $\Im(\gamma_i') \neq \Im(V_i)$ then $L|_{V_i M_k V_i}$ is not irreducible, by corollary 2.4, which is a contradiction. Therefore, $\Im(\gamma_i') = \Im(V_i)$ for $1 \leq i \leq m$ and $V_1 + \cdots + V_m = V'$.

Next, $VM_kV = V'M_kV' \oplus (V-V')M_k(V-V') \oplus R'$, where $R' = (V-V')M_kV' \oplus V'M_k(V-V')$. Notice that $R' \perp V'M_kV' \oplus (V-V')M_k(V-V')$.

Now, $V_1M_kV_1 \oplus \ldots \oplus V_mM_kV_m \subset V'M_kV'$ and $V_{m+1}M_kV_{m+1} \oplus \ldots \oplus V_sM_kV_s \subset (V-V')M_k(V-V')$, therefore $R' \perp V_1M_kV_1 \oplus \ldots \oplus V_sM_kV_s$ and $R' \subset R$. Therefore, $L|_{R'} \equiv 0$ and L has the decomposition property by definition 2.10.

Finally, if $L : VM_kV \to VM_kV$ is a self-adjoint completely reducible map then the non-null eigenvalues of L are the non-null eigenvalues of $L|_{V_iM_kV_i}$. Since $L|_{V_iM_kV_i}$ is irreducible then the multiplicity of the largest eigenvalue is 1 by lemma 2.11. Therefore each $L|_{V_iM_kV_i}$ has at most one largest eigenvalue of L. Thus, $s \geq$ the multiplicity of the largest eigenvalue of $L : VM_kV \to VM_kV$. Now, if $L(V''M_kV'') \subset V''M_kV''$ and $L|_{V''M_kV''}$ is irreducible then by lemma 2.11, there is $\gamma'' \in P_k \cap V''M_kV''$ such that $L(\gamma'') = \lambda\gamma''$, $\lambda > 0$ and $\Im(\gamma'') = \Im(V'')$. As we noticed in the second part of this proof, there is $V_iM_kV_i \subset V''M_kV''$ (V'' is a sum of some V_i's). Since $L(V_iM_kV_i) \subset V_iM_kV_i$ then $L|_{V''M_kV''}$ is irreducible if and only if $V'' = V_i$, for some $1 \leq i \leq s$. □

Chapter 3

Completely Reducible Maps in Quantum Information Theory

The results of this chapter were published in [14].

Let us identify the tensor product space $M_k \otimes M_m$ with M_{km}, via Kronecker product (i.e., if $C = (c_{ij}) \in M_k$ and $B \in M_m$ then $C \otimes B = (c_{ij}B) \in M_{km}$).

Let $A \in M_k \otimes M_m \simeq M_{km}$ be a Hermitian matrix. We can write $A = \sum_{i=1}^n A_i \otimes B_i$, where A_i, B_i are Hermitian matrices for every i. Let $F_A : M_m \to M_k$ be $F_A(X) = \sum_{i=1}^n tr(B_i X) A_i$ and $G_A : M_k \to M_m$ be $G_A(X) = \sum_{i=1}^n tr(A_i X) B_i$. These maps are adjoint with respect to the trace inner product (Since A_i, B_i are Hermitian matrices then $F_A(Y^*) = F_A(Y)^*$, for every $Y \in M_m$. Notice that if $X \in M_k$ and $Y \in M_m$ then $tr(A(X \otimes Y^*)) = tr(G_A(X)Y^*) = tr(XF_A(Y^*)) = tr(XF_A(Y)^*)$). Notice that if $\{\gamma_1, \dots, \gamma_{k^2}\}$ is an orthonormal basis of M_k formed by Hermitian matrices then $A = \sum_{i=1}^{k^2} \gamma_i \otimes G_A(\gamma_i)$.

Moreover, if A is positive semidefinite then $F_A : M_m \to M_k$ and $G_A : M_k \to M_m$ are also positive maps, since $0 \leq tr(A(X \otimes Y)) = tr(G_A(X)Y) = tr(XF_A(Y))$, when $X \in P_k$ and $Y \in P_m$. Thus, $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map.

In Section 3.1, we prove that if $A \in M_k \otimes M_m$ is positive under partial transposition or symmetric with positive coefficients or invariant under realignment then $F_A \circ G_A : M_k \to M_k$ is completely reducible (theorems 3.2, 3.12 and 3.13). These are our main results and we shall apply them to Quantum Information Theory.

In Section 3.2, we apply our main results to two problems in Quantum Information Theory. We reduce the separability problem to the weakly irreducible case. We provide a complete description of weakly irreducible PPT matrices. We also show that if $F_A \circ G_A : M_k \to M_k$ is completely reducible with eigenvalues equal to 1 or 0 then A is separable. We use this result in order to obtain a new proof of the following one: If $\mathbb{C}^k$ contains k mutually unbiased bases then $\mathbb{C}^k$ contains $k+1$. This is our last application to Quantum Information Theory.

We complete this chapter with some remarks on our main theorems and on the applications to Quantum Information Theory. We present a couple of examples of positive semidefinite Hermitian matrices A in $M_k \otimes M_k \simeq M_{k^2}$ such that $F_A \circ G_A : M_k \to M_k$ is not completely reducible. We also show that if $A \in M_2 \otimes M_2$ is symmetric with positive coefficients or invariant under realignment then A is positive under partial transposition, reinforcing the connection between these three types. Finally, we show that low tensor rank implies separability and we connect the aforementioned applications to Quantum Information Theory to this last result (see subsection 3.3.2).

Throughout this chapter we shall adopt the following notation: $\mathbb{C}^k$ is the set of column vectors with k complex entries. We shall also identify the tensor product space $\mathbb{C}^k \otimes \mathbb{C}^m$ with $\mathbb{C}^{km}$, via Kronecker product (i.e. if $v = (v_i) \in \mathbb{C}^k, w \in \mathbb{C}^m$ then $v \otimes w = (v_i w) \in \mathbb{C}^{km}$).

The identification of the tensor product space $\mathbb{C}^k \otimes \mathbb{C}^m$ with $\mathbb{C}^{km}$ and the tensor product space $M_k \otimes M_m$ with M_{km}, via Kronecker product, allow us to write $(v \otimes w)(r \otimes s)^t = vr^t \otimes ws^t$, where $v \otimes w \in \mathbb{C}^k \otimes \mathbb{C}^m$ is a column, $(v \otimes w)^t$ its transpose and $v, r \in \mathbb{C}^k$ and $w, s \in \mathbb{C}^m$. Therefore if $x, y \in \mathbb{C}^k \otimes \mathbb{C}^m \simeq \mathbb{C}^{km}$ we have $xy^t \in M_k \otimes M_m \simeq M_{km}$. Here, $tr(A)$ denotes the trace of a matrix A, $\overline{A}$ stands for the matrix whose entries are $\overline{a_{ij}}$, where $\overline{a_{ij}}$ is the complex conjugate of the entry a_{ij} of A and A^t stands for the transpose of A. We shall consider the usual inner product in M_k, $\langle A, B \rangle = tr(AB^*)$, and the usual inner product in $\mathbb{C}^k$, $\langle x, y \rangle = x^t \overline{y}$. If $A = \sum_{i=1}^n A_i \otimes B_i$, we shall denote by A^{t_2} the matrix $\sum_{i=1}^n A_i \otimes B_i^t$, which is called the partial transposition of A. The image (or the range) of the matrix $A \in M_k$ in $\mathbb{C}^k$ shall be denoted by $\mathfrak{I}(A)$.

3.1 Main Theorems: The Complete Reducibility of $F_A \circ G_A : M_k \to M_k$

3.1.1 Main Theorem for PPT matrices

Definition 3.1. (***PPT matrices***) *Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix. We say that A is positive under partial transposition or simply PPT, if $A^{t_2} = Id \otimes (\cdot)^t(A) = \sum_{i=1}^n A_i \otimes B_i^t$ is positive semidefinite.*

Theorem 3.2. *Let $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. If A is PPT then $F_A \circ G_A : M_k \to M_k$ is completely reducible.*

Proof. Let $\gamma \in P_k \cap VM_kV$ be such that $F_A(G_A(\gamma)) = \lambda\gamma$, $\lambda > 0$. Let $V_1 \in M_k$ be the orthogonal projection onto $\Im(\gamma)$. Let $W_1 \in M_m$ be the orthogonal projection onto $\Im(G_A(\gamma))$.

By lemma 2.3, we have $G_A(V_1M_kV_1) \subset W_1M_mW_1$ and $F_A(W_1M_mW_1) \subset V_1M_kV_1$.

If $V_2 = Id - V_1$ and $W_2 = Id - W_1$ then $A = \sum_{i,j,r,s=1}^{2}(V_i \otimes W_j)A(V_r \otimes W_s)$.

Notice that $tr(A(V_1 \otimes W_2)) = tr(G_A(V_1)W_2) = 0$. Thus, $A(V_1 \otimes W_2) = (V_1 \otimes W_2)A = 0$, since $A \in P_{km}$ and $V_1 \otimes W_2 \in P_{km}$. Notice that $tr(A(V_2 \otimes W_1)) = tr(V_2F_A(W_1)) = 0$. Thus, $A(V_2 \otimes W_1) = (V_2 \otimes W_1)A = 0$, since $A \in P_{km}$ and $V_2 \otimes W_1 \in P_{km}$.

Therefore, $A = \sum_{i,j=1}^{2}(V_i \otimes W_i)A(V_j \otimes W_j)$.

Next, $0 = (A(V_1 \otimes W_2))^{t_2} = (Id \otimes W_2^t)A^{t_2}(V_1 \otimes Id)$ and $0 = tr((Id \otimes W_2^t)A^{t_2}(V_1 \otimes Id)) = tr(A^{t_2}(V_1 \otimes W_2^t))$. Since A is PPT then A^{t_2} is positive semidefinite and $A^{t_2}(V_1 \otimes W_2^t) = (V_1 \otimes W_2^t)A^{t_2} = 0$. Analogously, we obtain $A^{t_2}(V_2 \otimes W_1^t) = (V_2 \otimes W_1^t)A^{t_2} = 0$.

Thus, $A^{t_2} = \sum_{i,j=1}^{2}(V_i \otimes W_j^t)A^{t_2}(V_j \otimes W_i^t)$ and $A^{t_2} = \sum_{i=1}^{2}(V_i \otimes W_i^t)A^{t_2}(V_i \otimes W_i^t)$. Hence, $A = \sum_{i=1}^{2}(V_i \otimes W_i)A(V_i \otimes W_i)$.

Notice also that if $X \in R = V_1M_kV_2 \oplus V_2M_kV_1$, which is the orthogonal complement of $V_1M_kV_1 \oplus V_2M_kV_2$ in M_k, then $G_A(X) = 0$ and $F_A \circ G_A|_R \equiv 0$. Thus, $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map with the decomposition property (definition 2.10). By proposition 2.13, $F_A \circ G_A : M_k \to M_k$ is completely reducible. □

3.1.2 Main Theorems for SPC matrices and Matrices Invariant under Realignment

In order to obtain our main theorems for SPC matrices and matrices invariant under realignment, we need some definitions and some preliminary results.

Definition 3.3. *Let $\{e_1, \dots, e_k\}$ be the canonical basis of $\mathbb{C}^k$.*

(1) *Let $T = \sum_{i,j=1}^{k} e_ie_j^t \otimes e_je_i^t \in M_k \otimes M_k \simeq M_{k^2}$. This matrix satisfies $Ta \otimes b = b \otimes a$, $(a \otimes b)^tT = (b \otimes a)^t$, for every $a, b \in \mathbb{C}^k$, where $a \otimes b$ is a column vector in $\mathbb{C}^{k^2}$ and $(a \otimes b)^t$ is its transpose. This matrix is usually called the flip operator (see* [45]*).*

(2) *Let $u = \sum_{i=1}^{k} e_i \otimes e_i \in \mathbb{C}^k \otimes \mathbb{C}^k$.*

(3) *Let $F : M_k \to \mathbb{C}^k \otimes \mathbb{C}^k$, $F(\sum_{i=1}^{n} a_ib_i^t) = \sum_{i=1}^{n} a_i \otimes b_i$.*

Remark 3.4. *Recall that F is an isometry, i.e., $F(A)^t\overline{F(B)} = tr(AB^*)$, for every $A, B \in M_k$, where $F(A), F(B) \in \mathbb{C}^k \otimes \mathbb{C}^k$ and $\overline{F(B)}$ is the conjugation of the column vector $F(B)$. We also have $tr(F^{-1}(v)F^{-1}(w)^*) = v^t\overline{w}$, for every $v, w \in \mathbb{C}^{k^2}$ (see [41]).*

Definition 3.5. *Let $S : M_k \otimes M_k \to M_k \otimes M_k$ be defined by*

$$S(\sum_{i=1}^{n} A_i \otimes B_i) = \sum_{i=1}^{n} F(A_i)F(B_i)^t,$$

where $F(A_i) \in \mathbb{C}^k \otimes \mathbb{C}^k$ is a column vector and $F(B_i)^t$ is a row vector (definition 3.3). This map is usually called the "realignment map" (see [18, 40, 41]).

Definition 3.6. (***SPC matrices***) *Let $A \in M_k \otimes M_k \simeq M_{k^2}$ be a positive semidefinite Hermitian matrix. We say that A is symmetric with positive coefficients or simply SPC, if $S(A^{t_2})$ is a positive semidefinite Hermitian matrix.*

Remark 3.7. *The name symmetric with positive coefficients (SPC) is justified by proposition 3.33: If $A \in P_{k^2}$ then A is SPC if and only if A has the following symmetric Hermitian Schmidt decomposition (definition 3.16) with positive coefficients: $\sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i$, with $\lambda_i > 0$, for every i.*

Definition 3.8. (***Matrices Invariant under Realignment***) *Let $A \in M_k \otimes M_k$ be a positive semidefinite Hermitian matrix. We say that A is invariant under realignment if $A = S(A)$.*

Examples 3.9. *a) Since $Id \otimes Id$ is invariant under partial transposition, $(Id \otimes Id)^{t_2} = Id \otimes Id$, then $Id \otimes Id$ is PPT. Since $S(Id \otimes Id) = uu^t$, by definitions 3.3 and 3.5, then $Id \otimes Id$ is also SPC.*

b) Since $uu^t = \sum_{i,j=1}^{k} e_ie_j^t \otimes e_ie_j^t$ then $S(uu^t) = \sum_{i,j=1}^{k} e_ie_i^t \otimes e_je_j^t = Id \otimes Id$. Observe that $S(Id \otimes Id + uu^t) = uu^t + Id \otimes Id$ and $Id \otimes Id + uu^t$ is positive semidefinite. Thus, $Id \otimes Id + uu^t$ is invariant under realignment.

c) Since $T = \sum_{i,j=1}^{k} e_ie_j^t \otimes e_je_i^t$ then $S(T) = T$. Since the eigenvalues of T are 1 and -1 then $Id \otimes Id - T$ is positive semidefinite. Hence, $Id \otimes Id + uu^t - T$ is invariant under realignment.

Lemma 3.10. (*Properties of the Realignment map*) *Let $S : M_k \otimes M_k \to M_k \otimes M_k$ be the realignment map defined in 3.5. Let $v, v_i, w_i \in \mathbb{C}^k \otimes \mathbb{C}^k$, $V, W, M, N \in M_k$. Then*

(1) $S(A^{t_2})v = F \circ F_A \circ F^*(v)$

(2) $S(\sum_{i=1}^{n} v_iw_i^t) = \sum_{i=1}^{n} F^{-1}(v_i) \otimes F^{-1}(w_i)$

(3) $S^2 = Id : M_k \otimes M_k \to M_k \otimes M_k$

(4) $S((V \otimes W)A(M \otimes N)) = (V \otimes M^t)S(A)(W^t \otimes N)$

(5) $S(AT)T = A^{t_2}$

(6) $S(A^{t_2}) = S(A)T$

(7) $S(AT) = S(A)^{t_2}$

Proof. Let $A = \sum_{i=1}^{n} A_i \otimes B_i$. Notice that $A^{t_2} = \sum_{i=1}^{n} A_i \otimes B_i^t$ and $S(A^{t_2})v = \sum_{i=1}^{n} F(A_i)F(B_i^t)^t v$. By remark 3.4, since $v = \overline{F(\overline{F^{-1}(v)})}$ then $F(B_i^t)^t v = tr(B_i^t F^{-1}(v)^t) = tr(B_i F^{-1}(v))$. Therefore, $S(A^{t_2})v = F(\sum_{i=1}^{n} A_i tr(B_i F^{-1}(v))) = F \circ F_A \circ F^{-1}(v)$. Since F is an isometry then $F^{-1} = F^*$, also by the same remark, and item 1 is proved.

Next, since $S(ab^t \otimes cd^t) = ac^t \otimes bd^t$ for every $a, b, c, d \in \mathbb{C}^k$ then $S^2(ab^t \otimes cd^t) = ab^t \otimes cd^t$. Since $\{ab^t \otimes cd^t,\ a, b, c, d \in \mathbb{C}^k\}$ is a set of generators of $M_k \otimes M_k$ then item 3 is proved. By definiton 3.5, item 2 is also proved.

In order to prove the other properties, since both sides of the equations are linear on A, we just need to prove for $A = ab^t \otimes cd^t$, where $a, b, c, d \in \mathbb{C}^k$.

Now, $S((V \otimes W)(ab^t \otimes cd^t)(M \otimes N)) = S((Va \otimes Wc)(M^t b \otimes N^t d)^t)$. By item (2), this is equal to $F^{-1}(Va \otimes Wc) \otimes F^{-1}(M^t b \otimes N^t d) = (Vac^t W^t) \otimes (M^t bd^t N) = (V \otimes M^t)(ac^t \otimes bd^t)(W^t \otimes N) = (V \otimes M^t)S(A)(W^t \otimes N)$. Thus, item 4 is proved.

The other properties are also straightforward. Just recall that $S(ab^t \otimes cd^t) = ac^t \otimes bd^t$, $(ab^t \otimes cd^t)T = ad^t \otimes cb^t$, $T(ab^t \otimes cd^t) = (cb^t \otimes ad^t)$ and $(ab^t \otimes cd^t)^{t_2} = ab^t \otimes dc^t$, for every $a, b, c, d \in \mathbb{C}^k$. □

Lemma 3.11. *Let $A \in M_k \otimes M_k$ be a Hermitian matrix.*

a) If $S(A^{t_2}) \in P_{k^2}$ and $F_A \circ G_A(\gamma) = \lambda\gamma$ then $G_A(\gamma) = \sqrt{\lambda}\gamma$.

b) If $S(A) \in P_{k^2}$ and $F_A \circ G_A(\gamma) = \lambda\gamma$ then $G_A(\gamma) = \sqrt{\lambda}\gamma^t$.

Proof. Since $S(A^{t_2}) \in P_{k^2}$ then $F_A : M_k \to M_k$ is a self-adjoint linear transformation with non negative eigenvalues, by item 1 of lemma 3.10 and by remark 3.4. Since A is Hermitian, G_A is the adjoint of F_A and $F_A = G_A$. Therefore, $F_A \circ G_A(\gamma) = \lambda\gamma$ if and only if $G_A^2(\gamma) = \lambda\gamma$ if and only if $G_A(\gamma) = \sqrt{\lambda}\gamma$, because G_A has only non negative eigenvalues. Thus, item $a)$ is proved.

Now, since A^{t_2} is Hermitian and $(A^{t_2})^{t_2} = A$ then, by item $a)$, $F_{A^{t_2}} \circ G_{A^{t_2}}(\gamma) = \lambda\gamma$ if and only if $G_{A^{t_2}}(\gamma) = \sqrt{\lambda}\gamma$. But $F_{A^{t_2}} \circ G_{A^{t_2}}(X) = F_A \circ G_A(X)$ and $G_{A^{t_2}}(X) = G_A(X)^t$. □

Theorem 3.12. *Let $A \in M_k \otimes M_k \simeq M_{k^2}$, $A \in P_{k^2}$. If A is SPC then $F_A \circ G_A : M_k \to M_k$ is completely reducible.*

Proof. By definition 3.6, $S(A^{t_2}) \in P_{k^2}$. Let $\gamma \in P_k \cap VM_kV$ be such that $F_A(G_A(\gamma)) = \lambda^2\gamma$, $\lambda > 0$. By item $a)$ of lemma 3.11, $G_A(\gamma) = \lambda\gamma$. Thus, $F_A(\gamma) = \lambda\gamma$.

Let $V_1 \in M_k$ be the orthogonal projection onto $\Im(\gamma)$. By lemma 2.3, we have $G_A(V_1M_kV_1) \subset V_1M_kV_1$ and $F_A(V_1M_kV_1) \subset V_1M_kV_1$. If $V_2 = Id - V_1$ then $A = \sum_{i,j,r,s=1}^{2}(V_i \otimes V_j)A(V_r \otimes V_s)$.

Notice that $tr(A(V_1 \otimes V_2)) = tr(G_A(V_1)V_2) = 0$. Thus, $A(V_1 \otimes V_2) = (V_1 \otimes V_2)A = 0$, since $A \in P_{k^2}$ and $V_1 \otimes V_2 \in P_{k^2}$. Notice that $tr(A(V_2 \otimes V_1)) = tr(V_2F_A(V_1)) = 0$. Thus, $A(V_2 \otimes V_1) = (V_2 \otimes V_1)A = 0$, since $A \in P_{k^2}$ and $V_2 \otimes V_1 \in P_{k^2}$.

Therefore, $A = \sum_{i,j=1}^{2}(V_i \otimes V_i)A(V_j \otimes V_j)$.

Next, $0 = (A(V_1 \otimes V_2))^{t_2} = (Id \otimes V_2^t)A^{t_2}(V_1 \otimes Id)$ and $0 = S((Id \otimes V_2^t)A^{t_2}(V_1 \otimes Id)) = (Id \otimes V_1^t)S(A^{t_2})(V_2 \otimes Id)$, by item 4 of lemma 3.10.

Now, $0 = tr((Id \otimes V_1^t)S(A^{t_2})(V_2 \otimes Id)) = tr(S(A^{t_2})(V_2 \otimes V_1^t))$. Since $S(A^{t_2}) \in P_{k^2}$ then $S(A^{t_2})(V_2 \otimes V_1^t) = (V_2 \otimes V_1^t)S(A^{t_2}) = 0$. Analogously, we obtain $S(A^{t_2})(V_1 \otimes V_2^t) = (V_1 \otimes V_2^t)S(A^{t_2}) = 0$.

Thus, $A^{t_2} = \sum_{i,j=1}^{2}(V_i \otimes V_j^t)A^{t_2}(V_j \otimes V_i^t)$ and $S(A^{t_2}) = \sum_{i,j=1}^{2}(V_i \otimes V_j^t)S(A^{t_2})(V_j \otimes V_i^t) = \sum_{i=1}^{2}(V_i \otimes V_i^t)S(A^{t_2})(V_i \otimes V_i^t)$, by item 4 of lemma 3.10.

So, $A^{t_2} = S^2(A^{t_2}) = \sum_{i=1}^{2}(V_i \otimes V_i^t)S^2(A^{t_2})(V_i \otimes V_i^t) = \sum_{i=1}^{2}(V_i \otimes V_i^t)A^{t_2}(V_i \otimes V_i^t)$, by items 3 and 4 of lemma 3.10. Therefore, $A = \sum_{i=1}^{2}(V_i \otimes V_i)A(V_i \otimes V_i)$.

Finally, notice that if $X \in R = V_1M_kV_2 \oplus V_2M_kV_1$, which is the orthogonal complement of $V_1M_kV_1 \oplus V_2M_kV_2$ in M_k, then $G_A(X) = 0$ and $F_A \circ G_A|_R \equiv 0$. Thus, $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map with the decomposition property (definition 2.10). By proposition 2.13, $F_A \circ G_A : M_k \to M_k$ is completely reducible. □

Theorem 3.13. *Let $A \in M_k \otimes M_k \simeq M_{k^2}$, $A \in P_{k^2}$. If A is invariant under realignment then $F_A \circ G_A : M_k \to M_k$ is completely reducible.*

Proof. Let $\gamma \in P_k \cap VM_kV$ be such that $F_A(G_A(\gamma)) = \lambda^2\gamma$, $\lambda > 0$. By item $b)$ of lemma 3.11, $G_A(\gamma) = \lambda\gamma^t$. Thus, $F_A(\gamma^t) = \lambda\gamma$. Let $V_1 \in M_k$ be the orthogonal projection onto $\Im(\gamma)$. By lemma 2.3, we have $G_A(V_1M_kV_1) \subset V_1^tM_kV_1^t$ and $F_A(V_1^tM_kV_1^t) \subset V_1M_kV_1$. Now, if $V_2 = Id - V_1$ then $A = \sum_{i,j,r,s=1}^{2}(V_i \otimes V_j^t)A(V_r \otimes V_s^t)$.

Notice that $tr(A(V_1 \otimes V_2^t)) = tr(G_A(V_1)V_2^t) = 0$. Thus, $A(V_1 \otimes V_2^t) = (V_1 \otimes V_2^t)A = 0$, since $A \in P_{k^2}$ and $V_1 \otimes V_2^t \in P_{k^2}$. Notice that $tr(A(V_2 \otimes V_1^t)) = tr(V_2F_A(V_1^t)) = 0$. Thus, $A(V_2 \otimes V_1^t) = (V_2 \otimes V_1^t)A = 0$, since $A \in P_{k^2}$ and $V_2 \otimes V_1^t \in P_{k^2}$.

Therefore, $A = \sum_{i,j=1}^{2}(V_i \otimes V_i^t)A(V_j \otimes V_j^t)$.

Next, $S(A) = \sum_{i,j=1}^{2} S((V_i \otimes V_i^t)A(V_j \otimes V_j^t)) = \sum_{i,j=1}^{2}(V_i \otimes V_j^t)S(A)(V_i \otimes V_j^t)$, by item 4 of lemma 3.10. Since $A = S(A)$, we have $A = \sum_{i,j=1}^{2}(V_i \otimes V_j^t)A(V_i \otimes V_j^t) = \sum_{i=1}^{2}(V_i \otimes V_i^t)A(V_i \otimes V_i^t)$.

Finally, notice that if $X \in R = V_1M_kV_2 \oplus V_2M_kV_1$, which is the orthogonal complement of $V_1M_kV_1 \oplus V_2M_kV_2$ in M_k, then $G_A(X) = 0$ and $F_A \circ G_A|_R \equiv 0$. Thus, $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map with the decomposition property (definition 2.10). By proposition 2.13, $F_A \circ G_A : M_k \to M_k$ is completely reducible. □

3.2 Applications to Quantum Information Theory

Throughout the following subsection we shall assume that $F_A \circ G_A : M_k \to M_k$ is completely reducible. This is a strong restriction. However, we know that if A is PPT or SPC or invariant under realignment then $F_A \circ G_A : M_k \to M_k$ is indeed completely reducible (theorems 3.2, 3.12, 3.13). Recall that a necessary condition for the separability of a matrix is to be PPT.

3.2.1 The Separability Problem

We assume that $F_A \circ G_A : M_k \to M_k$ is completely reducible and we give applications to Quantum Information Theory. The first application is the reduction of the Separability Problem to the weakly irreducible case (corollary 3.20) and the second is proposition 3.21 which grants the separability of A, if $F_A \circ G_A : M_k \to M_k$ has only eigenvalues 1 or 0.

Throughout the next section we present our last application concerning mutually unbiased bases using this proposition 3.21 for a matrix invariant under realignment (see proposition 3.22 and theorem 3.28).

We begin this section with a simple lemma that provides an equivalent way to prove that $F_A \circ G_A : M_k \to M_k$ is completely reducible.

Lemma 3.14. *Let $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. Thus, $F_A \circ G_A : M_k \to M_k$ is completely reducible if and only if for every $\gamma \in P_k$ such that $F_A \circ G_A(\gamma) = \lambda\gamma$, $\lambda > 0$, we have $A = (V_1 \otimes W_1)A(V_1 \otimes W_1) + (Id - V_1 \otimes Id - W_1)A(Id - V_1 \otimes Id - W_1)$, where $V_1 \in M_k, W_1 \in M_m$ are orthogonal projections onto $\Im(\gamma), \Im(G_A(\gamma))$, respectively.*

Proof. Suppose $F_A \circ G_A : M_k \to M_k$ is completely reducible then $F_A \circ G_A : M_k \to M_k$ has the decomposition property (definition 2.10) by proposition 2.13.

If $\gamma \in P_k$ is such that $F_A \circ G_A(\gamma) = \lambda\gamma$, $\lambda > 0$, then $M_k = V_1 M_k V_1 \oplus (Id - V_1)M_k(Id - V_1) \oplus R$, where $R \perp V_1 M_k V_1 \oplus (Id - V_1)M_k(Id - V_1)$ and $F_A \circ G_A|_R \equiv 0$, where $V_1 \in M_k$ is the orthogonal projection onto $\Im(\gamma)$.

Next, let $W_1 \in M_m$ be the orthogonal projection onto the $\Im(G_A(\gamma))$. By lemma 2.3, we have $G_A(V_1 M_k V_1) \subset W_1 M_m W_1$, because G_A is a positive map, since $A \in P_{km}$.

Now, $\langle G_A(Id - V_1), G_A(\gamma)\rangle = \langle Id - V_1, F_A \circ G_A(\gamma)\rangle = \lambda\langle Id - V_1, \gamma\rangle = 0$. Since $G_A(Id - V_1)$ and $G_A(\gamma)$ are positive semidefinite then $\Im(G_A(Id - V_1)) \perp \Im(G_A(\gamma)) = \Im(W_1)$. Thus, $\Im(G_A(Id - V_1)) \subset \Im(Id - W_1)$. Again by lemma 2.3, we have $G_A((Id - V_1)M_k(Id - V_1)) \subset (Id - W_1)M_m(Id - W_1)$.

Next, since $F_A \circ G_A|_R \equiv 0$ and F_A,G_A are adjoint maps then $G_A|_R \equiv 0$.

Let $\{\gamma_1, \dots, \gamma_r\}$ be an orthonormal basis of $V_1 M_k V_1$ formed by Hermitian matrices, $\{\delta_1, \dots, \delta_s\}$ be an orthonormal basis of $(Id - V_1)M_k(Id - V_1)$ formed by Hermitian matrices and $\{\alpha_1, \dots, \alpha_t\}$ be an orthonormal basis of R formed by Hermitian matrices. Then $A = \sum_{i=1}^r \gamma_i \otimes G_A(\gamma_i) + \sum_{i=1}^s \delta_i \otimes G_A(\delta_i) + \sum_{i=1}^s \alpha_i \otimes G_A(\alpha_i)$. Since $G_A(\alpha_i) = 0$ then $A = \sum_{i=1}^r \gamma_i \otimes G_A(\gamma_i) + \sum_{i=1}^s \delta_i \otimes G_A(\delta_i)$.

Since $\{\gamma_1, \dots, \gamma_r\} \subset V_1 M_k V_1$ then $G_A(\gamma_i) \in W_1 M_m W_1$ and since $\{\delta_1, \dots, \delta_s\} \subset (Id - V_1)M_k(Id - V_1)$ then $G_A(\delta_i) \in (Id - W_1)M_m(Id - W_1)$. Therefore, $(V_1 \otimes W_1)A(V_1 \otimes W_1) = \sum_{i=1}^r \gamma_i \otimes G_A(\gamma_i)$, $(Id - V_1 \otimes Id - W_1)A(Id - V_1 \otimes Id - W_1) = \sum_{i=1}^s \delta_i \otimes G_A(\delta_i)$ and $A = (V_1 \otimes W_1)A(V_1 \otimes W_1) + (Id - V_1 \otimes Id - W_1)A(Id - V_1 \otimes Id - W_1)$.

For the converse, assume that if $\gamma \in P_k$ is such that $F_A \circ G_A(\gamma) = \lambda\gamma$, $\lambda > 0$ then $A = (V_1 \otimes W_1)A(V_1 \otimes W_1) + (Id - V_1 \otimes Id - W_1)A(Id - V_1 \otimes Id - W_1)$, where V_1, W_1 are orthogonal projections onto $\Im(\gamma), \Im(G_A(\gamma))$, respectively.

Let $M_k = V_1 M_k V_1 \oplus (Id - V_1)M_k(Id - V_1) \oplus R$, $R \perp V_1 M_k V_1 \oplus (Id - V_1)M_k(Id - V_1)$.

Notice that $G_A|_R \equiv 0$ and $F_A \circ G_A|_R \equiv 0$. Therefore, $F_A \circ G_A : M_k \to M_k$ has the decomposition property (definition 2.10) and by proposition 2.13, $F_A \circ G_A : M_k \to M_k$ is completely reducible. □

Definition 3.15. *Let $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. We say that A is weakly irreducible if for every orthogonal projections $V_1, V_2 \in M_k$ and $W_1, W_2 \in M_m$ such that $V_2 = Id - V_1$, $W_2 = Id - W_1$ and $A = (V_1 \otimes W_1)A(V_1 \otimes W_1) + (V_2 \otimes W_2)A(V_2 \otimes W_2)$, we obtain $(V_1 \otimes W_1)A(V_1 \otimes W_1) = 0$ or $(V_2 \otimes W_2)A(V_2 \otimes W_2) = 0$.*

Definition 3.16. *A decomposition of a matrix $A \in M_k \otimes M_m$, $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$, is a Schmidt decomposition if $\{\gamma_i |\ 1 \leq i \leq n\} \subset M_k$, $\{\delta_i |\ 1 \leq i \leq n\} \subset M_m$ are orthonormal sets with respect to the trace inner product, $\lambda_i \in \mathbb{R}$ and $\lambda_i > 0$. Also, if γ_i and δ_i are Hermitian matrices for every i, then $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$ is a Hermitian Schmidt decomposition of A.*

Proposition 3.17. *Let $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. Let $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$ be a Hermitian Schmidt decomposition of A such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. If $F_A \circ G_A : M_k \to M_k$ is completely reducible then the following conditions are equivalent:*

(1) *A is weakly irreducible,*

(2) *$s = 1$ in definition 2.9 with $L = F_A \circ G_A : M_k \to M_k$,*

(3) *$\lambda_1 > \lambda_2$ and $\Im(\gamma_i) \subset \Im(\gamma_1)$, $\Im(\delta_i) \subset \Im(\delta_1)$, for $1 \leq i \leq n$.*

Proof. Notice that $F_A \circ G_A(\gamma_i) = \lambda_i^2 \gamma_i$ for $1 \leq i \leq n$. Thus, the largest eigenvalue of $F_A \circ G_A$ is λ_1^2. By definition 2.9, $M_k = V_1 M_k V_1 \oplus \ldots \oplus V_s M_k V_s \oplus R$, $F_A \circ G_A|_{V_i M_k V_i}$ is irreducible and $F_A \circ G_A|_R \equiv 0$, where $R \perp V_1 M_k V_1 \oplus \ldots \oplus V_s M_k V_s$. Since the non-null eigenvalues of $F_A \circ G_A : M_k \to M_k$ are eigenvalues of $F_A \circ G_A|_{V_i M_k V_i}$, for $1 \leq i \leq s$, then λ_1^2 is the largest eigenvalue of some $F_A \circ G_A|_{V_i M_k V_i}$. Without loss of generality we may assume that there exists $0 \neq \gamma \in P_k \cap V_1 M_k V_1$ such that $F_A \circ G_A(\gamma) = \lambda_1^2 \gamma$ and $\Im(\gamma) = \Im(V_1)$, by lemma 2.11. Thus, by lemma 3.14, $A = (V_1 \otimes W_1) A (V_1 \otimes W_1) + (Id - V_1 \otimes Id - W_1) A (Id - V_1 \otimes Id - W_1)$, where V_1, W_1 are orthogonal projections onto $\Im(\gamma), \Im(G_A(\gamma))$, respectively.

Firstly, let us assume that A is weakly irreducible, then or $(V_1 \otimes W_1) A (V_1 \otimes W_1) = 0$ or $(Id - V_1 \otimes Id - W_1) A (Id - V_1 \otimes Id - W_1) = 0$. Notice that if $(V_1 \otimes W_1) A (V_1 \otimes W_1) = 0$ then $A = (Id - V_1 \otimes Id - W_1) A (Id - V_1 \otimes Id - W_1)$ and $G_A(\gamma) = 0$, since $\gamma \in V_1 M_k V_1$. Therefore, $0 = F_A \circ G_A(\gamma) = \lambda_1^2 \gamma$, which is a contradiction. Therefore $(Id - V_1 \otimes Id - W_1) A (Id - V_1 \otimes Id - W_1) = 0$ and $A = (V_1 \otimes W_1) A (V_1 \otimes W_1)$. In this case, $G_A|_{(V_1 M_k V_1)^\perp} \equiv 0$ and $F_A \circ G_A|_{(V_1 M_k V_1)^\perp} \equiv 0$. Thus, $s = 1$ in definition 2.9.

Secondly, suppose that $s = 1$ in definition 2.9 then $M_k = V_1 M_k V_1 \oplus R$, $F_A \circ G_A|_{V_1 M_k V_1}$ is irreducible and $F_A \circ G_A|_R \equiv 0$, where $R = (V_1 M_k V_1)^\perp$. Thus, $\gamma_i \in V_1 M_k V_1$ for $1 \leq i \leq n$, since $F_A \circ G_A(\gamma_i) = \lambda_i^2 \gamma_i$ and $F_A \circ G_A(M_k) = F_A \circ G_A(V_1 M_k V_1) \subset V_1 M_k V_1$. By lemma 2.3, $G_A(V_1 M_k V_1) \subset W_1 M_m W_1$, since $\Im(G_A(\gamma)) = \Im(W_1)$. Thus, $\lambda_i \delta_i = G_A(\gamma_i) \in W_1 M_m W_1$ and $\Im(\delta_i) \subset \Im(W_1)$.

Next, since $F_A \circ G_A : V_1 M_k V_1 \to V_1 M_k V_1$ is irreducible then the multiplicity of the largest eigenvalue is 1 by lemma 2.11, thus $\lambda_1^2 > \lambda_2^2$ and $\lambda_1 > \lambda_2$. Moreover, γ must be a multiple of γ_1, because $F_A \circ G_A(\gamma_1) = \lambda_1^2 \gamma_1$.

Thus, $G_A(\gamma)$ is also a multiple of δ_1. Therefore, $\Im(\gamma_i) \subset \Im(V_1) = \Im(\gamma) = \Im(\gamma_1)$ and $\Im(\delta_i) \subset \Im(W_1) = \Im(G_A(\gamma)) = \Im(\delta_1)$.

Finally, let us assume that $\lambda_1 > \lambda_2$ and $\Im(\gamma_i) \subset \Im(\gamma_1)$, $\Im(\delta_i) \subset \Im(\delta_1)$, for $1 \leq i \leq n$. Let $V_j' \in M_k$ and $W_j' \in M_m$, $j = 1, 2$, be orthogonal projections such that $V_2' = Id - V_1'$, $W_2' = Id - W_1'$ and $A = (V_1' \otimes W_1') A (V_1' \otimes W_1') + (V_2' \otimes W_2') A (V_2' \otimes W_2')$. Thus, $G_A|_{V_1' M_k V_2' + V_2' M_k V_1'} \equiv 0$ and $F_A \circ G_A|_{V_1' M_k V_2' + V_2' M_k V_1'} \equiv 0$.

Next, notice that $G_A(V_j' M_k V_j') \subset W_j' M_m W_j'$ and $F_A(W_j' M_m W_j') \subset V_j' M_k V_j'$, $j = 1, 2$, since $V_1' V_2' = 0$ and $W_1' W_2' = 0$. Thus, $F_A \circ G_A(V_j' M_k V_j') \subset V_j' M_k V_j'$, for $j = 1, 2$.

Hence, the non-null eigenvalues of $F_A \circ G_A : M_k \to M_k$ are the non-null eigenvalues of $F_A \circ G_A|_{V_j' M_k V_j'}$, $j = 1, 2$. Without loss of generality, let us assume that λ_1^2 is an eigenvalue of $F_A \circ G_A|_{V_1' M_k V_1'}$. Since the multiplicity of λ_1^2 is 1 $(\lambda_1 > \lambda_2)$ then $\gamma_1 \in V_1' M_k V_1'$. Since $\Im(\gamma_j) \subset \Im(\gamma_1) \subset \Im(V_1) \perp \Im(V_2)$, $j = 1, 2$, then $(V_2' \otimes W_2') A (V_2' \otimes W_2') = 0$. Therefore, A is weakly irreducible. □

Proposition 3.18. *Let $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. If $F_A \circ G_A : M_k \to M_k$ is completely reducible then $A = \sum_{i=1}^{s} (V_i \otimes W_i) A (V_i \otimes W_i)$ such that*

(1) $V_1, \ldots, V_s \in M_k$ *are orthogonal projections such that* $V_iV_j = 0$

(2) $W_1, \ldots, W_s \in M_m$ *are orthogonal projections such that* $W_iW_j = 0$

(3) $(V_i \otimes W_i)A(V_i \otimes W_i)$ *is weakly irreducible and non-null for every* i.

(4) $s \geq$ *multiplicity of the largest eigenvalue of* $F_A \circ G_A : M_k \to M_k$.

Proof. Since $F_A \circ G_A : M_k \to M_k$ is completely reducible then $M_k = V_1M_kV_1 \oplus \ldots \oplus V_sM_kV_s \oplus R$, $F_A \circ G_A(V_iM_kV_i) \subset V_iM_kV_i$, $F_A \circ G_A|_{V_iM_kV_i}$ is irreducible, $F_A \circ G_A|_R \equiv 0$ and $s \geq$ multiplicity of the largest eigenvalue of $F_A \circ G_A : M_k \to M_k$, by proposition 2.13.

By lemma 2.11, there is $\gamma_1^j \in P_k \cap V_jM_kV_j$, $1 \leq j \leq s$, such that γ_1^j is an eigenvector of $F_A \circ G_A : V_jM_kV_j \to V_jM_kV_j$ associated to the unique largest eigenvalue and $\Im(\gamma_1^j) = \Im(V_j)$. Since G_A is a positive map then $G_A(\gamma_1^j) \in P_m$.

By lemma 2.3, $G_A(V_jM_kV_j) \subset W_jM_kW_j$, where W_j is the orthogonal projection onto $\Im(G_A(\gamma_1^j))$. Notice that $V_jM_kV_j \perp V_iM_kV_i$, for $i \neq j$, since $V_iV_j = 0$. Therefore, $\langle G_A(\gamma_1^j), G_A(\gamma_1^i)\rangle = \langle \gamma_1^j, F_A \circ G_A(\gamma_1^i)\rangle = 0$, for $i \neq j$. Thus, $W_iW_j = 0$ for $i \neq j$.

Let $\{\gamma_1^j, \ldots, \gamma_{r_j}^j\}$ be an orthonormal basis of $V_jM_kV_j$ formed by Hermitian matrices. Let $\{\delta_1, \ldots, \delta_r\}$ be an orthonormal basis of R formed by Hermitian matrices. Thus, $\bigcup_{j=1}^s \{\gamma_1^j, \ldots, \gamma_{r_j}^j\} \cup \{\delta_1, \ldots, \delta_r\}$ is an orthonormal basis of M_k formed by Hermitian matrices. Let $A_j = \gamma_1^j \otimes G_A(\gamma_1^j) + \ldots + \gamma_{r_j}^j \otimes G_A(\gamma_{r_j}^j)$. Thus, $A = \sum_{j=1}^s A_j + \delta_1 \otimes G_A(\delta_1) + \cdots + \delta_r \otimes G_A(\delta_r)$.

Now, since $F_A \circ G_A|_R \equiv 0$ and F_A, G_A are adjoint then $G_A|_R \equiv 0$ and $A = \sum_{j=1}^s A_j$.

Next, $(V_j \otimes W_j)A(V_j \otimes W_j) = A_j$, since $\gamma_l^j \in V_jM_kV_j$, $G_A(\gamma_l^j) \in W_jM_mW_j$, $V_iV_j = 0$ and $W_iW_j = 0$ for $i \neq j$. Therefore, $A_j \in P_{km}$.

Notice that, $F_{A_j} \circ G_{A_j}|_{V_jM_kV_j} = F_A \circ G_A|_{V_jM_kV_j}$ which is irreducible. Therefore $A_j \neq 0$ for $1 \leq j \leq s$. Next, $M_k = V_jM_kV_j \oplus (V_jM_kV_j)^\perp$ and $F_{A_j} \circ G_{A_j}((V_jM_kV_j)^\perp) = 0$. Therefore, by definition 2.9, $F_{A_j} \circ G_{A_j} : M_k \to M_k$ is completely reducible with $s = 1$. Finally, by item 2 of proposition 3.17, A_j is weakly irreducible. □

Definition 3.19. (***Separable Matrices***) *Let* $A \in M_k \otimes M_m$. *We say that* A *is separable if* $A = \sum_{i=1}^n C_i \otimes D_i$ *such that* $C_i \in M_k$ *and* $D_i \in M_m$ *are positive semidefinite Hermitian matrices for every* i.

Corollary 3.20. *Let* A *be the matrix of proposition 3.18. Then* A *is separable if and only if each* $(V_i \otimes W_i)A(V_i \otimes W_i)$ *is separable. Thus, for this type of* A *the Separability Problem is reduced to the weakly irreducible case.*

Proposition 3.21. *Let* $A \in M_k \otimes M_m \simeq M_{km}$, $A \in P_{km}$. *If* $F_A \circ G_A : M_k \to M_k$ *is completely reducible with all eigenvalues equal to* 1 *or* 0 *then there exists a unique Hermitian Schmidt decomposition of* A, $\sum_{i=1}^n \gamma_i \otimes \delta_i$, *such that* $\gamma_i \in P_k, \delta_i \in P_m$. *Therefore,* A *is separable.*

Proof. Suppose the multiplicity of the eigenvalue 1 is n. Since $F_A \circ G_A : M_k \to M_k$ is completely reducible then there are orthogonal projections $V_1, \dots, V_s$ such that $\Im(V_i) \perp \Im(V_j)$, $M_k = V_1 M_k V_1 \oplus \dots \oplus V_s M_k V_s \oplus R$, $F_A \circ G_A(V_i M_k V_i) \subset V_i M_k V_i$, $F_A \circ G_A|_{V_i M_k V_i}$ is irreducible, $F_A \circ G_A|_R \equiv 0$ and $s \geq n$, by definition 2.9 and proposition 2.13. Recall that each $F_A \circ G_A|_{V_i M_k V_i}$ has a unique largest eigenvalue, since $F_A \circ G_A|_{V_i M_k V_i}$ is irreducible by lemma 2.11. Moreover, the eigenvalues of $F_A \circ G_A|_{V_i M_k V_i}$ are 1 or 0. Thus, $s = n$ and for each $F_A \circ G_A|_{V_i M_k V_i}$ there exists a unique normalized eigenvector $\gamma_i \in P_k$ such that $F_A \circ G_A(\gamma_i) = \gamma_i$ and $\Im(\gamma_i) = \Im(V_i)$, by lemma 2.11.

Note that $\Im(\gamma_i) = \Im(V_i) \perp \Im(V_j) = \Im(\gamma_j)$, therefore $\gamma_1, \dots, \gamma_n$ are orthonormal. Complete this set to obtain an orthonormal basis $\{\gamma_1, \dots, \gamma_n, \gamma_{n+1}, \dots, \gamma_{k^2}\}$ of M_k formed by Hermitian matrices. Notice that $F_A \circ G_A(\gamma_j) = 0$, for $j > n$. Since F_A and G_A are adjoint maps, $G_A(\gamma_j) = 0$ for $j > n$.

Thus, $A = \gamma_1 \otimes G_A(\gamma_1) + \dots + \gamma_{k^2} \otimes G_A(\gamma_{k^2}) = \gamma_1 \otimes G_A(\gamma_1) + \dots + \gamma_n \otimes G_A(\gamma_n)$. Notice that $\langle G_A(\gamma_i), G_A(\gamma_j) \rangle = \langle \gamma_i, F_A \circ G_A(\gamma_j) \rangle = \langle \gamma_i, \gamma_j \rangle$, $1 \leq i, j \leq n$, therefore $G_A(\gamma_1), \dots, G_A(\gamma_n)$ are orthonormal too. Recall that G_A is a positive map then $G_A(\gamma_i) \in P_m$. Define $\delta_i = G_A(\gamma_i)$.

Finally, if $\sum_{i=1}^n \gamma_i' \otimes \delta_i'$ is a Hermitian Schmidt decomposition with $\gamma_i' \in P_k, \delta_i' \in P_m$ then $F_A \circ G_A(\gamma_i') = \gamma_i'$. Thus, $F_A \circ G_A(V_i' M_k V_i') \subset V_i' M_k V_i'$, $1 \leq i \leq n$, where V_i' is the orthogonal projection onto $\Im(\gamma_i')$, by corollary 2.4. Notice that each $F_A \circ G_A|_{V_i' M_k V_i'}$ has one eigenvalue equal to 1 and the others equal to 0. Thus, $F_A \circ G_A|_{V_i' M_k V_i'}$ is irreducible by lemma 2.11. Now, each V_i' must be equal to some V_j, by proposition 2.13.

Since each $F_A \circ G_A|_{V_j M_k V_j} = F_A \circ G_A|_{V_i' M_k V_i'}$ has only one eigenvalue equal to 1 then γ_i' is a multiple of γ_j, but both matrices are positive semidefinite Hermitian matrices and normalized then $\gamma_i' = \gamma_j$. Thus, each γ_i' is equal to some γ_j and this Hermitian Schmidt decomposition is unique. □

3.2.2 Extension of Mutually Unbiased Bases

In this subsection we obtain a new proof of the following theorem proved in [47]: If there is a set of k mutually unbiased bases of $\mathbb{C}^k$ then there exists another orthonormal basis which is mutually unbiased with these k bases. Our proof relies on proposition 3.22. We also proved that this additional basis is unique up to multiplication by complex numbers of norm 1.

Proposition 3.22. *Let $A \in M_k \otimes M_k \simeq M_{k^2}$, $A \in P_{k^2}$. If A is invariant under realignment and $F_A \circ G_A : M_k \to M_k$ has n eigenvalues equal to 1 and the others 0 then*

a) there exists an orthonormal set $\{v_1, \dots, v_n\} \subset \mathbb{C}^k$ such that $A = \sum_{i=1}^n v_i \overline{v_i}^t \otimes \overline{v_i} v_i^t$.

b) The orthonormal set of item a) is unique up to multiplication by complex numbers of norm one.

Proof. By theorem 3.13, $F_A \circ G_A : M_k \to M_k$ is completely reducible. By proposition 3.21, there exists a unique Hermitian Schmidt decomposition of A , $\sum_{i=1}^n \gamma_i \otimes \delta_i$, such that $\gamma_i \in P_k,\ \delta_i \in P_k$ for $1 \le i \le n$. Notice that $F_A(G_A(\gamma_i)) = \gamma_i$, for every i. So, by item $b)$ of lemma 3.11, $G_A(\gamma_i) = \gamma_i^t$. Thus, $\gamma_i^t = G_A(\gamma_i) = \delta_i$. Therefore, $A = \sum_{i=1}^n \gamma_i \otimes \gamma_i^t$ is the unique Hermitian Schmidt decomposition of A such that $\gamma_i \in P_k$ for $1 \le i \le n$.

Let V_i be the orthogonal projection onto $\Im(\gamma_i)$. Since $\{\gamma_1, \ldots, \gamma_n\}$ is an orthonormal set and each $\gamma_i \in P_k$ then $\Im(V_i) \perp \Im(V_j)$. Thus, $(V_i \otimes V_i^t)A(V_i \otimes V_i^t) = \gamma_i \otimes \gamma_i^t$.

Now, let $F(\gamma_i) = r_i$(see definition 3.3). By definition 3.5, $r_i\overline{r_i}^t = S(\gamma_i \otimes \overline{\gamma_i})$. Since γ_i is Hermitian then $\gamma_i \otimes \overline{\gamma_i} = \gamma_i \otimes \gamma_i^t$ and $r_i\overline{r_i}^t = S(\gamma_i \otimes \gamma_i^t) = S((V_i \otimes V_i^t)A(V_i \otimes V_i^t))$.

Next, by item 4 of lemma 3.10, $S((V_i \otimes V_i^t)A(V_i \otimes V_i^t)) = (V_i \otimes V_i^t)S(A)(V_i \otimes V_i^t)$. Since $S(A) = A$ then $r_i\overline{r_i}^t = (V_i \otimes V_i^t)A(V_i \otimes V_i^t) = \gamma_i \otimes \gamma_i^t$.

Therefore, $\gamma_i \otimes \gamma_i^t$ has rank 1 and γ_i has rank 1. Thus, $\gamma_i = v_i\overline{v_i}^t$ and $A = \sum_{i=1}^n v_i\overline{v_i}^t \otimes \overline{v_i}v_i^t$.

Since $tr(\gamma_i\gamma_j) = \delta_{ij}$ then $\{v_1, \ldots, v_n\}$ is an orthonormal set.

Finally, suppose $A = \sum_{j=1}^n w_j\overline{w_j}^t \otimes \overline{w_j}w_j^t$ for another orthonormal set $\{w_1, \ldots, w_n\}$. Since $\sum_{i=1}^n \gamma_i \otimes \gamma_i^t$ is unique (such that $\gamma_i \in P_k$ for $1 \le i \le n$) then for each p there is q such that $w_p\overline{w_p}^t = v_q\overline{v_q}^t$. Therefore $w_p = cv_q$ with $|c| = 1$. □

Definition 3.23. (***Mutually Unbiased Bases***) *Let $\{v_1, \ldots, v_k\}$ and $\{w_1, \ldots, w_k\}$ be orthonormal bases of $\mathbb{C}^k$. We say that they are mutually unbiased if $|\langle v_i, w_j\rangle|^2 = \frac{1}{k}$ for every i, j.*

Definition 3.24. *Let $\alpha = \{v_1, \ldots, v_k\}$ be an orthonormal basis of $\mathbb{C}^k$. Let us define $A_\alpha \in M_k \otimes M_k$ as $A_\alpha = \sum_{i=1}^k v_i\overline{v_i}^t \otimes \overline{v_i}v_i^t$. Notice that A_α is invariant under realignment.*

Lemma 3.25. *If α, β are orthonormal bases of $\mathbb{C}^k$ then they are mutually unbiased if and only if $A_\alpha A_\beta = A_\beta A_\alpha = \frac{1}{k}uu^t$ (Recall the definition of u in 3.3).*

Proof. Let $\alpha = \{v_1, \ldots, v_k\}$, $\beta = \{w_1, \ldots, w_k\}$ and $A_\alpha = \sum_{i=1}^k v_i\overline{v_i}^t \otimes \overline{v_i}v_i^t$, $A_\beta = \sum_{j=1}^k w_j\overline{w_j}^t \otimes \overline{w_j}w_j^t$. Notice that $A_\alpha A_\beta = \sum_{i,j=1}^k v_i\overline{w_j}^t \otimes \overline{v_i}w_j^t(\overline{v_i}^t w_j)(v_i^t\overline{w_j})$.

If α, β are mutually unbiased then for every i, j, we have $(\overline{v_i}^t w_j)(v_i^t\overline{w_j}) = |\langle v_i, w_j\rangle|^2 = \frac{1}{k}$. Therefore, $A_\alpha A_\beta = \sum_{i,j=1}^k \frac{1}{k} v_i\overline{w_j}^t \otimes \overline{v_i}w_j^t = \frac{1}{k}uu^t$, since $u = \sum_{i=1}^k v_i \otimes \overline{v_i} = \sum_{j=1}^k \overline{w_j} \otimes w_j$.

Now suppose that $A_\alpha A_\beta = \frac{1}{k}uu^t$. Therefore $\sum_{i,j=1}^k \lambda_{ij} v_i\overline{w_j}^t \otimes \overline{v_i}w_j^t = \frac{1}{k}uu^t$, where $\lambda_{ij} = |\langle v_i, w_j\rangle|^2$. Next, $\frac{1}{k}Id \otimes Id = S(\frac{1}{k}uu^t) = \sum_{i,j=1}^k \lambda_{ij} v_i\overline{v_i}^t \otimes \overline{w_j}w_j^t$. Notice that $\{v_i \otimes \overline{w_j}|\ 1 \le i, j \le k\}$ is an orthonormal basis of $\mathbb{C}^k \otimes \mathbb{C}^k$. Therefore λ_{ij} are the eigenvalues of $\frac{1}{k}Id \otimes Id$, thus $\lambda_{ij} = \frac{1}{k}$. □

Lemma 3.26. *Let $\alpha_1, \ldots, \alpha_{k+1}$ be orthonormal bases of $\mathbb{C}^k$. If they are pairwise mutually unbiased then $\sum_{i=1}^{k+1} A_{\alpha_i} = Id \otimes Id + uu^t \in M_k \otimes M_k$.*

Proof. Since $A_{\alpha_1}, \ldots, A_{\alpha_{k+1}}$ commute, by lemma 3.25, there is a common basis of $\mathbb{C}^k \otimes \mathbb{C}^k$ formed by orthonormal eigenvectors. Since $A_{\alpha_1}, \ldots, A_{\alpha_{k+1}}$ are orthogonal projections and their pairwise multiplications are equal to $\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}}$, by lemma 3.25, the intersection of their images is generated only by u. Notice that each A_{α_i} has rank k.

Thus, every A_{α_i} can be written as $\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{l=(i-1)(k-1)+1}^{i(k-1)} r_l\overline{r_l}^t$, where $r_1, \ldots, r_{k^2-1}, \frac{u}{\sqrt{k}}$ is a common orthonormal basis of eigenvectors.

Finally, $\sum_{i=1}^{k+1} A_{\alpha_i} = (k+1)\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{l=1}^{k^2-1} r_l\overline{r_l}^t = k\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{i=1}^{k^2-1} r_i\overline{r_i}^t = uu^t + Id\otimes Id.$ □

Remark 3.27. *Adapting the proof of the previous lemma, we can show the following: If $\alpha_1, \ldots, \alpha_{k+1}$ are pairwise mutually unbiased orthonormal bases of $\mathbb{R}^{2k}$ then $\sum_{i=1}^{k+1} A_{\alpha_i} = \frac{1}{2}(Id \otimes Id + T + uu^t)$, where T is the flip operator (see definition 3.3).*

Theorem 3.28. *If $\mathbb{C}^k$ contains k mutually unbiased bases then there exists another orthonormal basis which is mutually unbiased with these k bases. This additional one is unique up to multiplication by complex numbers of norm one.*

Proof. Let $\alpha_1, \ldots, \alpha_k$ be orthonormal bases of $\mathbb{C}^k$, which are pairwise mutually unbiased. Consider $B = Id \otimes Id + uu^t - (\sum_{i=1}^{k} A_{\alpha_i}) \in M_k \otimes M_k$. Recall $A_{\alpha_i} \in M_k \otimes M_k \simeq M_{k^2}$ from definition 3.24.

Since $A_{\alpha_1}, \ldots, A_{\alpha_k}$ are commuting orthogonal projections and their pairwise multiplications are equal to $\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}}$, by lemma 3.25, then every A_{α_i}, $1 \leq i \leq k$, can be written as $\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{l=(i-1)(k-1)+1}^{i(k-1)} r_l\overline{r_l}^t$, where $r_1, \ldots, r_{k^2-1}, \frac{u}{\sqrt{k}}$ is a common orthonormal basis of eigenvectors.

Therefore, $B = Id\otimes Id + uu^t - (\sum_{i=1}^{k} A_{\alpha_i}) = (k+1)\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{l=1}^{k^2-1} r_l\overline{r_l}^t - k\frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} - \sum_{l=1}^{k(k-1)} r_l\overline{r_l}^t = \frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}} + \sum_{l=k(k-1)+1}^{k^2-1} r_l\overline{r_l}^t$. Thus, B is an orthogonal projection with k eigenvalues equal to 1 and the others zero and $BA_{\alpha_i} = A_{\alpha_i}B = \frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}}$, $1 \leq i \leq k$.

In order to complete the proof, we must show that $B = A_{\alpha_{k+1}}$ for some orthonormal basis α_{k+1} of $\mathbb{C}^k$ (definition 3.24). Since $BA_{\alpha_i} = A_{\alpha_i}B = \frac{u}{\sqrt{k}}\frac{u^t}{\sqrt{k}}$, $1 \leq i \leq k$, then α_{k+1} is mutually unbiased with each α_i, $1 \leq i \leq k$, by lemma 3.25.

Next, $S(B) = S(Id\otimes Id + uu^t - \sum_{i=1}^{k} A_{\alpha_i}) = S(Id\otimes Id + uu^t) - \sum_{i=1}^{k} S(A_{\alpha_i}) = Id\otimes Id + uu^t - \sum_{i=1}^{k} A_{\alpha_i} = B$, by item $a)$ in example 3.9 and by definition 3.24. Thus B is invariant under realignment.

By item $b)$ of proposition 3.33, we know that B has a Hermitian Schmidt decomposition $\sum_{i=1}^{n}\lambda_i\gamma_i\otimes\gamma_i^t$ with $\lambda_i>0$. Therefore, $B=S(B)=\sum_{i=1}^{n}\lambda_i v_i\overline{v_i}^t$, where $v_i=F(\gamma_i)$. Since F is an isometry, by remark 3.4, $\sum_{i=1}^{n}\lambda_i v_i\overline{v_i}^t$ is a spectral decomposition of B and λ_i are the non-null eigenvalues of B. Then $n=k$ and $\lambda_i=1$, for $1\leq i\leq k$.

Thus, $\sum_{i=1}^{k}\gamma_i\otimes\gamma_i^t$ is a Hermitian Schmidt decomposition of B and $F_B\circ G_B:M_k\to M_k$ is $F_B\circ G_B(X)=\sum_{i=1}^{k}tr(\gamma_i X)\gamma_i$, where $\gamma_1,\dots,\gamma_k$ are orthonormal eigenvectors of $F_B\circ G_B$ associated to the eigenvalue 1. By proposition 3.22, there exists an orthonormal basis α_{k+1} of $\mathbb{C}^k$ such that $B=A_{\alpha_{k+1}}$ and this basis is unique up to multiplication by complex numbers of norm one. □

Remark 3.29. *Assume $\alpha_1,\dots,\alpha_k$ are pairwise mutually unbiased orthonormal bases of $\mathbb{R}^{2k}$ and define $B=\frac{1}{2}(Id\otimes Id+T+uu^t)-(\sum_{i=1}^{k}A_{\alpha_i})$. We can repeat the proof of the previous theorem in order to obtain $B=A_{\alpha_{k+1}}$, for some orthonormal basis α_{k+1} of $\mathbb{C}^{2k}$, since $Id\otimes Id+T+uu^t$ is invariant under realignment. The basis α_{k+1} is actually a basis of $\mathbb{R}^{2k}$ (up to multiplication by complex numbers of norm 1), because B is also invariant under partial transposition. Thus, if $\mathbb{R}^{2k}$ has k pairwise mutually unbiased bases then there exists another orthonormal basis which is mutually unbiased with these k bases.*

3.3 Remarks

3.3.1 Some Remarks on our Main Theorems

All the results within this subsetion were published in [13, 14].

Below we present a couple of easy examples showing that $F_A\circ G_A:M_k\to M_k$ is not completely reducible in general (lemmas 3.30, 3.31). The assumption that A is PPT or SPC or invariant under realignment is essential in order to obtain the complete reducibility of $F_A\circ G_A:M_k\to M_k$ (theorems 3.2, 3.12, 3.13).

Thus, these three types of matrices are connected and we can ask the following question: Is it possible that every SPC matrix or every matrix invariant under realignment is PPT? The answer is YES in $M_2\otimes M_2$ (see lemma 3.34) and NO in $M_k\otimes M_k$, $k>2$ (see examples 3.36).

Lemma 3.30. *Let $u\in\mathbb{C}^k\otimes\mathbb{C}^k$, $k\geq 2$, be the vector defined in 3.3 and $A=uu^t\in M_k\otimes M_k$. The linear transformation $F_A\circ G_A:M_k\to M_k$ is not completely reducible.*

Proof. By definition 3.3, $u=\sum_{i=1}^{k}e_i\otimes e_i$, where $\{e_1,\dots,e_k\}$ is the canonical basis of $\mathbb{C}^k$. Thus, $A=uu^t=\sum_{i,j=1}^{k}e_ie_j^t\otimes e_ie_j^t$ and $G_A(X)=F_A(X)=\sum_{i,j=1}^{k}e_ie_j^t tr(e_ie_j^tX)=X^t$.

Now, the identity map $Id=F_A\circ G_A:M_k\to M_k$ has null kernel and every matrix is an eigenvector. Thus, $Id:M_k\to M_k$ does not have the decomposition property (definition 2.10) and $F_A\circ G_A$ is not completely reducible by proposition 2.13. □

Lemma 3.31. *Let $k \geq 3$. Let $v_1, e_3 \in \mathbb{C}^k$ be such that $v_1^t = (\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0, \ldots, 0)$ and $e_3^t = (0,0,1,0,\ldots,0)$. Consider $v = v_1 \otimes \overline{v_1} + e_3 \otimes e_3 \in \mathbb{C}^k \otimes \mathbb{C}^k$. Let A be the positive semidefinite Hermitian matrix $A = v\overline{v}^t + S(\overline{v}v^t) \in M_k \otimes M_k$. The map $F_A \circ G_A : M_k \to M_k$ is not completely reducible.*

Proof. Let $\gamma_1 = v_1\overline{v_1}^t, \gamma_2 = \frac{e_3\overline{v_1}^t + v_1 e_3^t}{\sqrt{2}}, \gamma_3 = \frac{i(e_3\overline{v_1}^t - v_1 e_3^t)}{\sqrt{2}}, \gamma_4 = e_3 e_3^t$. Notice that $v\overline{v}^t = \sum_{i=1}^4 \gamma_i \otimes \gamma_i^t$.

Now, $S(\overline{v}v^t) = \overline{V} \otimes V$, where $V = F^{-1}(v) = v_1\overline{v_1}^t + e_3 e_3^t$, by item 2 of lemma 3.10. Thus, $A = \sum_{i=1}^4 \gamma_i \otimes \gamma_i^t + \overline{V} \otimes V$.

Since $0 = tr(\gamma_1\gamma_2) = tr(\gamma_1\gamma_3) = tr(\gamma_1\gamma_4) = tr(\gamma_1\overline{V})$ then $G_A(\gamma_1) = \gamma_1^t$ and $F_A(\gamma_1^t) = \gamma_1$. Therefore $F_A \circ G_A(\gamma_1) = \gamma_1$. Next, $0 = tr(\gamma_2\gamma_1) = tr(\gamma_2\gamma_3) = tr(\gamma_2\gamma_4) = tr(\gamma_2\overline{V})$. Thus, $G_A(\gamma_2) = \gamma_2^t$ and $F_A(\gamma_2^t) = \gamma_2$, thus $F_A \circ G_A(\gamma_2) = \gamma_2$.

Finally, notice that $\gamma_2 \in (Id - V_1)M_k V_1 \oplus V_1 M_k (Id - V_1) = R$, where V_1 is the orthogonal projection onto $\Im(\gamma_1)$. Therefore $F_A \circ G_A|_R \neq 0$. Thus, $F_A \circ G_A$ does not have the decomposition property(definition 2.10) and $F_A \circ G_A$ is not completely reducible by proposition 2.13. □

Lemma 3.32. *Let $A \in M_2 \otimes M_2 \simeq M_4$. Suppose A has a Hermitian Schmidt decomposition $\sum_{i=1}^m \lambda_i\gamma_i \otimes \gamma_i$ with $\lambda_i > 0$, for every i. If $\sum_{i=1}^m \lambda_i \det(\gamma_i) \geq 0$ then A is separable.*

Proof. If $\det(\gamma_i) = 0$, for every i, then γ_i has rank 1 and $\gamma_i \otimes \gamma_i \in P_4$. Therefore each $\gamma_i \otimes \gamma_i$ is separable and $\sum_{i=1}^m \lambda_i\gamma_i \otimes \gamma_i$ is separable too, since $\lambda_i > 0$ for every i .

If there is i such that $\det(\gamma_i) \neq 0$ then we must have at least one i such that $\det(\gamma_i) > 0$, because $\sum_{i=1}^m \lambda_i \det(\gamma_i) \geq 0$ and $\lambda_i > 0$ for every i . Let us assume $\det(\gamma_1) > 0$. It means that the eigenvalues of γ_1 are both positive or negative. Therefore $\gamma_1 \otimes \gamma_1$ is positive definite. Thus, we can assume that γ_1 is positive definite. Let $\gamma_1 = N^2$ for some invertible $N \in P_2$. Consider $B = (N^{-1} \otimes N^{-1})A(N^{-1} \otimes N^{-1}) = \lambda_1 Id \otimes Id + \sum_{i=2}^m \lambda_i N^{-1}\gamma_i N^{-1} \otimes N^{-1}\gamma_i N^{-1}$.

Now, since $tr(\gamma_i\gamma_1) = 0$, for every $i > 1$, then γ_i has a positive and a negative eigenvalue. Thus, $\det(\gamma_i) < 0$, for every $i > 1$.

Next, since $\lambda_1 + \sum_{i=2}^m \lambda_i \frac{\det(\gamma_i)}{\det(\gamma_1)} \geq 0$ and $\det(\gamma_i) < 0$, for every $i > 1$, then $\lambda_1 \geq \sum_{i=2}^m \lambda_i \frac{|\det(\gamma_i)|}{\det(\gamma_1)}$ and $B = (\lambda_1 - \sum_{i=2}^m \lambda_i \frac{|\det(\gamma_i)|}{\det(\gamma_1)})Id \otimes Id + \sum_{i=2}^m \lambda_i(\frac{|\det(\gamma_i)|}{\det(\gamma_1)} Id \otimes Id + N^{-1}\gamma_i N^{-1} \otimes N^{-1}\gamma_i N^{-1})$.

Now, the smallest eigenvalue of $N^{-1}\gamma_i N^{-1} \otimes N^{-1}\gamma_i N^{-1}$ is the product of the two distinct eigenvalues of $N^{-1}\gamma_i N^{-1}$ (since they have opposite signs), which is equal to $\det(N^{-1}\gamma_i N^{-1}) = \frac{\det(\gamma_i)}{\det(\gamma_1)}$. Therefore, $\frac{|\det(\gamma_i)|}{\det(\gamma_1)} Id \otimes Id + N^{-1}\gamma_i N^{-1} \otimes N^{-1}\gamma_i N^{-1} \in P_4$ and has tensor rank 2. Thus, by theorem 3.44, $\frac{|\det(\gamma_i)|}{\det(\gamma_1)} Id \otimes Id + N^{-1}\gamma_i N^{-1} \otimes N^{-1}\gamma_i N^{-1}$ is separable, for every i, and B is separable. Therefore A is separable. □

Proposition 3.33. *Let $A \in M_k \otimes M_k$ be a Hermitian matrix.*

a) $S(A^{t_2}) \in P_{k^2}$ *if and only if there is a Hermitian Schmidt decomposition of* A, $\sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i$, *such that* $\lambda_i > 0$ *for every* i.

b) $S(A) \in P_{k^2}$ *if and only if there is a Hermitian Schmidt decomposition of* A, $\sum_{i=1}^{n} \lambda_i \gamma_i \otimes \gamma_i^t$, *such that* $\lambda_i > 0$ *for every* i.

Proof. Since A is Hermitian then F_A and G_A are adjoint linear transformations. Therefore, $F_A \circ G_A : M_k \to M_k$ is a self-adjoint linear transformation with non negative eigenvalues. Moreover, the set of Hermitian matrices is left invariant by $F_A \circ G_A : M_k \to M_k$. Thus, there exists an orthonormal basis of Hermitian matrices of M_k, $\{\gamma_1, \dots, \gamma_{k^2}\}$, formed by eigenvectors of $F_A \circ G_A : M_k \to M_k$. Let $\{\lambda_1^2, \dots, \lambda_{k^2}^2\}$ be the corresponding eigenvalues such that $\lambda_i > 0$, for $i \le n$, and $\lambda_i = 0$, for $i > n$. Since $\{\gamma_1, \dots, \gamma_{k^2}\}$ is an orthonormal basis of Hermitian matrices of M_k then $A = \sum_{i=1}^{k^2} \gamma_i \otimes G_A(\gamma_i)$. Now, use lemma 3.11 to obtain the required Hermitian Schmidt decompositions for each item. The converse part of each item follows from definition 3.5. □

Lemma 3.34. *Let $A \in M_2 \otimes M_2 \simeq M_4$ and $A \in P_4$. If A is SPC then A is separable. If $A = S(A)$ or $A^t = S(A)$ then A is separable and therefore PPT.*

Proof. Let $B \in M_2 \otimes M_2 \simeq M_4$. Suppose B has a Hermitian Schmidt decomposition $\sum_{i=1}^{m} \lambda_i \gamma_i \otimes \gamma_i$ with $\lambda_i > 0$, for every i. Thus, the subspaces of symmetric and anti-symmetric tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2$ are left invariant by B.

Since the subspace of anti-symmetric tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is generated by $w = e_1 \otimes e_2 - e_2 \otimes e_1$, where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$, then $Bw = \lambda w$. Notice that $(\gamma_i \otimes \gamma_i)w = \det(\gamma_i)w$. Thus, $\lambda = \sum_{i=1}^{m} \lambda_i \det(\gamma_i)$.

If A is SPC then A has a Hermitian Schmidt decomposition $\sum_{i=1}^{m} \lambda_i \gamma_i \otimes \gamma_i$ with $\lambda_i > 0$, for every i, by item a) of proposition 3.33. Moreover, A is positive semidefinite. Thus, $Aw = \lambda w$ and $\lambda \ge 0$. By lemma 3.32, A is separable.

Now, if $S(A) = A$ or A^t then $S(A)$ is a positive semidefinite Hermitian matrix. Therefore, A has a Hermitian Schmidt decomposition $\sum_{i=1}^{n} \alpha_i \gamma_i \otimes \gamma_i^t$ with $\alpha_i > 0$, for every i, by item b) of proposition 3.33.

Thus, A^{t_2} satisfies the same conditions of B and $A^{t_2} w = \lambda w$. For these two cases, let us prove that $\lambda = 0$. By lemma 3.32, A^{t_2} is separable and A is separable.

First, suppose $A = S(A)$. Thus, $A^{t_2} = S(AT)T = S(A)^{t_2} T = A^{t_2} T$, by items 5 and 7 in lemma 3.10. Since T is the flip operator and w is an anti-symmetric tensor then $Tw = -w$. Therefore $A^{t_2} w = A^{t_2} T w = -A^{t_2} w$ and $A^{t_2} w = 0$. Therefore $\lambda = 0$.

Second, suppose $A^t = S(A)$. Thus, $A^{t_2} = S(AT)T = S(A)^{t_2} T = (A^t)^{t_2} T = \overline{(A^{t_2})^t T}$, by items 5 and 7 in lemma 3.10. Since A^{t_2} is hermitian, $(A^{t_2})^t = \overline{A^{t_2}}$ and $A^{t_2} = \overline{A^{t_2}} T$. Since $\overline{w} = w$ and $\lambda \in \mathbb{R}$ then $\overline{A^{t_2}} w = \overline{A^{t_2} w} = \lambda w$. Thus, $\lambda w = A^{t_2} w = \overline{A^{t_2}} T w = -\overline{A^{t_2}} w = -\lambda w$ and $\lambda = 0$. □

Remark 3.35. *In the proof of the previous theorem We saw that, if $A \in P_4$ and $S(A)$ is equal to A or A^t, then A^{t_2} has non null kernel. Remember that if $A \in P_4$ is not PPT then A^{t_2} has full rank and has only one negative eigenvalue (see proposition 1 in [3]). Thus, $A \in M_2 \otimes M_2$ must be PPT and separable by Horodecki's theorem (see [29]). However, this argument does not work for SPC matrices in $M_2 \otimes M_2$.*

Examples 3.36. *Counterexamples for lemma 3.34 in $M_k \otimes M_k, k \geq 3$:*

(1) *The matrix $C = |m_q| Id \otimes Id + D \otimes D + (iA) \otimes (iA) \in M_3 \otimes M_3$ of proposition 25 in [13] is SPC, but it is not PPT.*

(2) *As discussed in example 3.9, $Id \otimes Id + uu^t - T \in M_k \otimes M_k$ is invariant under realignment. Since its partial tranposition is $Id \otimes Id + T - uu^t$ and $(Id \otimes Id + T - uu^t)u = (2-k)u$ then it is not PPT for $k \geq 3$. Notice also that $S((Id \otimes Id + uu^t - T)^{t_2}) = S(Id \otimes Id + T - uu^t) = uu^t + T - Id \otimes Id$ and any anti-symmetric vector of $\mathbb{C}^k \otimes \mathbb{C}^k$ is an eigenvector of $uu^t + T - Id \otimes Id$ associated to -2. Thus, $S((Id \otimes Id + uu^t - T)^{t_2})$ is not positive semidefinite and $Id \otimes Id + uu^t - T$ is not SPC, by definiton 3.6.*

(3) *Let $A = v\overline{v}^t + S(\overline{v}v^t) \in M_k \otimes M_k$, $k \geq 3$, as in lemma 3.31. Notice that, by properties 2 and 3 in lemma 3.10 and since $V = F^{-1}(v)$ is Hermitian (definition 3.3), $S(A) = V \otimes \overline{V} + \overline{v}v^t = (\overline{V} \otimes V + v\overline{v}^t)^t = (S(\overline{v}v^t) + v\overline{v}^t)^t = A^t$. Now, by lemma 3.31, $F_A \circ G_A : M_k \to M_k$ is not completely reducible then A is not PPT or SPC, by theorems 3.2, 3.12.*

3.3.2 A Remark on the Application to the Separability Problem

All the results within this subsetion were published in [12].

A very useful tool to study separability in $M_k \otimes M_m$ is the so-called filter normal form (section IV.D of [23]): If $A \in M_k \otimes M_m \simeq M_{km}$ is a positive definite Hermitian matrix then there exist invertible matrices $R \in M_k$ and $S \in M_m$ such that $(R \otimes S)A(R^* \otimes S^*)$ has the following Hermitian Schmidt decomposition: $\sum_{i=1}^n \lambda_i \gamma_i \otimes \delta_i$, where $\gamma_1 = \frac{1}{\sqrt{k}} Id$ and $\delta_1 = \frac{1}{\sqrt{m}} Id$. The known proof of the existence of this canonical form depends heavily on the positive definiteness of A ([34, 46]).

Besides the positive definite case, there is another case where this filter normal form can be used. Assume A is PPT. By theorem 3.18, $A = \sum_{i=1}^s (V_i \otimes W_i)A(V_i \otimes W_i)$, where $V_iV_j = 0$ and $W_iW_j = 0$, for $i \neq j$. Notice that if $s > 1$ then $A(V_1 \otimes W_2) = 0$. Hence, A is not positive definite and we can not garantee the existence of the filter normal form for A.

Now, if $\text{rank}(V_i) = k_i$ and $\text{rank}(W_i) = m_i$ then we can embed $(V_i \otimes W_i)A(V_i \otimes W_i)$ in $M_{k_i} \otimes M_{m_i}$. If $(V_i \otimes W_i)A(V_i \otimes W_i)$ has rank $k_i m_i$ then its embedding in $M_{k_i} \otimes M_{m_i}$ is positive definite and we can obtain its filter normal form. So in this particular case, where rank

$(V_i \otimes W_i)A(V_i \otimes W_i)$ is $k_i m_i$ for every i, the filter normal form can still be used to study the separability of $(V_i \otimes W_i)A(V_i \otimes W_i)$. Recall that A is separable if and only if $(V_i \otimes W_i)A(V_i \otimes W_i)$ is separable for every i (corollary 3.20).

In theorem 3.18 it was shown that each $(V_i \otimes W_i)A(V_i \otimes W_i)$ is weakly irreducible. Thus, if we could prove the existence of the filter normal form for weakly irreducible PPT matrices then this canonical form would be useful to study separability of any PPT matrix. Recall that in order to be separable a matrix must be PPT.

We can obtain some inequalities for weakly irreducible PPT matrices that imply separability, even without the filter normal form. These inequalities are based on the fact that every positive semidefinite Hermitian matrix with tensor rank 2 is separable (see theorem 3.44). We want to emphasize that the filter normal form would also be useful to sharpen these inequalities (see example 3.38).

If $A \in M_k \otimes M_m$ is PPT or SPC or invariant under realignment then $F_A \circ G_A : M_k \to M_k$ is completely reducible. By corollary 3.20, the Separability Problem is reduced to the weakly irreducible case.

Let $A \in M_k \otimes M_m \simeq M_{km}$ be a weakly irreducible PPT or SPC or invariant under realignment matrix. By proposition 3.17, A has the following Hermitian Schmidt decomposition: $\sum_{i=1}^{n} \lambda_i \gamma_i \otimes \delta_i$, such that $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n > 0$ and $\Im(\gamma_i) \subset \Im(\gamma_1)$, $\Im(\delta_i) \subset \Im(\delta_1)$, for $1 \leq i \leq n$.

Proposition 3.37. *Let $A \in M_k \otimes M_m \simeq M_{km}$ and $A \in P_{km}$. Let A be a weakly irreducible PPT or SPC or matrix invariant under realignment. Let $\sum_{i=1}^{n} \lambda_i \gamma_i \otimes \delta_i$ be a Hermitian Schmidt decomposition of A such that $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n > 0$ and $\Im(\gamma_i) \subset \Im(\gamma_1)$, $\Im(\delta_i) \subset \Im(\delta_1)$, for $1 \leq i \leq n$, by proposition 3.17. Let μ be the smallest positive eigenvalue of $\gamma_1 \otimes \delta_1$.*

(1) *If $\frac{\lambda_1 \mu}{\lambda_2 + \ldots + \lambda_n} \geq 1$ then A is separable.*

(2) *If A is SPC or invariant under realignment and $\frac{\lambda_1 \mu}{\lambda_2 + \ldots + \lambda_n} \geq \frac{1}{2}$ then A is separable.*

Proof. (1) Notice that $F_A \circ G_A(\gamma_i) = \lambda_i^2 \gamma_i$ for $1 \leq i \leq n$. Thus, the eigenvalues of $F_A \circ G_A : M_k \to M_k$ are λ_i^2, $1 \leq i \leq n$, and possibly 0. Hence, the largest eigenvalue of $F_A \circ G_A$ is λ_1^2. Since $F_A \circ G_A : M_k \to M_k$ is a self-adjoint positive map then associated to this eigenvalue there exists $\gamma \in P_k$ such that $F_A \circ G_A(\gamma) = \lambda_1^2 \gamma$, by theorem 2.8 (or [21, Proposition 2.5]).

Since $\lambda_1^2 > \lambda_i^2 > 0$, for every $1 \leq i \leq n$, then the multiplicity of λ_1^2 is 1. Therefore, $\gamma_1 = \lambda \gamma$, for some $\lambda \in \mathbb{R}$, since γ_1 and γ are Hermitian. Hence, $\delta_1 = G_A(\gamma_1) = \lambda G_A(\gamma)$ and $\gamma_1 \otimes \delta_1 = \lambda^2 \gamma \otimes G_A(\gamma)$ is a positive semidefinite Hermitian matrix, since $\gamma \in P_k$ and $G_A : M_k \to M_m$ is a positive map.

Notice that the smallest positive eigenvalue of $\frac{1}{\mu} \gamma_1 \otimes \delta_1$ is 1 and, since $tr(\gamma_i^2) = tr(\delta_i^2) = 1$, the smallest eigenvalue of $\gamma_i \otimes \delta_i$ is greater or equal to -1. By hypothesis, $\Im(\gamma_i \otimes \delta_i) \subset \Im(\gamma_1 \otimes \delta_1)$

then $\frac{1}{\mu}\gamma_1 \otimes \delta_1 + \gamma_i \otimes \delta_i$ is positive semidefinite and, by theorem 3.44, it is separable. Now if $\lambda_1\mu \geq \lambda_2 + ... + \lambda_n$ then $A = (\lambda_1\mu - \sum_{i=2}^n \lambda_i)(\frac{1}{\mu}\gamma_1 \otimes \delta_1) + \sum_{i=2}^n \lambda_i(\frac{1}{\mu}\gamma_1 \otimes \delta_1 + \gamma_i \otimes \delta_i)$. Notice that all the matrices inside parentheses are separable.

(2) Since $F_A \circ G_A(\gamma_i) = \lambda_i^2\gamma_i$, for $1 \leq i \leq n$, then by lemma 3.11, $\delta_i = G_A(\gamma_i) = \gamma_i$, if A is SPC, and $\delta_i = G_A(\gamma_i) = \gamma_i^t$, if A is invariant under realignment. In any case, since $tr(\gamma_i^2) = 1$ then the smallest eigenvalue of $\gamma_i \otimes \delta_i$ ($\delta_i = \gamma_i$ or γ_i^t) is greater or equal to $-\frac{1}{2}$.

Finally, repeat the argument of item (1) and write $A = (\lambda_1\mu - \sum_{i=2}^n \frac{\lambda_i}{2})(\frac{1}{\mu}\gamma_1 \otimes \delta_1) + \sum_{i=2}^n \lambda_i(\frac{1}{2\mu}\gamma_1 \otimes \delta_1 + \gamma_i \otimes \delta_i)$. Note that if $\lambda_1\mu \geq \frac{1}{2}(\lambda_2 + ... + \lambda_n)$ then all the matrices inside parentheses are separable by theorem 3.44. □

Example 3.38. *Let $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ be the normalized Pauli's basis of M_2, where $\gamma_1 = \frac{1}{\sqrt{2}}Id$. It is known that a necessary and sufficient condition for the separability of $\sum_{i=1}^4 \lambda_i\gamma_i \otimes \gamma_i$, $\lambda_i \geq 0$, is the inequality of item* (2) (*see* [34]).

Tensor Rank 2 Implies Separability in $M_{k_1} \otimes \ldots \otimes M_{k_n}$

Here, we show that tensor rank 2 in $M_{k_1} \otimes \ldots \otimes M_{k_n}$ implies separability of positive semidefinite Hermitian matrices in $M_{k_1} \otimes \ldots \otimes M_{k_n} = M_{k_1} \otimes (M_{k_2} \otimes \ldots \otimes M_{k_n}) \simeq M_{k_1 \ldots k_n}$.

First, let us recall some definitions and some well known results regarding tensor rank.

Definition 3.39. *Let $V_1 \otimes \ldots \otimes V_n$ be the tensor product space of the complex vector spaces V_i $(1 \leq i \leq n)$ over the complex field. Let $r \in V_1 \otimes \ldots \otimes V_n$. The tensor rank of r is 1, if $r = v_1 \otimes \ldots \otimes v_n$ and $r \neq 0$. The tensor rank of r is the minimal number of tensors with tensor rank 1 that can be added to form r.*

Theorem 3.40. (*Marcus-Moyls* [37]) *Let V_1 and V_2 be complex vector spaces and let $v_i, r_j \in V_1$ and $w_i, s_j \in V_2$, for $1 \leq i \leq n$ and $1 \leq j \leq k$. Let $\sum_{i=1}^n v_i \otimes w_i = \sum_{j=1}^k r_j \otimes s_j \in V_1 \otimes V_2$.*
(i) If $\{v_1, ..., v_n\}$ is a linear independent set then $\text{span}\{w_1, ..., w_n\} \subset \text{span}\{s_1, ..., s_k\}$.
(ii) If $\{w_1, ..., w_n\}$ is a linear independent set then $\text{span}\{v_1, ..., v_n\} \subset \text{span}\{r_1, ..., r_k\}$.

Corollary 3.41. *Let $\sum_{i=1}^n v_i \otimes w_i = \sum_{j=1}^k r_j \otimes s_j$. If $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_n\}$ are linear independent sets then $k \geq n$. So the tensor rank of $\sum_{i=1}^n v_i \otimes w_i$ is n.*

Recall that M_k stands for the set of complex matrices of order k and P_k for the subset of positive semidefinite Hermitian matrices of M_k. We are also identifying $M_{k_1} \otimes \ldots \otimes M_{k_n}$ with $M_{k_1 \ldots k_n}$ via Kronecker product.

Lemma 3.42. *Let $A \in M_k \otimes M_m$, $A \in P_{km}$ and* tensor rank$(A) = n$. *We can write $A = \sum_{i=1}^n \gamma_i \otimes \delta_i$, where $\gamma_i \in M_k, \delta_i \in M_m$ are Hermitian matrices such that $\Im(\gamma_i) \subset \Im(\gamma_1)$ and $\Im(\delta_i) \subset \Im(\delta_1)$, for every i, and $\gamma_1 \in P_k$, $\delta_1 \in P_m$.*

Proof. Since $A \in M_k \otimes M_m$ is Hermitian with tensor rank n, we can write $A = \sum_{i=1}^n A_i \otimes B_i$, where $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$ are linear independent sets of Hermitian matrices (see minimal Hermitian decomposition in [12] or corollary 3.41). Let $\gamma_1 = \frac{\sum_{i=1}^n tr(B_i)A_i}{tr(A)}$ and $\delta_1 = \sum_{i=1}^n tr(A_i)B_i$. Since $A \in P_{km}$ then $F_A : M_m \to M_k$ and $G_A : M_k \to M_m$ are positive maps then $F_A(\frac{Id}{trA}) = \gamma_1 \in P_k$ and $G_A(Id) = \delta_1 \in P_m$.

First, let us prove that $B = A - \gamma_1 \otimes \delta_1$ has tensor rank $n-1$.

Notice that B is Hermitian and let $B = \sum_{j=1}^l \gamma_j' \otimes \delta_j'$, where $\{\gamma_1', \dots, \gamma_l'\}$ and $\{\delta_1', \dots, \delta_l'\}$ are linear independent sets of Hermitian matrices. Notice that $l \geq n-1$, otherwise A would have tensor rank smaller than n. Now, since $\{\gamma_1', \dots, \gamma_l'\}$ is a linear independent set, by theorem 3.40, we have $\text{span}\{\delta_1', \dots, \delta_l'\} \subset \text{span}\{B_1, \dots, B_n, \delta_1\} = \text{span}\{B_1, \dots, B_n\}$. Thus, $l \leq n$.

Next, consider the trace inner product. Let δ' be the projection of the Id inside the $\text{span}\{B_1, \dots, B_n\}$. Thus, $tr(B_i\delta') = tr(B_i Id) = tr(B_i)$ for every $1 \leq i \leq n$. Notice that $\delta' \neq 0$, otherwise $tr(B_i) = 0$ for every $1 \leq i \leq n$ and $tr(A) = 0$.

Now, $0 = \sum_{i=1}^n A_i tr(B_i) - \gamma_1 tr(\delta_1) = \sum_{i=1}^n A_i tr(B_i\delta') - \gamma_1 tr(\delta_1\delta') = F_B(\delta') = \sum_{j=1}^l \gamma_j' tr(\delta_j'\delta')$. Since $\{\gamma_1', \dots, \gamma_l'\}$ is a linear independent set then we get $tr(\delta_j'\delta') = 0$ for $1 \leq j \leq n$. Since $\delta' \in \text{span}\{B_1, \dots, B_n\}$ and δ' is orthogonal to $\text{span}\{\delta_1', \dots, \delta_l'\} \subset \text{span}\{B_1, \dots, B_n\}$ then $l \leq n-1$. Hence, $l = n-1$ and B has tensor rank $n-1$, by corollary 3.41.

Thus, let us write $A = \sum_{i=1}^n \gamma_i \otimes \delta_i$, such that $\{\gamma_1, \dots, \gamma_n\}$ and $\{\delta_1, \dots, \delta_n\}$ are linear independent sets of Hermitian matrices and γ_1, δ_1 as defined above.

Since $\{\gamma_1, \dots, \gamma_n\}$ is a linear independent set of Hermitian matrices then there exist Hermitian matrices $C_1, \dots, C_n$ such that $tr(\gamma_i C_j) = \delta_{ij}$. For each C_i there exists $\epsilon_i > 0$ such that $Id \pm \epsilon_i C_i \in P_k$. Since $A \in P_{km}$ then $G_A : M_k \to M_m$ is a positive map and $G_A(Id \pm \epsilon_i C_i) = \delta_1 \pm \epsilon_i \delta_i \in P_m$. By lemma 2.1, $\Im(\delta_i) \subset \Im(\delta_1)$, for every i. Analogously, we obtain $\Im(\gamma_i) \subset \Im(\gamma_1)$, for every i. □

Remark 3.43. *Notice that γ_1 and δ_1 in the proof of lemma 3.42 are multiples of the so-called marginal states of A (see [30]).*

Theorem 3.44. *Let $A \in M_{k_1} \otimes \dots \otimes M_{k_n}$ and $A \in P_{k_1 \dots k_n}$. If A has tensor rank smaller or equal to 2 then A is separable.*

Proof. Let $A = A_1 \otimes \dots \otimes A_n + B_1 \otimes \dots \otimes B_n$. Thus, A as an element of $M_{k_1} \otimes M_{k_2 \dots k_n}$ has tensor rank smaller or equal to 2. If A has tensor rank 1 in $M_{k_1} \otimes M_{k_2 \dots k_n}$ then $A = \gamma_1 \otimes \delta_1$, where $\gamma_1 \in P_{k_1}, \delta_1 \in P_{k_2 \dots k_n}$. By theorem 3.40, $\delta_1 \in \text{span}\{A_2 \otimes \dots \otimes A_n, B_2 \otimes \dots \otimes B_n\}$. So δ_1 has tensor rank smaller or equal to 2 in $M_{k_2} \otimes \dots \otimes M_{k_n}$ and by induction on n, δ_1 is separable in $M_{k_2} \otimes \dots \otimes M_{k_n}$. Therefore, A is separable in $M_{k_1} \otimes \dots \otimes M_{k_n}$.

Now, assume A has tensor rank 2 in $M_{k_1} \otimes M_{k_2 \dots k_n}$. By lemma 3.42, $A = \gamma_1 \otimes \delta_1 + \gamma_2 \otimes \delta_2$ such that $\gamma_1 \in P_{k_1}, \delta_1 \in P_{k_2 \dots k_n}$ and $\gamma_2 \in M_{k_1}, \delta_2 \in M_{k_2 \dots k_n}$ are Hermitian matrices such that $\Im(\gamma_2) \subset \Im(\gamma_1), \Im(\delta_2) \subset \Im(\delta_1)$.

Choose $0 \neq \lambda \in \mathbb{R}$ such that $\gamma_1 - \lambda\gamma_2 \in P_k$ and $0 \neq v \in \ker(\gamma_1 - \lambda\gamma_2) \cap \Im(\gamma_1)$, by lemma 2.2. Notice that $A = (\gamma_1 - \lambda\gamma_2) \otimes \delta_1 + \gamma_2 \otimes (\delta_2 + \lambda\delta_1)$.

Since $A \in P_{k_1 \dots k_n}$ then $G_A : M_{k_1} \to M_{k_2 \dots k_m}$ is a positive map. Since $tr((\gamma_1 - \lambda\gamma_2)v\overline{v}^t) = 0$ then $G_A(v\overline{v}^t) = tr(\gamma_2 v\overline{v}^t)(\delta_2 + \lambda\delta_1) \in P_m$. Notice that $0 \neq \frac{tr(\gamma_1 v\overline{v}^t)}{\lambda} = tr(\gamma_2 v\overline{v}^t)$, since $v \in \Im(\gamma_1)$ and $\gamma_1 \in P_k$.

Now, let $\beta_1 = \delta_1$, $\beta_2 = tr(\gamma_2 v\overline{v}^t)(\delta_2 + \lambda\delta_1)$, $\alpha_1 = \gamma_1 - \lambda\gamma_2$, $\alpha_2 = \frac{\gamma_2}{tr(\gamma_2 v\overline{v}^t)}$. Notice that $\alpha_1, \beta_2, \beta_1$ are positive semidefinite Hermitian matrices such that $\Im(\beta_2) \subset \Im(\beta_1)$ and $A = \alpha_1 \otimes \beta_1 + \alpha_2 \otimes \beta_2$.

Next choose $0 \neq \epsilon \in \mathbb{R}$ such that $\beta_1 - \epsilon\beta_2$ is positive semidefinite and $0 \neq w \in \ker(\beta_1 - \epsilon\beta_2) \cap \Im(\beta_1)$, by lemma 2.2. Notice that $A = \alpha_1 \otimes (\beta_1 - \epsilon\beta_2) + (\alpha_2 + \epsilon\alpha_1) \otimes \beta_2$.

Since $A \in P_{k_1 \dots k_n}$ then $F_A : M_{k_2 \dots k_m} \to M_{k_1}$ is a positive map. Since $tr((\beta_1 - \epsilon\beta_2)w\overline{w}^t) = 0$ then $F_A(w\overline{w}^t) = tr(\beta_2 w\overline{w}^t)(\alpha_2 + \epsilon\alpha_1) \in P_k$. Notice also that $0 \neq \frac{tr(\beta_1 w\overline{w}^t)}{\epsilon} = tr(\beta_2 w\overline{w}^t)$, since $\beta_1 \in P_m$ and $w \in \Im(\beta_1)$.

Since $tr(\beta_2 w\overline{w}^t) > 0$, by the positive semidefiniteness of β_2, we obtain the following minimal separable decomposition: $A = \alpha_1 \otimes (\beta_1 - \epsilon\beta_2) + tr(\beta_2 w\overline{w}^t)(\alpha_2 + \epsilon\alpha_1) \otimes \frac{\beta_2}{tr(\beta_2 w\overline{w}^t)}$.

Now since α_1 and $tr(\beta_2 w\overline{w}^t)(\alpha_2 + \epsilon\alpha_1)$ are linear independent, because A has tensor rank 2 in $M_{k_1} \otimes M_{k_2 \dots k_n}$, then $(\beta_1 - \epsilon\beta_2)$ and $\frac{\beta_2}{tr(\beta_2 w\overline{w}^t)}$ belong to the span$\{A_2 \otimes \dots \otimes A_n, B_2 \otimes \dots \otimes B_n\}$, by theorem 3.40. Thus $(\beta_1 - \epsilon\beta_2)$ and $\frac{\beta_2}{tr(\beta_2 w\overline{w}^t)}$ are positive semidefinite Hermitian matrices with tensor rank smaller or equal to 2 in $M_{k_2} \otimes \dots \otimes M_{k_n}$ and, by induction on n, $(\beta_1 - \epsilon\beta_2)$ and $\frac{\beta_2}{tr(\beta_2 w\overline{w}^t)}$ are separable in $M_{k_2} \otimes \dots \otimes M_{k_n}$. Therefore, A is separable in $M_{k_1} \otimes \dots \otimes M_{k_n}$. □

Remark 3.45. *There is a generalization of this result in $M_2 \otimes M_m$. Every $A \in P_{2m} \subset M_2 \otimes M_m$ with tensor rank 3 is separable (See theorem 19 in* [13]*). However, this is not true in $M_3 \otimes M_3$ (See proposition 25 in* [13]*).*

Chapter 4

An application of Borsuk-Ulam Theorem

All the results within this chapter were published in [11].

Given a topological space D, let us denote by $\widehat{C}(D)$ the subset of the vector space $C(D)$ of all real-valued continuous functions on D formed by the functions that attain the maximum exactly once in D. The set $\widehat{C}(D)$ fails to be a vector space for many reasons, for example the zero function does not belong to $\widehat{C}(D)$. Gurariy and Quarta asked the following question: Is it possible to find a linear subspace V of $C(D)$ such that $V \subset \widehat{C}(D) \cup \{0\}$? If so, how big can be the dimension of V?

The main results obtained by Gurariy and Quarta in this direction are the following:

(A) There is a 2-dimensional linear subspace of $C[a,b)$ contained in $\widehat{C}[a,b) \cup \{0\}$.

(B) There is a 2-dimensional linear subspace of $C(\mathbb{R})$ contained in $\widehat{C}(\mathbb{R}) \cup \{0\}$.

(C) There is no 2-dimensional linear subspace of $C[a,b]$ contained in $\widehat{C}[a,b] \cup \{0\}$.

The purpose of this chapter is to obtain far-reaching generalizations of the aforementioned results of Gurariy and Quarta. We investigate the existence of n-dimensional subspaces – instead of 2-dimensional subspaces – formed by functions that attain the maximum exactly once (question posed in [9, Problem 2.9]). While Gurariy and Quarta [27] used typical analytical techniques, the manifested nature of the problem led us to apply topological techniques, for example the Borsuk-Ulam theorem.

Our main result is theorem 4.1: If D is a compact subset of $\mathbb{R}^m$ and V is a linear subspace of $C(D)$ such that $V \subset \widehat{C}(D) \cup \{0\}$ then $\dim(V) \leq m$. Thus, we recover theorem (C) of Gurariy and Quarta. Moreover, this inequality is sharp.

Gurariy and Quarta also asked if there exists a 3-dimensional linear subspace of $C[a,b)$ contained in $\widehat{C}[a,b) \cup \{0\}$. We can not prove or disprove the existence of this subspace. Our approach seems to be useless when we replace the hypothesis of compactness of $D \subset \mathbb{R}^m$ by $\sigma-$compactness. However, it might be possible to use Borsuk-Ulam theorem locally and some other topological features in order to tackle this problem. We shall describe in the final section of this chapter an approach that seems promissing and one open question. An affirmative answer to this question would imply a complete solution for the problem.

4.1 Main Result

Theorem 4.1. *If D is a compact subset of $\mathbb{R}^n$ and V is a linear subspace of $C(D)$ such that $V \subset \widehat{C}(D) \cup \{0\}$ then $\dim(V) \leq n$.*

Proof. Let $f_1(x), \ldots, f_m(x)$ be a basis of V. Thus, any non null linear combination of these functions attains it maximum exactly once in K.

Define $F : D \to \mathbb{R}^m$, $F(x) = (f_1(x), \ldots, f_m(x))$. Notice that F is continuous since every f_i is continuous. Notice that every function of V can be written as $\langle v, F(x)\rangle$, where $v \in \mathbb{R}^m$.

Define $f : S^{m-1} \to D$ as $f(v)$ = the unique point of maximum of $\langle v, F(x)\rangle$ in D.

By contradiction assume that f is not continuous. Thus, there is a sequence $v_n \overset{n\to\infty}{\longrightarrow} v$ and $|f(v_n) - f(v)| > \epsilon$. Since $f(v_n) \in D$ and D is compact, there is a subsequence $f(v_{n_j}) \overset{j\to\infty}{\longrightarrow} y \in D$.

By definition of $f(v_{n_j})$, we have $\langle v_{n_j}, F(f(v_{n_j}))\rangle \geq \langle v_{n_j}, F(x)\rangle$, for every $x \in D$. If we fix x and let $j \to \infty$, since F is continuous, we get $\langle v, F(y)\rangle \geq \langle v, F(x)\rangle$, for every $x \in D$. Thus, $y = f(v)$ and $f(v_{n_j}) \overset{j\to\infty}{\longrightarrow} f(v) \in D$, but $|f(v_{n_j}) - f(v)| > \epsilon$. This is a contradiction. So $f : S^{m-1} \to D$ is continuous. Remind that $D \subset \mathbb{R}^n$.

Finally, if $m > n$ then by Borsuk-Ulam theorem (see, for example, [19]) there is a pair of antipodal points s and $-s$ in S^{m-1} such that $f(s) = f(-s)$. Hence, the point of maximum of $\langle s, F(x)\rangle$ in D is the point of maximum of $\langle -s, F(x)\rangle$ in D, which is the point of minimum of $\langle s, F(x)\rangle$ in D. Thus, $\langle s, F(x)\rangle$ is constant and does not belong to $\widehat{C}(D)$, which is a contradiction. □

Remark 4.2. *Since the Euclidean sphere $S^{m-1} \subset \mathbb{R}^m$ is compact and every linear functional defined on $\mathbb{R}^m$ restricted to S^{m-1} attains its maximum at only one point of S^{m-1} then there is a $m-$dimensional subspace V of $C(S^{m-1})$ such that $V \subset \widehat{C}(S^{m-1}) \cup \{0\}$. Thus, the upper bound for the dimension of the vector space V in the previous theorem is sharp.*

4.2 An Infinite Dimensional Example

Let D be a compact subset of $\mathbb{R}^n$. We saw above that, for $n < m$, there is no m-dimensional subspace of $C(D)$ formed, up to the origin, by functions that attain the maximum only at one point. In this section we show that if we allow D to be a compact subset of an infinite dimensional Banach space, $\widehat{C}(D)$ may contain, up to the origin, an infinite dimensional subspace of $C(D)$.

Example 4.3. Let D be the following subset of ℓ_2:

$$D = \left\{ \left(\frac{a_n}{n} \right)_{n=1}^{\infty} : (a_n)_{n=1}^{\infty} \in \ell_2 \text{ and } \|(a_n)_{n=1}^{\infty}\|_2 \leq 1 \right\}.$$

It is clear that D is a subset of the Hilbert cube $\prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n} \right]$. Since the Hilbert cube is compact, to prove that D is compact it is enough to show that it is closed. Let $(v_j)_{j=1}^{\infty} = \left(\left(\frac{v_n^j}{n} \right)_{n=1}^{\infty} \right)_{j=1}^{\infty}$ be a sequence in D converging to $w = (w_n)_{n=1}^{\infty} \in \ell_2$. Since convergence in ℓ_2 implies coordinatewise convergence, $w_n = \lim_j \frac{v_n^j}{n}$, so $nw_n = \lim_j v_n^j$, for every fixed n. For every k,

$$\sum_{n=1}^{k} n^2 |w_n|^2 = \sum_{n=1}^{k} \lim_j |v_n^j|^2 = \lim_j \sum_{n=1}^{k} |v_n^j|^2 \leq \limsup_j \|(v_n^j)_{n=1}^{\infty}\|_2^2 \leq 1.$$

This shows that $\|(nw_n)_{n=1}^{\infty}\|_2 \leq 1$, proving that $w \in D$. So D is a compact subset of ℓ_2.

Now we proceed to show that $\widehat{C}(D) \cup \{0\}$ contains an infinite dimensional subspace of $C(D)$. Consider the function

$$F\colon D \longrightarrow \ell_2 \ , \ F\left(\left(\frac{a_n}{n} \right)_{n=1}^{\infty} \right) = (a_n)_{n=1}^{\infty} .$$

Let $b = (c_1, c_2, \dots) \in \ell_2$, $b_n = (c_1, \dots, c_n, 0, 0, \dots)$ and $\phi_b\colon \ell_2 \longrightarrow \mathbb{R}$, $\phi_b(x) = \langle b, x \rangle$.

Consider $\phi_b \circ F : D \longrightarrow \mathbb{R}$ and note that $|\phi_{b_n} \circ F(x) - \phi_b \circ F(x)| \leq \|b_n - b\|_2 \|F(x)\|_2 \leq \|b_n - b\|_2$, since $\|F(x)\|_2 \leq 1$. Thus, $\phi_b \circ F : D \longrightarrow \mathbb{R}$ is continuous as a uniform limit of a sequence of continuous functions $(\phi_{b_n} \circ F(x))_{n=1}^{\infty}$.

Next, $\phi_b \circ F(x) = \langle b, F(x) \rangle < \left\langle b, \frac{b}{\|b\|_2} \right\rangle$ whenever $F(x) \neq \frac{b}{\|b\|_2}$. As F is a bijection onto the closed unit ball of ℓ_2, there is a unique $y \in D$ such that $F(y) = \frac{b}{\|b\|_2}$. This shows that $\phi_b \circ F(x)$ attains its maximum only at y. Finally $\{\phi_b \circ F : D \longrightarrow \mathbb{R},\ b \in \ell_2\}$ is a c–dimensional vector space.

4.3 Open Problem

Gurariy and Quarta asked if there exists a 3-dimensional linear subspace of $C[a,b)$ contained in $\widehat{C}[a,b) \cup \{0\}$. Here, we shall describe an approach that might be useful to disprove the existence of such subspace.

Assume that there is a 3 dimensional subspace of $C[a,b)$ contained in $\widehat{C}[a,b) \cup \{0\}$. Let $f_1(x), f_2(x), f_3(x)$ be a basis. Define $f : S^2 \to [a,b)$ as $f(v)$ = the unique point of maximum of $\langle v, F(x)\rangle$ in $[a,b)$, where $F(x) = (f_1(x), f_2(x), f_3(x))$. Let $[a,b) = \bigcup_{n>k, n\in\mathbb{N}} [a, b - \frac{1}{n}]$ for some suitable $k \in \mathbb{N}$.

Notice that $f^{-1}([a, b-\frac{1}{n}])$ is closed in S^2 and $f : f^{-1}([a, b-\frac{1}{n}]) \to [a, b-\frac{1}{n}]$ is continuous, by the same argument that was used in the proof of theorem 4.1.

Now, for every $c \in [a, b-\frac{1}{n}]$, we have $f^{-1}(c) \subset f^{-1}([a, b-\frac{1}{n}])$ and there is n such that $int(f^{-1}([a, b-\frac{1}{n}])) \neq \varnothing$ in S^2, since $S^2 = \bigcup_{n>k, n\in\mathbb{N}}^{\infty} f^{-1}([a, b-\frac{1}{n}])$, by Baire category theorem. But can we find $n \in \mathbb{N}$ and $c \in [a, b-\frac{1}{n}]$ such that $\partial(f^{-1}(c)) \subset int(f^{-1}([a, b-\frac{1}{n}]))$?

Let us assume that the following conjecture is true and let us obtain a contradiction. Thus, the key result to disprove the existence of this subspace is the following conjecture.

Conjecture 4.4. *There is $n \in \mathbb{N}$ and $c \in [a, b-\frac{1}{n}]$ such that $\partial(f^{-1}(c)) \subset int(f^{-1}([a, b-\frac{1}{n}]))$, where $\partial(A)$ and $int(A)$ mean the frontier and the interior of A in S^2, respectively.*

In order to obtain a contradiction, we need the following lemma :

Lemma 4.5. *If $x_1, x_2 \in f^{-1}(c) \subset S^2$ then the geodesic arc that connects x_1 and x_2, which shall be denoted by $\overline{x_1x_2}$, is also contained in $f^{-1}(c)$.*

Proof. Notice that $x_2 \neq -x_1$, otherwise c would be the point of maximum of $\langle x_1, F(x)\rangle$ and $\langle x_2, F(x)\rangle = -\langle x_1, F(x)\rangle$. Thus, $\langle x_1, F(x)\rangle$ would be constant and $\langle x_1, F(x)\rangle \notin \widehat{C}[a,b) \cup 0$.

The geodesic arc connecting x_1 to x_2 is $\frac{(1-t)x_1+tx_2}{|(1-t)x_1+tx_2|}$, $0 \le t \le 1$.

Notice that $\langle \frac{(1-t)x_1+tx_2}{|(1-t)x_1+tx_2|}, F(x)\rangle = \frac{1-t}{|(1-t)x_1+tx_2|}\langle x_1, F(x)\rangle + \frac{t}{|(1-t)x_1+tx_2|}\langle x_2, F(x)\rangle$

$\le \frac{1-t}{|(1-t)x_1+tx_2|}\langle x_1, F(c)\rangle + \frac{t}{|(1-t)x_1+tx_2|}\langle x_2, F(c)\rangle = \langle \frac{(1-t)x_1+tx_2}{|(1-t)x_1+tx_2|}, F(c)\rangle$, for every $x \in [a,b)$.

Thus, $f(\frac{(1-t)x_1+tx_2}{|(1-t)x_1+tx_2|}) = c$ for every $0 \le t \le 1$. ∎

Next, let $x_0 \in \partial(f^{-1}(c)) \subset int(f^{-1}([a, b - \frac{1}{n}]))$. There is a small circle $S \subset int(f^{-1}([a, b - \frac{1}{n}]))$ around x_0.

The function $f : S \to [a, b - \frac{1}{n}]$ is continuous, by Borsuk-Ulam theorem, there is a pair of antipodal points in S, x_1 and x_2, such that $f(x_1) = f(x_2) = d$. By the previous lemma, the geodesic arc $\overline{x_1x_2} \subset f^{-1}(d)$. Since $x_0 \in \overline{x_1x_2}$ then $c = d$, because $f^{-1}(c)$ is closed and $x_0 \in \partial(f^{-1}(c)) \subset f^{-1}(c)$. Thus, $\overline{x_1x_2} \subset f^{-1}(c)$ (see figure 1 below).

If $i \in \overline{x_1x_2} \cap int(f^{-1}(c))$ then $x_0 \in int(f^{-1}(c))$ (We can connect, by geodesic arcs, all the points of a neighborhood of i within $int(f^{-1}(c))$ to x_1 and x_2. Thus, $x_0 \in int(f^{-1}(c))$. See figure 2 below). So $\overline{x_1x_2} \subset \partial(f^{-1}(c))$.

Now, consider the circle S^1, centered in $C = (0,0,0)$, which contains the geodesic arc $\overline{x_1x_2}$. Notice that if $z \in \partial(f^{-1}(c)) \cap S^1$ then we can repeat the argument and obtain a geodesic arc $\overline{z_1z_2} \subset \partial(f^{-1}(c))$ such that $z \in \overline{z_1z_2}$ (see figure 3 below).

If $\overline{z_1z_2}$ is not contained in S^1 then we can prove that x_1 or x_2 belongs to $int(f^{-1}(c))$ (by connecting the points of $\overline{z_1z_2}$ and $\overline{x_1x_2}$ via geodesic arcs), which is a contradiction. Thus, $\overline{z_1z_2} \subset \partial(f^{-1}(c)) \cap S^1$.

We have just proved that $\partial(f^{-1}(c)) \cap S^1$ is open in S^1, but it is also closed as an intersection of closed sets. Since S^1 is connected then $\partial(f^{-1}(c)) \cap S^1 = S^1$. Thus, there are antipodal points in S^1, s and $-s$, such that $f(s) = f(-s) = c$, but this is a contradiction.

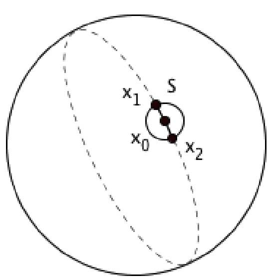

Figure 1.

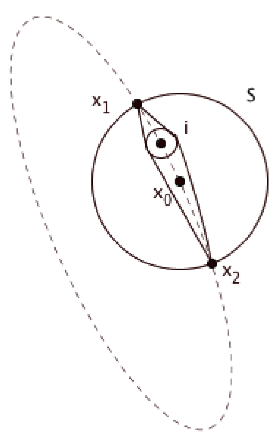

Figure 2.

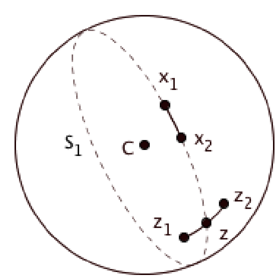

Figure 3.

Chapter 5

Basic Sequences in ℓ_p spaces

All the results of this chapter were published in [15].

During a *Non-linear Analysis Seminar* at Kent State University (Kent, Ohio, USA) in 2003, Richard M. Aron and Vladimir I. Gurariy posed the following question:

Question 5.1 (R. Aron & V. Gurariy, 2003)**.**
Is there an infinite dimensional closed subspace of ℓ_∞ every nonzero element of which has only a finite number of zero coordinates?

Question 5.1 has also appeared in several recent works (see, e.g., [9, 20, 22, 38]) and, for the last decade, there have been several attempts to partially answer it, although nothing conclusive in relation to the original problem has been obtained so far.

Throughout this chapter, and if X denotes a sequence space, we shall denote by $Z(X)$ the subset of X formed by sequences having only a finite number of zero coordinates. Here, we shall provide (among other results) the definitive answer to Question 5.1. Namely, if X stands for c_0, or ℓ_p, with $p \in [1, \infty]$, we prove the following:

(*i*) There is no infinite dimensional closed subspace of X inside $Z(X) \cup \{0\}$(Corollaries 5.7 and 5.16).

(*ii*) There exists an infinite dimensional closed subspace of X inside $V \setminus Z(V) \cup \{0\}$, for any infinite dimensional closed subspace V of X (Theorem 5.18).

In order to obtain the results above we shall make use of Functional Analysis techniques, basic sequences, complemented subspaces, and some classical Linear Algebra and Real Analysis approaches. From now on, if Y is any sequence space and $y \in Y$, then $y(j)$ shall denote the j−th coordinate of y with respect to the canonical basis $(e_j)_j$. Also, if $(m_k)_{k\in\mathbb{N}}$ is a subsequence of $(n_k)_{k\in\mathbb{N}}$, we shall write $(m_k)_{k\in\mathbb{N}} \subset (n_k)_{k\in\mathbb{N}}$. If V is a normed space and $(v_k)_{k\in\mathbb{N}} \subset V$, we denote by $\langle v_1, v_2, \dots \rangle$ the linear span of $\{v_1, v_2, \dots\}$ and by $[v_1, v_2, \dots]$ the closed linear span of $\{v_1, v_2, \dots\}$. If $W \subset V$, we denote $S_1(W) = \{w \in W, \|w\| = 1\}$. The rest of the notation shall be rather usual.

5.1 The case: $X = \ell_p,\ p \in [1, \infty[$

We need a series of technical lemmas in order to achieve the main result of this section. We believe that these lemmas are of independent interest.

Lemma 5.2. *Let V be an infinite dimensional closed subspace of ℓ_p, $p \in [1, \infty[$. Given $0 < \epsilon < \frac{4}{33}$ there is an increasing sequence of natural numbers $(s_k)_{k\in\mathbb{N}}$ and a normalized basic sequence $(f_k)_{k\in\mathbb{N}} \subset V$ such that*

(1) $f_k(s_j) = 0$ *for* $1 \le j \le k-1$.

(2) $f_1(s_1) \ne 0$.

(3) $|f_1(s_{k+1})| + \ldots + |f_k(s_{k+1})| < \frac{\epsilon}{2^{k+1}} |f_{k+1}(s_{k+1})|$ *for every* k
(thus $f_k(s_k) \ne 0$ *for every* $k \in \mathbb{N}$*).*

(4) $(f_k)_{k\in\mathbb{N}}$ *has basis constant smaller than* $\frac{8-2\epsilon}{4-9\epsilon}$.

(5) $[f_1, f_2, \ldots]$ *is complemented in* ℓ_p *with a projection* $Q : \ell_p \to \ell_p$ *of norm* $\|Q\| \le \frac{8-2\epsilon}{4-33\epsilon}$.

Proof. Let $f_1 \in V$ be such that $|f_1|_p = 1$. Let $N_1 \in \mathbb{N}$ be such that

(1) $f_1(N_1) \ne 0$.

(12 $|(f_1(n))_{n=N_1+1}^{\infty}|_p < \frac{\epsilon}{2^2}$.

Let $s_1 = N_1$. Suppose we have defined $f_2, \ldots, f_t \in V$ and

$$s_1 = N_1 < s_2 < N_2 < \ldots < s_t < N_t$$

such that

(1) $|f_k|_p = 1$ for $1 < k \le t$

(2) $f_k(n) = 0$ for $1 \le n \le N_{k-1}$ for every $1 < k \le t$
(Thus $f_k(s_j) = 0$ for $1 \le j \le k-1$ since $s_{k-1} < N_{k-1}$).

(3) $|(|f_1(n)| + \ldots + |f_k(n)|)_{n=N_k+1}^{\infty}|_p < \frac{\epsilon}{2^{k+1}}$ for $1 < k \le t$

(4) $|f_1(s_{k+1})| + \ldots + |f_k(s_{k+1})| < \frac{\epsilon}{2^{k+1}} |f_{k+1}(s_{k+1})|$ for $1 < k \le t-1$.
(Thus $f_k(s_k) \ne 0$ for $1 < k \le t$)

Since V is an infinite dimensional closed subspace of ℓ_p, there exists $f_{t+1} \in V$ such that $|f_{t+1}|_p = 1$ and $f_{t+1}(1) = \ldots = f_{t+1}(N_t) = 0$.

Now, if there is no $n > N_t$ such that $|f_1(n)| + \ldots + |f_t(n)| < \frac{\epsilon}{2^{t+1}}|f_{t+1}(n)|$ then

$$\frac{\epsilon}{2^{t+1}} > |(|f_1(n)| + \ldots + |f_t(n)|)_{n=N_t+1}^{\infty}|_p \geq \frac{\epsilon}{2^{t+1}}|f_{t+1}|_p = \frac{\epsilon}{2^{t+1}},$$

which is absurd. Therefore there exist $s_{t+1} > N_t$ such that

$$|f_1(s_{t+1})| + \ldots + |f_t(s_{t+1})| < \frac{\epsilon}{2^{t+1}}|f_{t+1}(s_{t+1})|.$$

Next, since $(|f_1(n)| + \ldots + |f_{t+1}(n)|)_{n\in\mathbb{N}} \in \ell_p$ then there exist $N_{t+1} > s_{t+1}$ such that

$$|(|f_1(n)| + \ldots + |f_{t+1}(n)|)_{n=N_{t+1}+1}^{\infty}|_p < \frac{\epsilon}{2^{t+2}}.$$

The induction to construct $(f_k)_{k\in\mathbb{N}}$ enjoying the four properties above is now complete. Now, in order to show that $(f_k)_{k\in\mathbb{N}}$ is a basic sequence, let us define

$$\widetilde{f_1}(n) = \begin{cases} f_1(n), & \text{if } 1 \leq n \leq N_1 \\ 0, & \text{otherwise} \end{cases} \qquad \widetilde{f_k}(n) = \begin{cases} f_k(n), & \text{if } N_{k-1} < n \leq N_k \\ 0, & \text{otherwise} \end{cases}$$

Notice that $\widetilde{f_k} \neq 0$, since $N_{k-1} < s_k < N_k$ and $\widetilde{f_k}(s_k) = f_k(s_k) \neq 0$. Note also that $(\widetilde{f_k})_{k\in\mathbb{N}}$ is a block basis of the canonical basis of ℓ_p.

Since

$$|(|f_1(n)| + \ldots + |f_k(n)|)_{n=N_k+1}^{\infty}|_p < \frac{\epsilon}{2^{k+1}},$$

then

$$|(f_k(n))_{n=N_k+1}^{\infty}|_p < \frac{\epsilon}{2^{k+1}}.$$

Now since $f_k(n) = 0$ for $1 \leq n \leq N_{k-1}$ we obtain

$$1 - \frac{\epsilon}{2^{k+1}} \leq |\widetilde{f_k}|_p \leq 1 \text{ and } |f_k - \widetilde{f_k}|_p < \frac{\epsilon}{2^{k+1}}$$

for $k \in \mathbb{N}$. In particular, $\frac{4-\epsilon}{4} = 1 - \frac{\epsilon}{4} \leq |\widetilde{f_k}|_p \leq 1$ for every $k \in \mathbb{N}$. Let $g_k = \frac{\widetilde{f_k}}{|\widetilde{f_k}|_p}$ for every k. Notice that $(g_k)_{k\in\mathbb{N}}$ is a normalized block basis of the canonical basis of ℓ_p. So $|\sum_{k=1}^{\infty} a_k g_k|_p = |(a_k)_{k\in\mathbb{N}}|_p$ and $(g_k)_{k\in\mathbb{N}}$ has basis constant $K = 1$. Let $\{\sigma_k,\ k \in \mathbb{N}\}$ be the following partition of $\mathbb{N}$:

$$\sigma_1 = \{1, \ldots, N_1\} \text{ and } \sigma_k = \{N_{k-1} + 1, \ldots, N_k\}.$$

Next, let $E_k = \{f \in \ell_p,\ f(i) = 0,\ \text{for } i \notin \sigma_k\}$. Thus, $g_k \in E_k$ and by [32, Theorem 30.18] the closed subspace $[g_1, g_2, \ldots]$ is complemented in ℓ_p with a projection $P : \ell_p \to \ell_p$ of norm 1.

Let us now prove that $(f_k)_{k\in\mathbb{N}}$ is equivalent to $(g_k)_{k\in\mathbb{N}}$ and $[f_1, f_2, \ldots]$ is also complemented in ℓ_p. Indeed,

$$\begin{aligned}|f_k - g_k|_p = |f_k - \frac{\widetilde{f}_k}{|\widetilde{f}_{kp}|}|_p &\le |f_k - \frac{f_k}{|\widetilde{f}_k|_p}|_p + |\frac{f_k}{|\widetilde{f}_k|_p} - \frac{\widetilde{f}_k}{|\widetilde{f}_k|_p}|_p \\ &\le \frac{1 - |\widetilde{f}_k|_p}{|\widetilde{f}_k|_p} + \frac{1}{|\widetilde{f}_k|_p}\frac{\epsilon}{2^{k+1}} \le \frac{4}{4-\epsilon}\left(1 - |\widetilde{f}_k|_p + \frac{\epsilon}{2^{k+1}}\right) \\ &\le \frac{4}{4-\epsilon}\left(\frac{2\epsilon}{2^{k+1}}\right).\end{aligned}$$

Thus, $(g_k)_{k\in\mathbb{N}}$ is a normalized basic sequence such that $[g_1, g_2, \ldots]$ is complemented in ℓ_p with a projection $P : \ell_p \to \ell_p$ of norm 1 and

$$\delta := \sum_{k=1}^{\infty} |f_k - g_k|_p \le \sum_{k=1}^{\infty} \frac{4}{4-\epsilon}\frac{\epsilon}{2^k} = \frac{4\epsilon}{4-\epsilon}.$$

Since $0 < \epsilon < \frac{4}{33}$, we obtain $8K\delta\|P\| = 8\delta \le 8\frac{4\epsilon}{4-\epsilon} < 1$. By the *principle of small perturbation* ([16, Theorem 4.5]) the sequence $(f_k)_{k\in\mathbb{N}}$ is equivalent to $(g_k)_{k\in\mathbb{N}}$ and $[f_1, f_2, \ldots]$ is also complemented in ℓ_p.

Finally, let us compute an upper bound for the basis constant of $(f_k)_{k\in\mathbb{N}}$ and for the norm of the projection $Q : \ell_p \to \ell_p$ onto $[f_1, f_2, \ldots]$.

First, the linear transformation $T(\sum_{k=1}^{\infty} a_k g_k) = \sum_{k=1}^{\infty} a_k f_k$ is an invertible continuous linear transformation from the closed span of $(g_k)_{k\in\mathbb{N}}$ to the closed span of $(f_k)_{k\in\mathbb{N}}$.

In the proof [16, Theorem 4.5] it is shown that $\|T\| \le (1 + 2K\delta) \le (1 + 8\delta) \le 2$ and $\|T^{-1}\| \le (1 - 2K\delta)^{-1}$. Let $P_n(\sum_{k=1}^{\infty} a_k g_k) = \sum_{k=1}^{n} a_k g_k$. Notice that $\|P_n\| = 1$.

Thus, for $n \le m$,

$$\begin{aligned}|\sum_{k=1}^{n} a_k f_k|_p = |T \circ P_n \circ T^{-1}(\sum_{k=1}^{m} a_k f_k)|_p &\le \|T\|\ \|P_n\|\ \|T^{-1}\|| \sum_{k=1}^{m} a_k f_k|_p \\ &\le \frac{2}{1 - 2K\delta}| \sum_{k=1}^{m} a_k f_k|_p.\end{aligned}$$

Then, the basis constant of $(f_k)_{k\in\mathbb{N}}$ is smaller than $\frac{2}{1-2K\delta} \le \frac{8-2\epsilon}{4-9\epsilon}$, since $K = 1$ and $\delta \le \frac{4\epsilon}{4-\epsilon}$. Again, using [16, Theorem 4.5], the linear transformation

$$Id - (T \circ P) : [f_1, f_2, \ldots] \to [f_1, f_2, \ldots]$$

is invertible and has norm smaller than $8K\delta\|P\| = 8\delta < 1$.

Therefore, there exists an inverse for $S = T \circ P : [f_1, f_2, \ldots] \to [f_1, f_2, \ldots]$ with norm $\|S^{-1}\| \le \frac{1}{1-8\delta}$. Now $Q = S^{-1} \circ (T \circ P) : \ell_p \to \ell_p$ is a projection onto $[f_1, f_2, \ldots]$ with norm $\|Q\| \le \|S^{-1}\|\ \|T\|\ \|P\| = \frac{1}{1-8\delta} \times 2 \times 1 \le \frac{8-2\epsilon}{4-33\epsilon}$, since $\delta \le \frac{4\epsilon}{4-\epsilon}$. □

Remark 5.3. *In the previous theorem, note that the convergence of* $\sum_{k=1}^{\infty} a_k f_k$ *implies the convergence* $\sum_{k=1}^{\infty} a_{2k} f_{2k}$ *and* $\sum_{k=1}^{\infty} a_{2k-1} f_{2k-1}$, *since* $(g_k)_{k=1}^{\infty}$ *is a block basis of the canonical basis of* ℓ_p *and* $(f_k)_{k\in\mathbb{N}}$*is equivalent to* $(g_k)_{k\in\mathbb{N}}$.

Lemma 5.4. *Let* V *be an infinite dimensional closed subspace of* ℓ_p, $p \in [1, \infty[$. *There exist an increasing sequence of natural numbers* $(s_k)_{k\in\mathbb{N}}$ *and a basic sequence* $(l_{s_k})_{k\in\mathbb{N}} \subset V$ *such that*

(1) $l_{s_k}(s_k) \neq 0$

(2) $l_{s_k}(s_j) = 0$ *for* $k \neq j$

(3) $[l_{s_1}, l_{s_2}, \dots]$ *is complemented in* ℓ_p.

Proof. Let $0 < \epsilon < \frac{1}{512}$. Then $4 - 9\epsilon > 1$, $4 - 33\epsilon > 1$ and

$$8\epsilon \left(\frac{8 - 2\epsilon}{4 - 9\epsilon}\right)\left(\frac{8 - 2\epsilon}{4 - 33\epsilon}\right) < 512\epsilon < 1.$$

Let $(s_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ be as in Lemma 5.2, using this ϵ.

Define $l_{0,k} = f_k$. Notice that $l_{0,k}(s_k) = f_k(s_k) \neq 0$ and $l_{0,k}(s_j) = 0$ for $s_j \in \{s_1, \dots, s_k\} \setminus \{s_k\}$. Define

$$l_{1,k} = l_{0,k} - \frac{l_{0,k}(s_{k+1})}{f_{k+1}(s_{k+1})} f_{k+1}.$$

Notice that

(1) $l_{1,k}(s_j) = 0$ for $s_j \in \{s_1, \dots, s_k, s_{k+1}\} \setminus \{s_k\}$.

(2) $l_{1,k}(s_k) = f_k(s_k) \neq 0$.

(3) Since $|l_{0,k}(s_{k+1})| = |f_k(s_{k+1})| < \frac{\epsilon}{2^{k+1}} |f_{k+1}(s_{k+1})|$ thus $\frac{|f_k(s_{k+1})|}{|f_{k+1}(s_{k+1})|} < \frac{\epsilon}{2^{k+1}} < 1$ and $|l_{1,k}(n)| \leq |f_k(n)| + |f_{k+1}(n)|$ for every $n \in \mathbb{N}$.

(4) $|l_{1,k} - l_{0,k}|_p < \frac{\epsilon}{2^{k+1}} |f_{k+1}|_p = \frac{\epsilon}{2^{k+1}}$.

Suppose we have already defined $l_{0,k}, \dots, l_{t,k}$ such that

(1) $l_{i,k}(s_j) = 0$ for $s_j \in \{s_1, \dots, s_{k+i}\} \setminus \{s_k\}$, for $1 \leq i \leq t$

(2) $l_{i,k}(s_k) = f_k(s_k) \neq 0$, for $1 \leq i \leq t$

(3) $|l_{i,k}(n)| \le |f_k(n)| + \ldots + |f_{k+i}(n)|$, for every $n \in \mathbb{N}$ and for $1 \le i \le t$

(4) $|l_{i,k} - l_{i-1,k}|_p < \frac{\epsilon}{2^{k+i}}$, for $1 \le i \le t$.

Define $l_{t+1,k} = l_{t,k} - \frac{l_{t,k}(s_{k+t+1})}{f_{k+t+1}(s_{k+t+1})} f_{k+t+1}$. Since $f_{k+t+1}(s_j) = 0$ for $1 \le j \le k+t$ then $l_{t+1,k}(s_j) = l_{t,k}(s_j)$ for $1 \le j \le k+t$. Since $l_{t+1,k}(s_{k+t+1}) = 0$ then

(1) $l_{t+1,k}(s_j) = 0$ for $s_j \in \{s_1, \ldots, s_{k+t+1}\} \setminus \{s_k\}$

(2) $l_{t+1,k}(s_k) = l_{t,k}(s_k) = f_k(s_k) \ne 0$

(3) $|l_{t,k}(s_{k+t+1})| \le |f_k(s_{k+t+1})| + \ldots + |f_{k+t}(s_{k+t+1})|$
$\le |f_1(s_{k+t+1})| + \ldots + |f_{k+t}(s_{k+t+1})| < \frac{\epsilon}{2^{k+t+1}} |f_{k+t+1}(s_{k+t+1})|$.
Therefore $\frac{|l_{t,k}(s_{k+t+1})|}{|f_{k+t+1}(s_{k+t+1})|} < \frac{\epsilon}{2^{k+t+1}} < 1$ and
$|l_{t+1,k}(n)| \le |l_{t,k}(n)| + |f_{k+t+1}(n)| \le |f_k(n)| + \ldots + |f_{k+t+1}(n)|$
for every $n \in \mathbb{N}$.

(4) $|l_{t+1,k} - l_{t,k}|_p < \frac{\epsilon}{2^{k+t+1}} |f_{k+t+1}|_p = \frac{\epsilon}{2^{k+t+1}}$.

The induction to construct $(l_{t,k})_{t=0}^{\infty}$ for each $k \in \mathbb{N}$ is completed. Next, let $t > m$ and notice that

$$|l_{t,k} - l_{m,k}|_p = |l_{t,k} - l_{t-1,k}|_p + \ldots + |l_{m+1,k} - l_{m,k}|_p \le \frac{\epsilon}{2^{k+t}} + \ldots + \frac{\epsilon}{2^{k+m+1}} \le \frac{\epsilon}{2^{k+m}}.$$

Therefore $(l_{t,k})_{t=0}^{\infty}$ is a Cauchy sequence in V, for each k. Let $\lim_{t\to\infty} l_{t,k} = l_k \in V$. Now notice that

(1) Since for every t, we have $l_{t,k}(s_k) = f_k(s_k) \ne 0$, then
$l_k(s_k) = \lim_{t\to\infty} l_{t,k}(s_k) = f_k(s_k) \ne 0$

(2) Since for $t > j$ and $j \ne k$, we have $l_{t,k}(s_j) = 0$, then
$l_k(s_j) = \lim_{t\to\infty} l_{t,k}(s_j) = 0$.

(3) Since $|l_{t,k} - l_{0,k}|_p \le \frac{\epsilon}{2^k}$ then $|l_k - f_k|_p = \lim_{t\to\infty} |l_{t,k} - l_{0,k}|_p \le \dfrac{\epsilon}{2^k}$.

Thus, $(f_k)_{k\in\mathbb{N}}$ is a normalized basic sequence with basis constant $K \le \frac{8-2\epsilon}{4-9\epsilon}$ such that $[f_1, f_2, \ldots]$ is complemented in ℓ_p with a projection $P : \ell_p \to \ell_p$ with norm $\|P\| \le \frac{8-2\epsilon}{4-33\epsilon}$ and

$$\delta = \sum_{k=1}^{\infty} |l_k - f_k|_p \le \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

Finally $8K\delta\|P\| \le 8\epsilon\left(\frac{8-2\epsilon}{4-9\epsilon}\right)\left(\frac{8-2\epsilon}{4-33\epsilon}\right) < 512\epsilon < 1$. By the principle of small pertubation [16, Theorem 4.5] the sequence $(l_k)_{k\in\mathbb{N}}$ is equivalent to $(f_k)_{k\in\mathbb{N}}$ and $[l_1, l_2, \ldots]$ is complemented in ℓ_p. Finally define $l_{s_k} = l_k$ for $k \in \mathbb{N}$. □

Remark 5.5. *Since $(f_k)_{k\in\mathbb{N}}$ is equivalent to $(l_{s_k})_{k\in\mathbb{N}}$ then the convergence of $\sum_{k=1}^{\infty} a_k l_{s_k}$ implies the convergence of $\sum_{k=1}^{\infty} a_{2k} l_{s_{2k}}$ and the convergence of $\sum_{k=1}^{\infty} a_{2k-1} l_{s_{2k-1}}$, by remark 5.3. Therefore $\sum_{k=1}^{\infty} a_k l_{s_k} = \sum_{k=1}^{\infty} a_{2k} l_{s_{2k}} + \sum_{k=1}^{\infty} a_{2k-1} l_{s_{2k-1}}$.*

Proposition 5.6. *Let V be an infinite dimensional closed subspace of ℓ_p, $p \in [1, \infty[$. There exists $0 \neq h \in V \setminus Z(V)$.*

Proof. Consider any l_k from Lemma 5.4. Notice that any $l_k \in V \setminus Z(V)$. □

Corollary 5.7. *There is no infinite dimensional closed subspace V of ℓ_p, $p \in [1, \infty[$, such that $V \subset Z(\ell_p) \cup \{0\}$.*

Corollary 5.8. *Let V be an infinite dimensional closed subspace of ℓ_p. Then $V \setminus Z(V)$ is dense in V.*

Proof. Let $0 \neq f \in V$. Define $f_1 = \frac{f}{|f|_p}$. We can start the proof of Lemma 5.2 using this f_1. Consider the proof of Lemma 5.4. For a sufficiently small ϵ (independent of $|f|_p$), we found a $l_1 \in V \setminus Z(V)$ such that $|f_1 - l_1|_p < \frac{\epsilon}{2^1}$ then

$$|f - |f|_p \, l_1| < \frac{|f|_p \epsilon}{2}.$$

Now $0 \neq |f|_p l_1 \in V \setminus Z(V)$. □

5.2 The case: $X = c_0$ or ℓ_∞

This section shall provide the definitive answer to Question 5.1 by showing that ℓ_∞ does not contain infinite dimensional Banach subspaces every nonzero element of which has only a finite number of zero coordinates. In order to achieve this we shall need to obtain a sequence $(l_{s_k})_{k\in\mathbb{N}}$ similar to that from Lemma 5.4 (see Lemma 5.14). Despite losing the hypothesis of the closed span of $(l_{s_k})_{k\in\mathbb{N}}$ being complemented, we gain the property $l_{s_k}(s_k) = 1$, obtaining still a basic sequence.

Definition 5.9. *Let V be an infinite dimensional closed subspace of ℓ_∞. Let $s \in \mathbb{N}$ and define*

$$V_s = \left\{ f \in V, \; f \neq 0, \; |f(s)| \geq \frac{|f|_\infty}{2} \right\}.$$

Lemma 5.10. *Let V be an infinite dimensional closed subspace of ℓ_∞.*
For every $K \subset V$, $K \neq \{0\}$, there exists $s \in \mathbb{N}$ such that

$$V_s \cap K \neq \varnothing.$$

Proof. Let $f \in K$, $f \neq 0$. Since $|f|_\infty = \sup_{k\in\mathbb{N}} |f(k)|$ there is $s \in \mathbb{N}$ such that $|f(s)| \geq \frac{|f|_\infty}{2}$. So $f \in V_s \cap K$. □

Lemma 5.11. *Let V be an infinite dimensional closed subspace of ℓ_∞. There exist an increasing sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ and a basic sequence $(f_{n_k})_{k\in\mathbb{N}} \subset V$ with:*

(1) $f_{n_k}(n_k) = 1$,

(2) $f_{n_j}(n_i) = 0$ *for* $j > i$, *and*

(3) $|f_{n_k}|_\infty \leq 2$ *for every* $k \in \mathbb{N}$.

Proof. This proof is a variation of Mazur's lemma ([16, Proposition 4.1]).

Let $\epsilon_1 = 1$ and $\epsilon_i \in \,]0,1[$ be such that $\prod_{i=1}^{\infty}(1+\epsilon_i) < \infty$.

By Lemma 5.10, there exists $s \in \mathbb{N}$ such that $V_s = V_s \cap V \neq \varnothing$.

Let $n_1 = \min\{s \in \mathbb{N}, V_s \neq \varnothing\}$ and let $f_1 \in V_{n_1}$. Define

$$f_{n_1} = \frac{f_1}{f_1(n_1)}.$$

Notice that

$$f_{n_1}(n_1) = 1 \text{ and } 1 \leq |f_{n_1}|_\infty = \frac{|f_1|_\infty}{|f_1(n_1)|} \leq 2.$$

Consider the projection $\pi_{n_1} : V \to \mathbb{C}$, $\pi_{n_1}(f) = f(n_1)$. Let $W_1 = \ker(\pi_{n_1})$. Since $\operatorname{codim}(W_1)$ in V is finite then $\dim(W_1) = \infty$, by Lemma 5.10 there exists $s \in \mathbb{N}$ such that $V_s \cap W_1 \neq \varnothing$.

Let $n_2 = \min\{s \in \mathbb{N}, V_s \cap W_1 \neq \varnothing\}$. Since $V_s \supset V_s \cap W_1$ then $n_2 \geq n_1$.

Now for every $f \in W_1$, $f(n_1) = 0$ then $V_{n_1} \cap W_1 = \varnothing$ then $n_2 > n_1$. Next, let $f_2 \in V_{n_2} \cap W_1$ and define

$$f_{n_2} = \frac{f_2}{f_2(n_2)}.$$

Notice that

$$f_{n_2}(n_2) = 1,\ f_{n_2}(n_1) = 0, \text{ and } 1 \leq |f_{n_2}|_\infty = \frac{|f_2|_\infty}{|f_2(n_2)|} \leq 2.$$

Next, for $a_1, a_2 \in \mathbb{C}$, $|a_1 f_{n_1} + a_2 f_{n_2}|_\infty \geq |\pi_{n_1}(a_1 f_{n_1} + a_2 f_{n_2})| = |a_1|$ and $1 + \epsilon_1 = 2 \geq |f_{n_1}|_\infty$, so

$$|a_1 f_{n_1} + a_2 f_{n_2}|_\infty (1 + \epsilon_1) \geq |a_1||f_{n_1}|_\infty = |a_1 f_{n_1}|_\infty.$$

Consider now the compact set $S_1(\langle f_{n_1}, f_{n_2}\rangle)$ and let $\{y_1, \ldots, y_k\} \subset S_1(\langle f_{n_1}, f_{n_2}\rangle)$ be such that if $y \in S_1(\langle f_{n_1}, f_{n_2}\rangle)$ then there exists y_i such that $|y - y_i|_\infty < \frac{\epsilon_2}{2}$. Consider $\{\phi_1, \ldots, \phi_k\} \subset S_1(V^*)$ such that $\phi_i(y_i) = 1$.

Take $\pi_{n_2} : V \to \mathbb{C}$, $T_{n_2}(f) = f(n_2)$. Let

$$W_2 = \bigcap_{i=1}^{k} \ker(\phi_i) \cap \ker(\pi_{n_2}) \cap W_1.$$

Since $\operatorname{codim}(\ker(\phi_i))$, $\operatorname{codim}(\ker(\pi_{n_2}))$, and $\operatorname{codim}(W_1)$ are finite in V then $\operatorname{codim}(W_2)$ is finite and $\dim(W_2) = \infty$. By lemma 5.10 there exists $s \in \mathbb{N}$ such that $V_s \cap W_2 \neq \varnothing$.

Let $n_3 = \min\{s \in \mathbb{N},\ V_s \cap W_2 \neq \varnothing\}$. Since $V_s \cap W_1 \supset V_s \cap W_2$ then $n_3 \geq n_2$.

Now, for all $f \in W_2$, $f(n_2) = 0$ then $V_{n_2} \cap W_2 = \varnothing$ then $n_3 > n_2$. Next, let $f_3 \in V_{n_3} \cap W_2$ and define

$$f_{n_3} = \frac{f_3}{f_3(n_3)}.$$

Notice that

$$f_{n_3}(n_3) = 1,\ f_{n_3}(n_2) = f_{n_3}(n_1) = 0, \text{ and } 1 \leq |f_{n_3}|_\infty = \frac{|f_3|_\infty}{|f_3(n_3)|} \leq 2.$$

Now, let $y \in S_1(\langle f_{n_1}, f_{n_2}\rangle)$. Notice that

$$\begin{aligned}
|y + \lambda f_{n_3}|_\infty &\geq |y_i + \lambda f_{n_3}|_\infty - |y_i - y|_\infty \\
&\geq |y_i + \lambda f_{n_3}|_\infty - \frac{\epsilon_2}{2} \quad (\text{for some } i \in \{1, \ldots, k\}) \\
&\geq \phi_i(y_i + \lambda f_{n_3}) - \frac{\epsilon_2}{2} \\
&\geq \phi_i(y_i) - \frac{\epsilon_2}{2} \\
&\geq 1 - \frac{\epsilon_2}{2} \geq \frac{1}{1 + \epsilon_2}.
\end{aligned}$$

Thus, for every $y \in S_1(\langle f_{n_1}, f_{n_2}\rangle)$ and any $\lambda \in \mathbb{C}$ we have

$$|y + \lambda f_{n_3}|_\infty (1 + \epsilon_2) \geq |y|_\infty.$$

Then

$$|a_1 f_{n_1} + a_2 f_{n_2} + a_3 f_{n_3}|_\infty (1 + \epsilon_2) \geq |a_1 f_{n_1} + a_2 f_{n_2}|_\infty$$

for all a_1, a_2, a_3 in $\mathbb{C}$. We can repeat the procedure to build $f_{n_4}, f_{n_5}, \ldots$ satisfying

$$|a_1 f_{n_1} + \ldots + a_k f_{n_m}|_\infty (1 + \epsilon_{m-1}) \ldots (1 + \epsilon_k) \geq |a_1 f_{n_1} + \ldots + a_k f_{n_k}|_\infty$$

for every $a_1, \ldots, a_m \in \mathbb{C}$ and $m \geq k$ and by Banach's criterion $(f_{n_k})_{k \in \mathbb{N}} \subset V$ is a basic sequence. Note that $(f_{n_k})_{k \in \mathbb{N}}$ satisfies the desired conditions. □

Lemma 5.12. *Let $g_1, g_2 \in \ell_\infty$ and let $(m_k)_{k \in \mathbb{N}}$ be an increasing sequence of natural numbers. There exists $(m_k^1)_{k \in \mathbb{N}} \subset (m_k)_{k \in \mathbb{N}}$ such that*

(1) *There exists* $\lim\limits_{k \to \infty} g_1(m_k^1) = L_1$,

(2) *There exists* $\lim\limits_{k \to \infty} g_2(m_k^1) = L_2$, *and*

(3) $m_2^1 > m_1^1 > m_2 > m_1$.

Proof. The sequence $(g_1(m_k))_{k \in \mathbb{N}}$ is bounded since $g_1 \in \ell_\infty$, therefore there is a subsequence $(m_k^0)_{k \in \mathbb{N}} \subset (m_k)_{k \in \mathbb{N}}$ and $L_1 \in \mathbb{C}$ such that $\lim\limits_{k \to \infty} g_1(m_k^0) = L_1$, and by same reasoning, there is a subsequence $(m_k^1)_{k \in \mathbb{N}} \subset (m_k^0)_{k \in \mathbb{N}}$ and L_2 such that $\lim\limits_{k \to \infty} g_2(m_k^1) = L_2$. Therefore $\lim\limits_{k \to \infty} g_1(m_k^1) = L_1$ and $\lim\limits_{k \to \infty} g_2(m_k^1) = L_2$. Removing, if necessary, the first two terms in the sequence $(m_k^1)_{k \in \mathbb{N}}$ we may assume that $m_2^1 > m_1^1 > m_2 > m_1$. □

Lemma 5.13. *Let V be an infinite dimensional closed subspace of ℓ_∞ and let $(n_k)_{k \in \mathbb{N}}$ be as in Lemma 5.11. For every $(m_k)_{k \in \mathbb{N}} \subset (n_k)_{k \in \mathbb{N}}$ there exist $(t_k)_{k \in \mathbb{N}} \subset (m_k)_{k \in \mathbb{N}}$ and basic sequence $(h_{t_k})_{k \in \mathbb{N}} \subset V$ satisfying*

a) $h_{t_k}(t_s) = 0$ *for* $s < k$,

b) $h_{t_k}(t_k) = 1$,

c) $|h_{t_k}|_\infty \leq 8$, *and*

d) $\lim\limits_{s \to \infty} h_{t_k}(t_s) = 0$.

Proof. Let $(f_{n_k})_k$ be as in Lemma 5.11. Define $g_1 = f_{m_1} - f_{m_1}(m_2) f_{m_2}$ and $g_2 = f_{m_2}$. Notice that $g_1(m_1) = 1$, $g_1(m_2) = 0$, $g_2(m_1) = 0$ and $g_2(m_2) = 1$. Now by Lemma 5.12 there exist $(m_k^1)_{k \in \mathbb{N}} \subset (m_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \to \infty} g_1(m_k^1) = L_1, \quad \lim_{k \to \infty} g_2(m_k^1) = L_2, \text{ and } m_2^1 > m_1^1 > m_2 > m_1.$$

We now have the following possibilities.

(1) If $L_1 = 0$, let $h_1 = g_1$. Notice that, since $|f_{m_i}|_\infty \le 2$ $(1 \le i \le 2)$ we have $|h_1|_\infty \le 6$. Notice also that $h_1(m_1) = 1$.

(2) If $L_1 \ne 0$ and $L_2 = 0$, let $h_1 = g_2$. We have $|h_1|_\infty \le 2$ and $h_1(m_2) = 1$.

(3) If $L_1 \ne 0$, $L_2 \ne 0$ and $|L_1| \le |L_2|$, define $h_1 = g_1 - \dfrac{L_1}{L_2} g_2$. Notice that $|h_1|_\infty \le |g_1|_\infty + \dfrac{|L_1|}{|L_2|}|g_2|_\infty \le 8$. Also, $h_1(m_1) = 1$.

(4) Finally, if $L_1 \ne 0$, $L_2 \ne 0$ and $|L_2| \le |L_1|$, let $h_1 = g_2 - \dfrac{L_2}{L_1} g_1$, having now that $|h_1|_\infty \le |g_2|_\infty + \dfrac{|L_2|}{|L_1|}|g_1|_\infty \le 8$. Also, note that $h_1(m_2) = 1$.

Next, if $h_1(m_1) = 1$, define $t_1 = m_1$ and, if $h_1(m_1) \ne 1$, then $h_1(m_2) = 1$ and we let $t_1 = m_2$. In any case, note that $\lim_{k\to\infty} h_1(m_k^1) = 0$. Let us now suppose that, by induction, we have already defined

(1) $(m_k^i)_{k\in\mathbb{N}} \subset \cdots \subset (m_k^1)_{k\in\mathbb{N}} \subset (m_k)_{k\in\mathbb{N}}$ with

$$m_2^i > m_1^i > m_2^{i-1} > m_1^{i-1} > \ldots > m_2^1 > m_1^1 > m_2 > m_1,$$

(2) $t_1 = m_1$ or m_2 and $t_j = m_1^{j-1}$ or m_2^{j-1}, $2 \le j \le i$.

(3) $h_j \in V$, $1 \le j \le i$, verifying conditions $a), b)$ and $c)$ of this lemma, and

(4) $\lim_{k\to\infty} h_j(m_k^j) = 0$, $1 \le j \le i$.

Next, repeat the construction of h_1 in order to obtain h_{i+1} by means of $f_{m_1^i}, f_{m_2^i}$ instead of f_{m_1}, f_{m_2}.

Using the sequence $(m_k^i)_{k\in\mathbb{N}}$, instead of $(m_k)_{k\in\mathbb{N}}$ in the previous construction, we obtain $(m_k^{i+1})_{k\in\mathbb{N}} \subset (m_k^i)_{k\in\mathbb{N}}$ such that

$$m_2^{i+1} > m_1^{i+1} > m_2^i > m_1^i \quad \text{and} \quad \lim_{k\to\infty} h_{i+1}(m_k^{i+1}) = 0.$$

Define now $t_{i+1} = m_1^i$ or m_2^i, depending on whether $h_{i+1}(m_1^i) = 1$ or $h_{i+1}(m_2^i) = 1$, as we previously did for t_1. Therefore we have $h_{i+1}(t_{i+1}) = 1$. Next, since h_{i+1} is a linear combination of $f_{m_1^i}, f_{m_2^i}$, and

$$m_2^i > m_1^i > m_2^{i-1} > m_1^{i-1} > \ldots > m_2^1 > m_1^1 > m_2 > m_1,$$

we obtain that $h_{i+1}(m_1) = h_{i+1}(m_2) = h_{i+1}(m_1^{j-1}) = h_{i+1}(m_2^{j-1}) = 0$ (for $2 \leq j \leq i$), but $t_1 = m_1$ or m_2, $t_j = m_1^{j-1}$ or m_2^{j-1} (for $2 \leq j \leq i$), which implies that $h_{i+1}(t_j) = 0$ for $1 \leq j \leq i$.

Finally, notice that $(t_s)_{s=i+1}^{\infty} \subset (m_k^i)_{k\in\mathbb{N}}$, thus $\lim_{s\to\infty} h_i(t_s) = 0$ (for every $i \in \mathbb{N}$).

Notice that $(f_{m_k})_{k\in\mathbb{N}}$ is a basic sequence as subsequence of the basic sequence $(f_{n_k})_{k\in\mathbb{N}}$. Notice also that h_k is a linear combination of $f_{m_1^{k-1}}$ and $f_{m_2^{k-1}}$, h_1 is a linear combination of f_{m_1} and f_{m_2} and $m_2^{k-1} > m_1^{k-1} > \ldots > m_2^1 > m_1^1 > m_2 > m_1$ for every k. Therefore $(h_k)_{k\in\mathbb{N}}$ is a block sequence of the basic sequence $(f_{m_k})_{k\in\mathbb{N}}$. Therefore $(h_k)_{k\in\mathbb{N}}$ is also a basic sequence. Finally, let $h_{t_k} = h_k$. □

Lemma 5.14. *Let V be an infinite dimensional closed subspace of ℓ_∞ and let $(n_k)_{k\in\mathbb{N}}$ be as in Lemma 5.11. For every $(m_k)_{k\in\mathbb{N}} \subset (n_k)_{k\in\mathbb{N}}$ there exist $(s_k)_{k\in\mathbb{N}} \subset (m_k)_{k\in\mathbb{N}}$ and a basic sequence $(l_{s_k})_{k\in\mathbb{N}} \in V$, satisfying*

a) $l_{s_k}(s_k) = 1$,

b) $l_{s_k}(s_j) = 0$, *for* $j \neq k$.

c) $|l_{s_k}|_\infty \leq 9$, *for every* $k \in \mathbb{N}$.

Proof. Consider $(t_k)_{k\in\mathbb{N}} \subset (m_k)_{k\in\mathbb{N}}$ and $(h_{t_k})_{k\in\mathbb{N}} \subset V$ as in Lemma 5.13. Let K be the basis constant of the basic sequence $(h_{t_k})_{k\in\mathbb{N}}$ and let $0 < \epsilon < \frac{1}{2K}$. (Recall that K is always equal or bigger than 1, therefore $\epsilon < 1$). Let $s_1 = t_1$. Suppose defined, by induction, $\{s_1, \ldots, s_n\} \subset \{t_1, t_2, \ldots\}$. Since $\lim_{j\to\infty} |h_{s_1}(t_j)| + \ldots + |h_{s_n}(t_j)| = 0$, there exists $s_{n+1} \in \{t_1, t_2, \ldots\}$, $s_{n+1} > s_n$, such that

$$|h_{s_1}(s_{n+1})| + \ldots + |h_{s_n}(s_{n+1})| \leq \frac{\epsilon}{2^{n+1} \times 8}.$$

The induction to construct $(s_k)_{k\in\mathbb{N}} \subset (m_k)_{k\in\mathbb{N}}$ is completed.

Now define $l_{0,k} = h_{s_k}$. Notice that $l_{0,k}(s_k) = 1$ and $l_{0,k}(s_j) = 0$ for $s_j \in \{s_1, ..., s_k\} \setminus \{s_k\}$. Define $l_{1,k} = l_{0,k} - l_{0,k}(s_{k+1})h_{s_{k+1}}$.

Notice that

- $l_{1,k}(s_k) = 1$ and $l_{1,k}(s_j) = 0$ for $s_j \in \{s_1, ..., s_{k+1}\} \setminus \{s_k\}$,
- since $|h_{s_k}(s_{k+1})| \leq \frac{\epsilon}{2^{k+1}\times 8}$ then

$$|l_{1,k}(s_j)| \leq |l_{0,k}(s_j)| + |h_{s_{k+1}}(s_j)| = |h_{s_k}(s_j)| + |h_{s_{k+1}}(s_j)|$$

  for every $j \in \mathbb{N}$.

- $|l_{1,k} - l_{0,k}|_\infty = |l_{0,k}(s_{k+1})||h_{s_{k+1}}|_\infty \leq \frac{\epsilon}{2^{k+1}\times 8} \times 8 = \frac{\epsilon}{2^{k+1}}$

Suppose we have already defined, by induction, $l_{0,k}, l_{1,k}, \ldots, l_{t,k} \in V$ such that

- $l_{n,k}(s_k) = 1$ for $0 \leq n \leq t$,
- $l_{n,k}(s_j) = 0$ for $s_j \in \{s_1, ..., s_{k+n}\} \setminus \{s_k\}$ and $0 \leq n \leq t$,
- $|l_{n,k}(s_j)| \leq |h_{s_k}(s_j)| + |h_{s_{k+1}}(s_j)| + \ldots + |h_{s_{k+n}}(s_j)|$, for every $j \in \mathbb{N}$ and $0 \leq n \leq t$,
- $|l_{n,k} - l_{n-1,k}|_\infty \leq \frac{\epsilon}{2^{k+n}}$ for $1 \leq n \leq t$,

Next, define $l_{t+1,k} = l_{t,k} - l_{t,k}(s_{k+t+1})h_{s_{k+t+1}}$. Notice that

- $l_{t+1,k}(s_k) = 1$,
- $l_{t+1,k}(s_j) = 0$ for $s_j \in \{s_1, ..., s_{k+t+1}\} \setminus \{s_k\}$,
- since $|l_{t,k}(s_{k+t+1})| \leq |h_{s_k}(s_{k+t+1})| + |h_{s_{k+1}}(s_{k+t+1})| + \ldots + |h_{s_{k+t}}(s_{k+t+1})| \leq$

  $$|h_{s_1}(s_{k+t+1})| + |h_{s_2}(s_{k+t+1})| + \ldots + |h_{s_{k+t}}(s_{k+t+1})| \leq \frac{\epsilon}{2^{k+t+1} \times 8}$$

  then $|l_{t+1,k}(s_j)| \leq |l_{t,k}(s_j)| + |h_{s_{k+t+1}}(s_j)|$ for every $j \in \mathbb{N}$ and by induction hypothesis

  $$|l_{t+1,k}(s_j)| \leq |h_{s_k}(s_j)| + |h_{s_{k+1}}(s_j)| + \ldots + |h_{s_{k+t+1}}(s_j)|,$$

  for every $j \in \mathbb{N}$.
- $|l_{t+1,k} - l_{t,k}|_\infty = |l_{t,k}(s_{k+t+1})||h_{s_{k+t+1}}|_\infty \leq \frac{\epsilon}{2^{k+t+1}\times 8} \times 8 = \frac{\epsilon}{2^{k+t+1}}$

The induction to construct $(l_{t,k})_{t=0}^\infty \subset V$, for every $k \in \mathbb{N}$, is completed. Now

$$|l_{0,k}|_\infty + |l_{1,k} - l_{0,k}|_\infty + |l_{2,k} - l_{1,k}|_\infty + \ldots \leq |l_{0,k}|_\infty + \frac{\epsilon}{2^{k+1}} + \frac{\epsilon}{2^{k+2}} + \ldots \leq |l_{0,k}|_\infty + \epsilon.$$

Thus, for each $k \in \mathbb{N}$, the series $\lim_{t\to\infty} l_{t,k} = l_{0,k} + (l_{1,k} - l_{0,k}) + (l_{2,k} - l_{1,k}) + \ldots$ is absolutely and coordinatewise convergent to some $l_k \in V$. Notice that $l_{t,k}(s_k) = 1$ for every t then $\lim_{t\to\infty} l_{t,k}(s_k) = l_k(s_k) = 1$. Next $l_{t,k}(s_j) = 0$ for $t > j$ and $j \neq k$ then $\lim_{t\to\infty} l_{t,k}(s_j) = l_k(s_j) = 0$. Now, $l_{t,k} - l_{0,k} = (l_{t,k} - l_{t-1,k}) + \ldots + (l_{1,k} - l_{0,k})$ then

$$|l_{t,k} - l_{0,k}|_\infty \leq \frac{\epsilon}{2^{k+t}} + \frac{\epsilon}{2^{k+t-1}} + \ldots + \frac{\epsilon}{2^{k+1}} \leq \frac{\epsilon}{2^k},$$

then $\lim_{t\to\infty} |l_{t,k} - l_{0,k}|_\infty = |l_k - h_{s_k}|_\infty \le \frac{\epsilon}{2^k}$, for every $k \in \mathbb{N}$, so

$$|l_k|_\infty \le |h_{s_k}|_\infty + \frac{\epsilon}{2^k} \le 8 + 1 = 9.$$

Since $h_{s_k}(s_k) = 1$ then $|h_{s_k}|_\infty \ge 1$ and we have

$$\left| \frac{l_k}{|h_{s_k}|_\infty} - \frac{h_{s_k}}{|h_{s_k}|_\infty} \right|_\infty \le \frac{\epsilon}{2^k}.$$

Then $\delta := \sum_{k=1}^{\infty} \left| \frac{l_k}{|h_{s_k}|_\infty} - \frac{h_{s_k}}{|h_{s_k}|_\infty} \right|_\infty \le \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$

Now the normalized sequence $\left(\frac{h_{s_k}}{|h_{s_k}|_\infty}\right)_{k\in\mathbb{N}}$ as a block basis of the basic sequence $(h_{t_k})_{k\in\mathbb{N}}$ is also a basic sequence with basic constant $K' \le K$. Then $2K'\delta \le 2K\delta \le 2K\epsilon < 1$.

By the *principle of small pertubation* [16, Theorem 4.5] the sequence $\left(\frac{l_k}{|h_{s_k}|_\infty}\right)_{k\in\mathbb{N}}$ is a basic sequence equivalent to the normalized basic sequence $\left(\frac{h_{s_k}}{|h_{s_k}|_\infty}\right)_{k\in\mathbb{N}}$. Notice that $(l_k)_{k\in\mathbb{N}}$ is a block basis of $\left(\frac{l_k}{|h_{s_k}|_\infty}\right)_{k\in\mathbb{N}}$, therefore it is also a basic sequence. Finally define $l_{s_k} = l_k$. □

From the previous lemma, we can now infer the following.

Proposition 5.15. *Let V be an infinite dimensional closed subspace of ℓ_∞. There exists $0 \ne h \in V \setminus Z(V)$.*

Proof. Consider l_{s_k} from Lemma 5.14. We have that $l_{s_k} \in V \setminus Z(V)$. □

Corollary 5.16. *There is no infinite dimensional closed subspace V of ℓ_∞, such that $V \subset Z(\ell_\infty) \cup \{0\}$.*

As a consequence of Lemma 5.14 we also have the following result, whose proof is simple.

Corollary 5.17. *Let V be an infinite dimensional closed subspace of c_0. Then $V \setminus Z(V)$ is dense in V.*

Proof. Let $(n_k)_k$ be as in Lemma 5.11. Every $0 \ne f \in V \subset c_0$ satisfies $\lim_{k\to\infty} f(n_k) = 0$. Let $\epsilon > 0$. There exists $(m_k)_{k\in\mathbb{N}} \subset (n_k)_{k\in\mathbb{N}}$ such that $(f(m_k))_{k\in\mathbb{N}} \in l_1$ and $|(f(m_k))_{k\in\mathbb{N}}|_1 \le \frac{\epsilon}{9}$.

By Lemma 5.14, there exist $(s_k)_{k\in\mathbb{N}} \subset (m_k)_{k\in\mathbb{N}}$ and $l_{s_k} \in V$ such that *a)* $l_{s_k}(s_k) = 1$,

b) $l_{s_k}(s_j) = 0$, for $j \neq k$.

c) $|l_{s_k}|_\infty \leq 9$ for every $k \in \mathbb{N}$.

Notice that $|f(s_1)l_{s_1}|_\infty + |f(s_2)l_{s_2}|_\infty + \ldots \leq (|f(s_1)| + |f(s_2)| + \ldots)\ 9 \leq \epsilon$.

Therefore $f - f(s_1)l_{s_1} - f(s_2)l_{s_2} - \ldots$ converges absolutely and coordinatewise to some $g \in V$. Notice that for every $k \in \mathbb{N}$

$$g(s_k) = f(s_k) - f(s_1)l_{s_1}(s_k) - f(s_2)l_{s_2}(s_k) - \ldots = f(s_k) - f(s_k)l_{s_k}(s_k) = 0$$

and $|g - f|_\infty \leq \epsilon$. □

Theorem 5.18. *Let V be an infinite dimensional closed subspace of ℓ_p, $p \in [1, \infty]$. There exists an infinite dimensional closed subspace of ℓ_p, $p \in [1, \infty]$ inside $V \setminus Z(V) \cup \{0\}$.*

Proof. By Lemmas 5.4 and 5.14, there is an increasing sequence of natural numbers $(s_k)_{k\in\mathbb{N}}$ and a sequence $(l_{s_k})_{k\in\mathbb{N}} \subset V$ such that

a) $l_{s_k}(s_k) \neq 0$,

b) $l_{s_k}(s_j) = 0$, for $j \neq k$.

Let $W = \langle l_{s_2}, l_{s_4}, l_{s_6}, \ldots\rangle$ and notice that every $f \in W$ satisfies $f(s_{2k-1}) = 0$ for every $k \in \mathbb{N}$. Since convergence in norm implies coordinatewise convergence in ℓ_p, $p \in [1, \infty]$ then for every $f \in \overline{W}$, we obtain $f(s_{2k-1}) = 0$ for every $k \in \mathbb{N}$.

Notice that $\{l_{2k} \in \overline{W},\ k \in \mathbb{N}\}$ is a linear independent set then $\overline{W}$ is a infinite dimensional closed subspace of V with $\overline{W} \subset V \setminus Z(V) \cup \{0\}$. □

Remark 5.19. *If V of theorem 5.18 is also an algebra with the coordinatewise product then every element of the closed subalgebra generated by W has zeros at coordinates $s_1, s_3, s_5 \ldots$*

Corollary 5.20. *Let V be an infinite dimensional closed subspace of ℓ_p, $p \in [1, \infty[$. Then the infinite dimensional closed subspace $\overline{W} \subset V \setminus Z(V) \cup \{0\}$, obtained in Theorem 5.18, is complemented in ℓ_p.*

Proof. Notice that the sequence $(l_{s_k})_{k\in\mathbb{N}} \subset V$ used in the proof of Theorem 5.18 is the basic sequence constructed in lemma 5.4, when $p \in [1, \infty[$. Thus, $[l_{s_1}, l_{s_2}, \ldots]$ is complemented in ℓ_p. Since $\overline{W} = [l_{s_2}, l_{s_4}, l_{s_6}, \ldots]$ is complemented in $[l_{s_1}, l_{s_2}, \ldots]$ by $[l_{s_1}, l_{s_3}, l_{s_5}, \ldots]$, by remark 5.5. We got the result. □

List of Symbols and Notation

Chapters 2 and 3:

$\mathbb{C}^k$ – The set of column vectors with k complex entries.
M_k – The set of complex matrices of order k.
P_k – The set of positive semidefinite Hermitian matrices of order k.
$X \otimes Y$ – The Kronecker product of the matrices X, Y.
$\mathbb{C}^k \otimes \mathbb{C}^m$ – The tensor product space of $\mathbb{C}^k$ and $\mathbb{C}^m$.
$M_k \otimes M_m$ – The tensor product space of M_k and M_m.
$M_{k_1} \otimes \ldots \otimes M_{k_n}$ – The tensor product space $M_{k_1} \otimes (M_{k_2} \otimes \ldots \otimes M_{k_n})$.
Id – The Identity matrix.
VM_kW – The set $\{VXW, X \in M_k\}$, where $V, W \in M_k$ are orthogonal projections.
$tr(X)$ – The trace of the matrix X.
X^t – The transpose of the matrix X.
$\overline{X}$ – The matrix whose entries are the complex conjugate of the entries of the matrix X.
X^* – The conjugate transpose of the matrix X, i.e., $X^* = \overline{X}^t$.
$\langle X, Y\rangle$ – The trace inner product of the square matrices X, Y, i.e., $tr(XY^*)$.
$\mathfrak{I}(\gamma)$ – The image (or the range) of the matrix γ.
$T^* : WM_mW \to VM_kV$ – The adjoint of $T : VM_kV \to WM_mW$ with respect to $\langle X, Y\rangle$.
$\|X\|_2$ – The spectral norm of the matrix $X \in M_k$.
X^+ – The pseudo-inverse of the matrix X.
$R' + R$ – The sum of the spaces R', R.
$R' \oplus R$ – The direct sum of the spaces R', R.
$R' \perp R$ – The orthogonality of the subspaces $R' \subset M_k$ and $R \subset M_k$ with respect to $\langle X, Y\rangle$.
$L|_R$ – The restriction of the map $L : VM_kV \to VM_kV$ to $R \subset VM_kV$.
$F_A : M_m \to M_k$ – The map $F_A(X) = \sum_{i=1}^n tr(B_iX)A_i$, where $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m$.
$G_A : M_k \to M_m$ – The map $G_A(X) = \sum_{i=1}^n tr(A_iX)B_i$, where $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m$.
x^t – The transposition of the column vector $x \in \mathbb{C}^k$.
$\overline{x}$ – The column vector whose entries are the complex conjugate of the entries of x.
$\langle x, y\rangle$ – The usual inner product of the column vectors $x, y \in \mathbb{C}^k$, i.e., $\langle x, y\rangle = x^t\overline{y}$.

A^{t_2} – The partial transposition of $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m$, i.e., $A^{t_2} = \sum_{i=1}^n A_i \otimes B_i^t$.
$\det(\gamma)$ – The determinant of the matrix γ.

Chapter 4 and 5:

$C(D)$– The set of real-valued continuous functions on a topological space D.
$\widehat{C}(D)$ – The subset of $C(D)$ formed by the functions that attain the maximum exactly once in D.
$\dim(V)$ – The dimension of the vector space V.
S^k – The Euclidean sphere of radius 1 of $\mathbb{R}^{k+1}$.
$\partial(A)$ – The frontier of $A \subset S^2$.
$int(A)$ – The interior of $A \subset S^2$.
$y(j)$ – The j–th coordinate of the sequence y, i.e., $y = (y(j))_{j\in\mathbb{N}}$.
$(m_k)_{k\in\mathbb{N}} \subset (n_k)_{k\in\mathbb{N}}$ – This symbol means that $(m_k)_{k\in\mathbb{N}}$ is a subsequence of $(n_k)_{k\in\mathbb{N}}$.
$\langle v_1, v_2, \ldots \rangle$ – The linear space spanned by $(v_k)_{k\in\mathbb{N}}$.
$[v_1, v_2, \ldots\ ]$ – The closed linear space spanned by $(v_k)_{k\in\mathbb{N}}$.
$S_1(W)$ – The sphere of radius 1 of the normed set W, i.e., $S_1(W) = \{w \in W,\ |w| = 1\}$.

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