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SOME CLASSICAL INEQUALITIES AND SUMMABILITY OF MULTILINEAR OPERATORS

João Pessoa-PB
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Gustavo da Silva Araújo

Some classical inequalities and summability of multilinear operators

Tese apresentada ao Corpo Docente do Programa Associado de Pós-Graduação em Matemática - UFPB/UFCG, como requisito parcial para a obtenção do título de Doutor em Matemática.

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Resumo

Este trabalho está dividido em três partes. Na primeira parte, investigamos o comportamento das constantes das desigualdes polinomial e multilinear de Hardy–Littlewood. Na segunda parte, apresentamos uma nova classe de operadores multilineares somantes, a qual recupera as classes dos operadores multilineares absolutamente e multiplo somantes. Além disso, mostramos um resultado ótimo de espaçabilidade para o complementar de uma classe de operadores multiplo somantes em ℓ_p e também generalizamos um resultado relacionado a cotipo (de 2010) devido a G. Botelho, C. Michels, and D. Pellegrino. Finalmente, provamos novos resultados de coincidência para as classes de operadores multilineares absolutamente e multiplo somantes. Em particular, mostramos que o famoso teorema de Defant–Voigt é ótimo. Na terceira parte, provamos várias desigualdades ótimas para o espaço $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ de polinômios 2-homogêneos em $\mathbb{R}^2$ dotados com a norma do supremo no setor $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. Além dos resultados principais, encontramos desigualdades ótimas de Bernstein e Markov e calculamos as constantes incondicional e de polarização da base canônica do espaço $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Palavras-chave: Constante incondicional, constante de polarization, desigualdade de Bernstein, desigualdade de Bohnenblust–Hille, desigualdade de Hardy–Littlewood, desigualdade de Markov, espaçabilidade, operadores multilineares somantes.

Abstract

This work is divided into three parts. In the first part, we investigate the behaviour of the constants of the Hardy–Littlewood polynomial and multilinear inequalities. In the second part, we present a new class of summing multilinear operators, which recovers the class of absolutely and multiple summing operators. Moreover, we show an optimal spaceability result for a set of non-multiple summing forms on ℓ_p and we also generalize a result related to cotype (from 2010) as highlighted by G. Botelho, C. Michels, and D. Pellegrino. Lastly, we prove new coincidence results for the class of absolutely and multiple summing multilinear operators. In particular, we show that the well-known Defant–Voigt theorem is optimal. In the third part, a number of sharp inequalities are proved for the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ of 2-homogeneous polynomials on $\mathbb{R}^2$, endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. Among the main results we can find sharp Bernstein and Markov inequalities and the calculation of the unconditional and polarization constants of the canonical basis of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Key-words: Bernstein inequality, Bohnenblust–Hille inequality, Hardy–Littlewood inequality, Markov inequality, polarization constant, spaceability, summing multilinear operatos, unconditional constant.

Contents

Preliminaries and Notation

For any function f, whenever it makes sense we formally define $f(\infty) = \lim_{p\to\infty} f(p)$. Throughout this, $E, E_1, E_2, \ldots, F$ shall denote Banach spaces over $\mathbb{K}$, which shall stands for the complex $\mathbb{C}$ or real $\mathbb{R}$ fields. $\mathcal{L}(E_1, ..., E_m; F)$ stand for the Banach space of all bounded m-linear operators from $E_1 \times \cdots \times E_m$ to F under the supremum norm and when $E_1 = \cdots = E_m = E$ we denote $\mathcal{L}(E_1, ..., E_m; F)$ by $\mathcal{L}(^m E; F)$. The topological dual of E shall be denoted by E^* and for any $p \geq 1$ its conjugate is represented by p^*, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. For $p \in [1, \infty]$, as usual, we consider the Banach spaces of weakly and strongly p-summable sequences, respectively, as bellow:

$$\ell_p^w(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \left\| (x_j)_{j=1}^\infty \right\|_{w,p} := \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^\infty |\varphi(x_j)|^p \right)^{1/p} < \infty \right\}$$

and

$$\ell_p(E) := \left\{ (x_j)_{j=1}^\infty \subset E : \left\| (x_j)_{j=1}^\infty \right\|_p := \left(\sum_{j=1}^\infty \|x_j\|^p \right)^{1/p} < \infty \right\}$$

(naturally, the sum $\sum$ should be replaced by the supremum if $p = \infty$). Besides, we set $X_\infty := c_0$ and $X_p := \ell_p := \ell_p(\mathbb{K})$. For a positive integer m, $\mathbf{p}$ stands for a multiple exponent $(p_1, \ldots, p_m) \in [1, \infty]^m$ and

$$\left| \frac{1}{\mathbf{p}} \right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

The Khinchine inequality (see [62]) asserts that for any $0 < q < \infty$, there are positive constants A_q, B_q such that regardless of the scalar sequence $(a_j)_{j=1}^\infty$ in ℓ_2 we have

$$A_q \left(\sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{j=1}^\infty a_j r_j(t) \right|^q dt \right)^{\frac{1}{q}} \leq B_q \left(\sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}},$$

where r_j are the Rademacher functions. More generally, from the above inequality together with the Minkowski inequality we know that (see [16], for instance, and the references therein)

$$
\begin{aligned}
A_q^m\left(\sum_{j_1,\ldots,j_m=1}^{\infty}|a_{j_1\cdots j_m}|^2\right)^{\frac{1}{2}} &\le \left(\int_I\left|\sum_{j_1,\ldots,j_m=1}^{\infty}a_{j_1\cdots j_m}r_{j_1}(t_1)\cdots r_{j_m}(t_m)\right|^q dt\right)^{\frac{1}{q}} \\
&\le B_q^m\left(\sum_{j_1,\ldots,j_m=1}^{\infty}|a_{j_1\cdots j_m}|^2\right)^{\frac{1}{2}},
\end{aligned}
\tag{1}
$$

where $I=[0,1]^m$ and $dt=dt_1\cdots dt_m$, for all scalar sequences $(a_{j_1\cdots j_m})_{j_1,\ldots,j_m=1}^{\infty}$ in ℓ_2.

The optimal constants A_q of the Khinchine inequality (these constants are due to U. Haagerup [72]) are:

- $A_q=\sqrt{2}\left(\frac{\Gamma\left(\frac{1+q}{2}\right)}{\sqrt{\pi}}\right)^{\frac{1}{q}}$ if $q>q_0\cong 1.8474$;

- $A_q=2^{\frac{1}{2}-\frac{1}{q}}$ if $q<q_0$.

The definition of the number q_0 above is the following: $q_0\in(1,2)$ is the unique real number with

$$\Gamma\left(\frac{p_0+1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

For complex scalars, using Steinhaus variables instead of Rademacher functions it is well known that a similar inequality holds, but with better constants (see [81, 124]). In this case the optimal constant is

- $A_q=\Gamma\left(\frac{q+2}{2}\right)^{\frac{1}{q}}$ if $q\in[1,2]$.

The notation of the constant A_q shown above will be employed throughout thesis.

Using the argument introduced in [29, Theorem 4] we present a variant of result by Boas, that first appeared in [5, Lemma 6.1], and that is proved in [1].

Kahane–Salem–Zygmund's inequality. *Let $m,n\ge 1$, $p_1,...,p_m\in[1,+\infty]^m$ and, for $p\ge 1$, define*

$$\alpha(p)=\begin{cases}\frac{1}{2}-\frac{1}{p}, & \text{if } p\ge 2;\\ 0, & \text{otherwise.}\end{cases}$$

Then there exists a m-linear map $A:\ell_{p_1}^n\times\cdots\times\ell_{p_m}^n\to\mathbb{K}$ of the form

$$A(z_1,\ldots,z_m)=\sum_{j_1,\ldots,j_m=1}^{n}\epsilon_{j_1\ldots j_m}z_{j_1}^1\cdots z_{j_m}^d$$

with $\epsilon_{j_1\cdots j_m}\in\{-1,1\}$, such that

$$\|A\|\le C_m\cdot n^{\frac{1}{2}+\alpha(p_1)+\cdots+\alpha(p_m)} \tag{2}$$

where $C_m=(m!)^{1-\frac{1}{\min\{p,2\}}}\sqrt{32m\log(6m)}$ and $p=\max\{p_1,\ldots,p_m\}$.

The essence of the Kahane–Salem–Zygmund inequalities probably appeared for the first time in [79], but our approach follows the lines of Boas' paper [29]. Paraphrasing Boas, the Kahane–Salem–Zygmund inequalities use probabilistic methods to construct a homogeneous polynomial (or multilinear operator) with a relatively small supremum norm but relatively large majorant function (we refer [1, Appendix B] for a more detailed study of the Kahane–Salem–Zygmund inequalities).

Introduction

Part I: On the Bohnenblust–Hille and Hardy–Littlewood inequalities

To solve a problem posed by P.J. Daniell, Littlewood [84] proved in 1930 his famous 4/3-inequality, which asserts that

$$\left(\sum_{i,j=1}^{\infty} |T(e_i, e_j)|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\,\|U\|$$

for every continuous bilinear form $T : c_0 \times c_0 \to \mathbb{K}$. One year later, and due to his interest in solving a long standing problem on Dirichlet series, H.F. Bohnenblust and E. Hille proved in *Annals of Mathematics* (see [32]) a generalization of Littlewood's 4/3 inequality to m-linear forms: there exists a (optimal) constant $B_{\mathbb{K},m}^{\text{mult}} \geq 1$ such that for all continuous m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$, and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \|T\|\,.$$

The problem was posed by H. Bohr and consisted in determining the width of the maximal strips on which a Dirichlet series can converge absolutely but non uniformly. More precisely, for a Dirichlet series $\sum_n a_n n^{-s}$, Bohr defined

$$\sigma_a = \inf\left\{r : \sum_n a_n n^{-s} \text{ converges for } Re(s) > r\right\},$$

$$\sigma_u = \inf\left\{r : \sum_n a_n n^{-s} \text{ converges uniformly in } Re\,(s) > r + \varepsilon \text{ for every } \varepsilon > 0\right\},$$

and

$$S := \sup\left\{\sigma_a - \sigma_u\right\}.$$

Bohr's question asked for the precise value of S. The answer came from H.F. Bohnenblust and E. Hille (1931):

$$S = 1/2.$$

The main tool is the, by now, so-called Bohnenblust–Hille inequality. The precise growth of the constants $B_{\mathbb{K},m}^{\text{mult}}$ is important for applications and is nowadays a challenging problem in Mathematical Analysis. For real scalars, the estimates of $B_{\mathbb{R},m}^{\text{mult}}$ are important in Quantum Information Theory (see [92]). In the last years a series of papers related to the Bohnenblust–Hille inequality have been published and several advances were achieved (see [5, 52, 55, 58, 104, 112, 119] and the references therein). Only very recently, in [23, 104] it was shown that the constants $B_{\mathbb{K},m}^{\text{mult}}$ have a subpolynomial growth, which is *quite surprising* because all previous estimates (from 1931 up to 2011) predicted an exponential growth. For real scalars, in 2014 (see [65]) it was shown that the optimal constant for $m = 2$ is $\sqrt{2}$ and in general $B_{\mathbb{R},m}^{\text{mult}} \geq 2^{1-\frac{1}{m}}$. In the case of complex scalars it is still an open problem whether the optimal constants are strictly greater than 1.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality (see [5, 32] and the references therein) ensures that, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_∞^n given by

$$P(x_1, ..., x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{pol}} \|P\|$$

for some constant $B_{\mathbb{K},m}^{\text{pol}} \geq 1$ which does not depend on n (the exponent $\frac{2m}{m+1}$ is optimal), where $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$. The search of precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\text{pol}}$ is crucial for different applications and remains an important open problem (see [23] and the references therein). For real scalars, it was shown in [45] that the hypercontractivity of $B_{\mathbb{R},m}^{\text{pol}}$ is optimal. For complex scalars the behavior of $B_{\mathbb{K},m}^{\text{pol}}$ is still unknown. Moreover, in the complex scalar case, having good estimates for $B_{\mathbb{C},m}^{\text{pol}}$ is crucial in applications in Complex Analysis and Analytic Number Theory (see [55]); for instance, the subexponentiality of the constants of the polynomial version of the Bohnenblust–Hille inequality (complex scalars case) was recenly used in [23] in order to obtain the asymptotic growth of the Bohr radius of the n-dimensional polydisk. More precisely, according to Boas and Khavinson [31], the Bohr radius K_n of the n-dimensional polydisk is the largest positive number r such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|.$$

The Bohr radius K_1 was estimated by H. Bohr, and it was later shown (independently) by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$ (see [31, 33] and the references therein). For $n \geq 2$, exact values of K_n are unknown. In [23], the subexponentiality of the constants of the complex polynomial version of the Bohnenblust–Hille inequality was proved and using this fact it was finally proved that

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{\frac{\log n}{n}}} = 1,$$

solving a challenging problem that many researchers have been struggling for several years.

The Hardy-Littlewood inequality is a natural generalization of the Bohnenblust–Hille inequality for ℓ_p spaces. The bilinear case was proved by Hardy and Littlewood in 1934 (see [73]) and in 1981 it was extended to multilinear operators by Praciano-Pereira (see [118]). More precisely, the classical Hardy–Littlewood inequality asserts that for $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ there exists a (optimal) constant $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ such that, for all positive integers n and all continuous m-linear forms $T : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$,

$$\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}} \right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \|T\| .$$

When $\left|\frac{1}{\mathbf{p}}\right| = 0$ (or equivalently $p_1 = \cdots = p_m = \infty$) since $\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|} = \frac{2m}{m+1}$, we recover the classical Bohnenblust–Hille inequality (see [32]).

When replacing ℓ^n_∞ by ℓ^n_p the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality. More precisely, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ^n_p with $2m \leq p \leq \infty$ given by $P(x_1,\dots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$, then there is a constant $C^{\text{pol}}_{\mathbb{K},m,p} \geq 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C^{\text{pol}}_{\mathbb{K},m,p} \|P\| ,$$

and $C^{\text{pol}}_{\mathbb{K},m,p}$ does not depend on n, where $\|P\| := \sup_{z \in B_{\ell^n_p}} |P(z)|$.

When $p = \infty$ we recover the polynomial Bohnenblust–Hille inequality. Using the generalized Kahane–Salem–Zygmund inequality (2) (see, for instance, [5]) we can verify that the exponents in the above inequalities are optimal.

The precise estimates of the constants of the Hardy–Littlewood inequalities are unknown and even its asymptotic growth is a mystery (as it happens with the Bohnenblust–Hille inequality).

Very recently an extended version of the Hardy–Littlewood inequality was presented in [5] (see also [63]). Let $X_p := \ell_p$ (for $1 \leq p < \infty$) and also $X_\infty := c_0$.

Theorem 0.1 (Generalized Hardy–Littlewood inequality for $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ [5])**.** *Let* $\mathbf{p} := (p_1,\dots,p_m) \in [1,+\infty]^m$ *such that* $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$. *Let also* $\mathbf{q} := (q_1,\dots,q_m) \in \left[\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^m$. *The following are equivalent:*

(1) *There is a (optimal) constant* $C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}}\|T\|$$

for all continuous m-linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$.

For the case $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$ there is also a version of the multilinear Hardy–Littlewood inequality, which is an immediate consequence of Theorem 1.2 from [4] (see also [63]).

Theorem 0.2 (Hardy–Littlewood inequality for $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$)**.** *Let* $m \geq 1$ *and* $\mathbf{p} = (p_1,\ldots,p_m) \in [1,\infty]^m$ *be such that* $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$. *Then there is a (optimal) constant* $D_{\mathbb{K},m,\mathbf{p}}^{\text{mult}} \geq 1$ *such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|T(e_{i_1},\ldots,e_{i_m})\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}\|T\|$$

for every continuous m-linear operator $T : \ell_{p_1}^N \times \cdots \times \ell_{p_m}^N \to \mathbb{K}$. *Moreover, the exponent* $\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}$ *is optimal.*

In this part of the work, we investigate the behavior of the constants $C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}}$, $D_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}$ (Chapter 1) and $C_{\mathbb{K},m,p}^{\text{pol}}$ (Chapter 3). In Chapter 2 we answer, for $1 \leq p \leq m$, the question on how the Hardy–Littlewood multilinear inequalities behave if we replace the exponents $2mp/(mp+p-2m)$ and $p/(p-m)$ by a smaller value r (see Theorem 2.1). This case ($1 \leq p \leq m$) was only explored for the case of Hilbert spaces ($p = 2$, see [37, Corollary 5.20] and [51]) and the case $p = \infty$ was explored in [46].

Part II: Summability of multilinear operators

In 1950, A. Dvoretzky and C. A. Rogers [66] solved a long standing problem in Banach Space Theory when they proved that in every infinite-dimensional Banach space there exists an unconditionally convergent series which is not absolutely convergent. This result is the answer to Problem 122 of the Scottish Book [88], addressed by S. Banach in [21, page 40]). It was the starting point of the theory of absolutely summing operators.

A. Grothendieck, in [71], presented a different proof of the Dvoretzky-Rogers theorem and his "Résumé de la théorie métrique des produits tensoriels topologiques" brought many illuminating insights to the theory of absolutely summing operators.

The notion of absolutely p-summing linear operators is credited to A. Pietsch [117] and the notion of (q,p)-summing operator is credited to B. Mitiagin and A. Pełczyński [91]. In 1968 J. Lindenstrauss and A. Pełczyński's seminal paper [83], re-wrote Grothendieck's Résumé in a more comprehensive form, putting the subject in the spotlight. In 2003, Matos [86] and, independently, Bombal, Pérez-García and Villanueva [34] introduced a more general notion of absolutely summing operators called multiple summing multilinear operators, which has gained special attention, being considered by several authors as the most important multilinear generalization of absolutely summing operators: let $1 \leq p_1, \ldots, p_m \leq q < \infty$. A bounded m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is multiple $(q; p_1, \ldots, p_m)$-summing if there exists $C_m > 0$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left\| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)} \right) \right\|^q \right)^{\frac{1}{q}} \leq C_m \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k}$$

for every $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^w(E_k)$, $k = 1, \ldots, m$. The class of all multiple $(q; p_1, \ldots, p_m)$-summing operators from $E_1 \times \cdots \times E_m$ to F will be denoted by $\Pi^m_{\mathrm{mult}(q;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F)$.

The roots of the subject could probably be traced back to 1930, when Littlewood [84] proved his famous 4/3-inequality to solve a problem posed by P.J. Daniell. One year later, interested in solving a long standing problem on Dirichlet series, H.F. Bohnenblust and E. Hille generalized Littlewood's 4/3 inequality to m-linear forms. Using that $\mathcal{L}(c_0; E)$ is isometrically isomorphic to $\ell_1^w(E)$ (see [62]), the Bohnenblust–Hille inequality can be interpreted as the beginning of the notion of multiple summing operators, because in the modern terminology, the classical Bohnenblust–Hille inequality [32] ensures that, for all $m \geq 2$ and all Banach spaces $E_1, ..., E_m$,

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};1,\ldots,1\right)}(E_1, ..., E_m; \mathbb{K}).$$

In Chapter 4, we prove that, if $1 < s < p^*$, the set $(\mathcal{L}(^m\ell_p; \mathbb{K}) \smallsetminus \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};s\right)}(^m\ell_p; \mathbb{K})) \cup \{0\}$ contains a closed infinite-dimensional Banach space with the same dimension of $\mathcal{L}(^m\ell_p; \mathbb{K})$. As a consequence, we observe, for instance, a new optimal component of the Bohnenblust–Hille inequality: the terms 1 from the tuple $\left(\frac{2m}{m+1}; 1, ..., 1\right)$ is also optimal. Moreover, we generalize a result related to cotype (from 2010) credited to G. Botelho, C. Michels, and D. Pellegrino, and we investigate the optimality of coincidence results for multiple summing operators in c_0 and in the framework of absolutely summing multilinear operators. As a result, we observe that the Defant–Voigt theorem is optimal. In Chapter 5 we present a new class of summing multilinear operators, which recovers the class of absolutely (and multiple) summing operators.

Part III: Classical inequalities for polynomials on circle sectors

The study of low dimensional spaces of polynomials can be an interesting source of examples and counterexamples related to more general questions. In this chapter, we mind 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$. The space of such polynomials is represented by $\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$. This chapter can be seen as a continuation of [77] and [93]. Other publications within the same direction of research are [69, 70, 94, 95, 96, 98].

In order to obtain sharp polynomial inequalities in $\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$ we will use the so called Krein-Milman approach, which is based on the fact that norm attaining convex functions attain their norm at an extreme point of their domain.

Let us describe now the four inequalities that will be studied in this chapter. Section 6.1 is devoted to obtain a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Namely, for a fixed $(x,y) \in D\left(\frac{\pi}{4}\right)$, we find the best (smallest) constant $\Phi(x,y)$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. Similarly, we also obtain a Markov global estimate on the gradient of polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, or in other words, the smallest constant $M>0$ in the inequality

$$\|\nabla P(x,y)\|_2 \leq M\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y) \in D\left(\frac{\pi}{4}\right)$. It is necessary to mention that the study of Bernstein and Markov type inequalities has a longstanding tradition. The interested reader can find further information on this classical topic in [28, 74, 75, 82, 89, 90, 97, 100, 122, 123, 125, 127].

In Section 6.2 we find the smallest constant $K>0$ in the inequality

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq K\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where P is an arbitrary polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and L is the polar of P. Observe that here $\|L\|_{D\left(\frac{\pi}{4}\right)}$ stands for the supremum norm of L over $D\left(\frac{\pi}{4}\right)^2$. Hence, what we do is to provide the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. The calculation of polarization constants in various polynomial spaces is largely motivated by the extensive research on the topic (for examples, you can observe at [64, 75, 85, 121]).

Finally, Section 6.3 focuses on obtaining the smallest constant $C>0$ in the inequality

$$\||P|\|_{D\left(\frac{\pi}{4}\right)} \leq C\|P\|_{D\left(\frac{\pi}{4}\right)}, \tag{3}$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $|P|$ is the modulus of P, i.e., if $P(x,y) = ax^2 + by^2 + cxy$, then $|P|(x,y) = |a|x^2 + |b|y^2 + |c|xy$. The constant C turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. It is interesting to note that (in 1914) H. Bohr [33] studied this type of inequalities for infinite complex power series. The study of Bohr radii is nowadays a fruitful field (see for instance [23, 29, 54, 56, 57, 59]). It can be observed that the relationship between unconditional constants in polynomial spaces and inequalities of the type (3) was already noticed in [56].

Part I

On the Bohnenblust–Hille and Hardy–Littlewood inequalities

Chapter 1

The multilinear Bohnenblust–Hille and Hardy–Littlewood inequalities

In this chapter we present the results from the following research papers:

[12] G. Araújo, and D. Pellegrino, *Lower bounds for the constants of the Hardy-Littlewood inequalities*, Linear Algebra Appl. **463** (2014), 10-15.

[13] G. Araújo, and D. Pellegrino, *On the constants of the Bohnenblust-Hille and Hardy–Littlewood inequalities*, arXiv:1407.7120 [math.FA].

[16] G. Araújo, D. Pellegrino and D.D.P. Silva e Silva, *On the upper bounds for the constants of the Hardy-Littlewood inequality*, J. Funct. Anal. **267** (2014), no. 6, 1878-1888.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $m \geq 2$ be a positive integer. In 1931, F. Bohnenblust and E. Hille (see [32]) proved in the *Annals of Mathematics* that there exists a (optimal) constant $B_{\mathbb{K},m}^{\text{mult}} \geq 1$ such that for all continuous m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$, and all positive integers n,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \left\| T \right\|. \tag{1.1}$$

The precise growth of the constants $B_{\mathbb{K},m}^{\text{mult}}$ is important for many applications (see, e.g., [92]) and remains a big open problem. Only very recently, in [23, 104] it was shown that the constants have a subpolynomial growth. For real scalars (2014, see [65]) it was shown that the optimal constant for $m = 2$ is $\sqrt{2}$ and in general $B_{\mathbb{R},m}^{\text{mult}} \geq 2^{1-\frac{1}{m}}$. In the case of complex scalars it is still an open problem whether the optimal constants are strictly grater than 1; in the polynomial case, in 2013 D. Núñez-Alarcón proved that the complex constants are strictly greater than 1 (see [101]). Even basic questions related to the constants $B_{\mathbb{K},m}^{\text{mult}}$ remain unsolved. For instance:

- Is the sequence of optimal constants $\left(B_{\mathbb{K},m}^{\text{mult}}\right)_{m=1}^{\infty}$ increasing?
- Is the sequence of optimal constants $\left(B_{\mathbb{K},m}^{\text{mult}}\right)_{m=1}^{\infty}$ bounded?

- Is $B^{\text{mult}}_{\mathbb{C},m} = 1$?

The best known estimates for the constants in (1.1), which are recently presented in [23], are ($B^{\text{mult}}_{\mathbb{K},1} = 1$ is obvious)

$$B^{\text{mult}}_{\mathbb{K},m} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}},$$

where $A_{\frac{2j-2}{j}}$ are the respective constants of the Khnichine inequality, i.e.,

$$\begin{aligned}
B^{\text{mult}}_{\mathbb{C},m} &\leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}, && \\
B^{\text{mult}}_{\mathbb{R},m} &\leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, && \text{for } m \geq 14, \\
B^{\text{mult}}_{\mathbb{R},m} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, && \text{for } 2 \leq m \leq 13.
\end{aligned} \tag{1.2}$$

In a more friendly presentation the above formulas tell us that the growth of the constants $\mathrm{B}^{\text{mult}}_{\mathbb{K},m}$ is subpolynomial (in fact, sublinear) since, from the above estimates it can be proved that (see [23])

$$\begin{aligned}
B^{\text{mult}}_{\mathbb{C},m} &< m^{\frac{1-\gamma}{2}} < m^{0.21139}, \\
B^{\text{mult}}_{\mathbb{R},m} &< 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} < 1.3 \cdot m^{0.36482},
\end{aligned}$$

where γ denotes the Euler–Mascheroni constant. The above estimates are *quite surprising* because all previous estimates (from 1931 up to 2011) predicted an exponential growth. It was only in 2012, with [112] (motivated by [58]), when the perspective on the subject changed entirely.

The Hardy-Littlewood inequality is a natural generalization of the Bohnenblust–Hille inequality to ℓ_p spaces. More precisely, the classical Hardy–Littlewood inequality asserts that for $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ there exists a (optimal) constant $C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ such that, for all positive integers n and all continuous m-linear forms $T : \ell^n_{p_1} \times \cdots \times \ell^n_{p_m} \to \mathbb{K}$,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}}\right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \|T\|. \tag{1.3}$$

Using the generalized Kahane-Salem-Zygmund inequality (2) (see [5]) one can easily verify that the exponents $\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}$ are optimal. When $\left|\frac{1}{\mathbf{p}}\right| = 0$ (or equivalently $p_1 = \cdots = p_m = \infty$) since $\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|} = \frac{2m}{m+1}$, we recover the classical Bohnenblust–Hille inequality (see [32]).

The precise estimates of the constants of the Hardy–Littlewood inequalities are unknown and even its asymptotic growth is a mystery (as it happens with the Bohnenblust–

Hille inequality). The original estimates for $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}}$ (see [5]) were of the form

$$C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}} \leq \left(\sqrt{2}\right)^{m-1}. \tag{1.4}$$

Very recently an extended version of the Hardy–Littlewood inequality was presented in [5] (see also [63]). Consider $X_p := \ell_p$, for $1 \leq p < \infty$, and also $X_\infty := c_0$.

Theorem 1.1 (Generalized Hardy–Littlewood inequality for $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ [5])**.** *Let* $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$ *such that* $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$. *Let also* $\mathbf{q} := (q_1, \ldots, q_m) \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^m$. *The following are equivalent:*

(1) There is a (optimal) constant $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}} \geq 1$ *such that*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left|T\left(e_{j_1}, \ldots, e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}}\|T\|$$

for all continuous m*-linear forms* $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$.

(2) $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$.

Some particular cases of $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p},\mathbf{q}}$ will be used throughout this chapter, therefore, we will establish notations for the (optimal) constants in some special cases:

- If $p_1 = \cdots = p_m = \infty$ we recover the generalized Bohnenblust–Hille inequality and we will denote $C^{\mathrm{mult}}_{\mathbb{K},m,(\infty,\ldots,\infty),\mathbf{q}}$ by $B^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{q}}$. Moreover, if $q_1 = \cdots = q_m = \frac{2m}{m+1}$ we recover the classical Bohnenblust–Hille inequality and we will denote $B^{\mathrm{mult}}_{\mathbb{K},m,\left(\frac{2m}{m+1},\ldots,\frac{2m}{m+1}\right)}$ by $B^{\mathrm{mult}}_{\mathbb{K},m}$;
- If $q_1 = \cdots = q_m = \frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}$ we recover the classical Hardy–Littlewood inequality and we will denote $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p},\left(\frac{2m}{m+1-2|1/\mathbf{p}|},\ldots,\frac{2m}{m+1-2|1/\mathbf{p}|}\right)}$ by $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}}$. Moreover, if $p_1 = \cdots = p_m = p$ we will denote $C^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}}$ by $C^{\mathrm{mult}}_{\mathbb{K},m,p}$.

For the case $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$ there is also a version of the multilinear Hardy–Littlewood inequality, which is an immediate consequence of Theorem 1.2 from [4] (see also [63]).

Theorem 1.2 (Hardy–Littlewood inequality for $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$)**.** *Let* $m \geq 1$ *and* $\mathbf{p} = (p_1, \ldots, p_m) \in [1, \infty]^m$ *be such that* $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$. *Then there is a (optimal) constant* $D^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}} \geq 1$ *such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|T(e_{i_1}, \ldots, e_{i_m})\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D^{\mathrm{mult}}_{\mathbb{K},m,\mathbf{p}}\|T\|$$

for every continuous m*-linear operator* $T : \ell^N_{p_1} \times \cdots \times \ell^N_{p_m} \to \mathbb{K}$. *Moreover, the exponent* $\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}$ *is optimal.*

The best known upper bounds for the constants on the previous result are $D^{\text{mult}}_{\mathbb{R},m,\mathbf{p}} \leq \left(\sqrt{2}\right)^{m-1}$ and $D^{\text{mult}}_{\mathbb{C},m,\mathbf{p}} \leq \left(2/\sqrt{\pi}\right)^{m-1}$ (see [4, 63]).

We will only deal with this second case of the Hardy–Littlewood inequality (for $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$) in Chapter **??**. Again we will establish notations for the (optimal) constants $D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ in some special cases:

- When $p_1 = \cdots = p_m = p$ we denote $D^{\text{mult}}_{\mathbb{K},m,\mathbf{p}}$ by $D^{\text{mult}}_{\mathbb{K},m,p}$.

Our main contributions regarding the constants of the multilinear case of the Hardy–Littlewood inequality can be summarized in the following result, which is a direct consequence of the forthcoming sections 1.1 and 1.2.

Theorem 1.3. *Let $m \geq 2$ and let $\sigma_{\mathbb{R}} = \sqrt{2}$ and $\sigma_{\mathbb{C}} = 2/\sqrt{\pi}$. Then,*

(1) Let $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$ such that $\left|\frac{1}{\mathbf{q}}\right| = \frac{m+1}{2}$ and $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then

$$B^{\text{mult}}_{\mathbb{K},m,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

(2) $C^{\text{mult}}_{\mathbb{R},m,p} \geq 2^{\frac{mp+2m-2m^2-p}{mp}}$ for $2m < p \leq \infty$ and $C^{\text{mult}}_{\mathbb{R},m,2m} > 1$.

(3) (i) For $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$,

$$C^{\text{mult}}_{\mathbb{K},m,\mathbf{p}} \leq (\sigma_{\mathbb{K}})^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B^{\text{mult}}_{\mathbb{K},m}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}.$$

In particular, $\left(C^{\text{mult}}_{\mathbb{K},m,p}\right)_{m=1}^{\infty}$ is sublinear if $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{m}$.

(ii) For $2m^3 - 4m^2 + 2m < p \leq \infty$,

$$C^{\text{mult}}_{\mathbb{K},m,p} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

(4) Let $2m < p \leq \infty$ and let $\mathbf{q} := (q_1, ..., q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ such that $\left|\frac{1}{\mathbf{q}}\right| = \frac{mp+p-2m}{2p}$. If $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then

$$C^{\text{mult}}_{\mathbb{K},m,p,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

Note that, for instance, if $2m^3 - 4m^2 + 2m < p \leq \infty$, the formula of item *(3)(ii)* is not dependent on p, contrary to what happens in item *(3)(i)*, where we can see a dependence on p but, paradoxically, it is *worse* than the formula from item *(3)(ii)*. This suggests the following problems:

- Are the optimal constants of the Bohnenblust–Hille and Hardy–Littlewood inequalities the same?

- Are the optimal constants of the Hardy–Littlewood inequality independent of p (at least for large p)?

Several advances and improvements have been obtained by various authors in this context. We can highlight and summarize these findings in the following remarks:

Remark 1.4. *D. Pellegrino proved in [111] that*

$$B^{\text{mult}}_{\mathbb{R},m,(1,2,...,2)} = (\sqrt{2})^{m-1} \tag{1.5}$$

and, for $i \in \{1, ..., m\}$,

$$B^{\text{mult}}_{\mathbb{R},m,\left(\frac{2(m-1)q_i}{(m+1)q_i-2},...,\frac{2(m-1)q_i}{(m+1)q_i-2},q_i,\frac{2(m-1)q_i}{(m+1)q_i-2},...,\frac{2(m-1)q_i}{(m+1)q_i-2}\right)} \geq 2^{\frac{2m-q_im-4+3q_i}{2q_i}},$$

with $q_i \in [1,2]$ *in the i-th position. In [44], when* $i = m$*, the above estimate it was improved for J. Campos, W. Cavalcante, V.V. Fávaro, D. Núñez-Alarcón, D. Pellegrino and D.M. Serrano-Rodríguez to*

$$B^{\text{mult}}_{\mathbb{R},m,\left(\frac{2(m-1)q_m}{(m+1)q_m-2},...,\frac{2(m-1)q_m}{(m+1)q_m-2},q_m\right)} \geq 2^{\frac{3q_mm-2m-5q_m+4}{2q_m(m-1)}}.$$

In particular, it was possible to conclude that

$$B^{\text{mult}}_{\mathbb{R},3,(4/3,4/3,2)} = B^{\text{mult}}_{\mathbb{R},3,(4/3,8/5,8/5)} = B^{\text{mult}}_{\mathbb{R},3,(4/3,2,4/3)} = 2^{3/4}.$$

D. Pellegrino and D.M. Serrano-Rodríguez proved in [113] the following (in some sense) more general result: if $m \geq 2$ *is a positive integer, and* $\mathbf{q} = (q_1, ..., q_m) \in [1,2]^m$ *are such that* $|1/\mathbf{q}| = (m+1)/2$*, then, for* $j = 1, 2$,

$$B^{\text{mult}}_{\mathbb{R},m,\mathbf{q}} \geq 2^{\frac{(m-1)(1-q_j)\hat{q}_j+\sum_{i\neq j}^{m}\hat{q}_i}{q_1\cdots q_m}},$$

with $\hat{q}_i = \frac{q_1\cdots q_m}{q_i}$, $i = 1, ..., m$*. In particular, they proved that* (1.5) *also is true for the exponent (2,1,2,...,2).*

Remark 1.5. *Very recently, D. Pellegrino presented*[1] *new lower bounds for the real case of the Hady–Littlewood inequalities, which improve the so far best known lower estimates (item (2) of the previous theorem) and provide a closed formula even for the case* $p = 2m$ *(see [44]). Pellegrino's approach is very interesting because even with a simple argument, he "finds an overlooked connection between the Clarkson's inequalities and Hardy–Littlewood's constants which helps to find analytical lower estimates for these constants". More precisely, using Clarkson's inequalities, D. Pellegrino proved that for* $m \geq 2$ *and* $p \geq 2m$*, we have*

$$C^{\text{mult}}_{\mathbb{R},m,p} \geq \frac{2^{\frac{2mp+2m-p-2m^2}{mp}}}{\sup_{x\in[0,1]} \frac{((1+x)^{p^*}+(1-x)^{p^*})^{1/p^*}}{(1+x^p)^{1/p}}}.$$

Remark 1.6. *If* $\mathbf{p} = (p, ..., p)$ *in Theorem 1.3 (3)(i) we have the following estimate for* $C^{\text{mult}}_{\mathbb{K},m,p}$ *with* $2m \leq p \leq 2m^3 - 4m^2 + 2m$:

$$C^{\text{mult}}_{\mathbb{K},m,p} \leq (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} \left(B^{\text{mult}}_{\mathbb{K},m}\right)^{\frac{p-2m}{p}}. \tag{1.6}$$

[1]The original paper that D. Pellegrino presented the new lower bounds for the real case of the Hardy–Littlewood inequalities has been withdrawn by the author (see [105]). This arXiv preprint is now incorporated to [44].

Very recently, D. Pellegrino in [106] proved that, for $m \geq 3$ *and* $2m \leq p \leq 2m^3 - 4m^2 + 2m$, *we can improve* (1.6) *to*

$$C_{\mathbb{K},m,p}^{\text{mult}} \leq (B_{\mathbb{K},m}^{\text{mult}})^{(m-1)\left(\frac{2m-p+mp-2m^2}{m^2p-2mp}\right)} (\sigma_{\mathbb{K}})^{\frac{p-2m-mp+6m^2-6m^3+2m^4}{mp(m-2)}}.$$

When $p = 2m^3 - 4m^2 + 2m$ *this formula coincides with Theorem 1.3 (3)(ii) when* $p \to 2m^3 - 4m^2 + 2m$.

Remark 1.7. *Let* $p_0 \in (1,2)$ *be the unique real number satisfying*

$$\Gamma\left(\frac{p_0+1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

D. Núñez-Alarcón and D. Pellegrino in [102] found the exact value of the constant in the particular case $\mathbb{K} = \mathbb{R}$, $m = 2$, $\mathbf{q} = \left(\frac{p}{p-1}, 2\right)$ *and* $\mathbf{p} = (p, \infty)$ *with* $p \geq \frac{p_0}{p_0-1}$. *More precisely, they showed that*

$$C_{\mathbb{R},2,(p,\infty),\left(\frac{p}{p-1},2\right)}^{\text{mult}} = 2^{\frac{1}{2}-\frac{1}{p}}$$

whenever $p \geq \frac{p_0}{p_0-1}$. *For* $2 < p < p_0$, *they found almost optimal constants, with better precision than* 4×10^{-4}.

Remark 1.8. *D. Pellegrino proved in [109] that for* $m \geq 2$, $p \geq 2m$ *and* $\mathbf{q} := (q_1, ..., q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ *such that* $\left|\frac{1}{\mathbf{q}}\right| = \frac{mp+p-2m}{2p}$ *and* $\max q_i \geq \frac{2m^2-4m+2}{m^2-m-1}$, *we have*

$$C_{\mathbb{K},m,p,\mathbf{q}}^{\text{mult}} \leq (\sigma_{\mathbb{K}})^{(m-1)\left(1-\frac{(m+1)(2-\max q_i)(m-1)^2}{(m^2-m-2)\max q_i}\right)} \left(\prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}\right)^{\frac{(m+1)(2-\max q_i)(m-1)^2}{(m^2-m-2)\max q_i}}. \tag{1.7}$$

The estimates (1.7) *behaves continuously when compared with Theorem 1.3 (4)(i).*

1.1 Lower and upper bounds for the constants of the classical Hardy–Littlewood inequality

From [23, 104] we know that $B_{\mathbb{K},m}^{\text{mult}}$ has a subpolynomial growth. On the other hand, the best known upper bounds for the constants $C_{\mathbb{K},m,p}^{\text{mult}}$ are $\left(\sqrt{2}\right)^{m-1}$ (see [4, 5, 63]). In this section we show that $\left(\sqrt{2}\right)^{m-1}$ can be improved to

$$\begin{aligned} C_{\mathbb{C},m,p}^{\text{mult}} &\leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2m(m-1)}{p}} \left(B_{\mathbb{C},m}^{\text{mult}}\right)^{\frac{p-2m}{p}}, \\ C_{\mathbb{R},m,p}^{\text{mult}} &\leq \left(\sqrt{2}\right)^{\frac{2m(m-1)}{p}} \left(B_{\mathbb{R},m,p}^{\text{mult}}\right)^{\frac{p-2m}{p}}. \end{aligned} \tag{1.8}$$

These estimates are quite better than $\left(\sqrt{2}\right)^{m-1}$ because $B_{\mathbb{K},m}^{\text{mult}}$ is sublinear. Moreover, our estimates depend on p and m and catch more subtle information since now it is clear that the estimates improve as p grows. As p goes to infinity we note that the above estimates tend to the best known estimates for $B_{\mathbb{K},m}^{\text{mult}}$ (see (1.2)) and, for instance, if $p \geq m^2$ we conclude that $\left(C_{\mathbb{K},m,p}^{\text{mult}}\right)_{m=1}^{\infty}$ has a subpolynomial growth. One of our main result in this section is the following:

Theorem 1.9. *Let $m \geq 2$ be a positive integer and $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$. Then, for all continuous m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n, we have*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}}\right)^{\frac{m+1-2\left|\frac{1}{\mathbf{p}}\right|}{2m}} \leq C_{\mathbb{K},m,\mathbf{p}}^{\text{mult}} \|T\| \tag{1.9}$$

with

$$C_{\mathbb{R},m,\mathbf{p}}^{\text{mult}} \leq \left(\sqrt{2}\right)^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B_{\mathbb{R},m}^{\text{mult}}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}$$

and

$$C_{\mathbb{C},m,\mathbf{p}}^{\text{mult}} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{2(m-1)\left|\frac{1}{\mathbf{p}}\right|} \left(B_{\mathbb{C},m}^{\text{mult}}\right)^{1-2\left|\frac{1}{\mathbf{p}}\right|}.$$

In particular, $\left(C_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}\right)_{m=1}^{\infty}$ has a subpolynomial growth if $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{m}$.

Remark 1.10. *If $p_1 = \cdots = p_m = p$ and $2m^3 - 4m^2 + 2m < p \leq \infty$, we already have better information for $C_{\mathbb{K},m,p}^{\text{mult}}$ when compared to the previous theorem (see Theorem 1.16).*

Proof of Theorem 1.9. For the sake of simplicity we shall deal with the case $p_1 = \cdots = p_m = p$. The case $p = \infty$ in (1.9) is precisely the Bohnenblust–Hille inequality, so we just need to consider $2m \leq p < \infty$. Let $\frac{2m-2}{m} \leq s \leq 2$ and

$$\lambda_0 = \frac{2s}{ms + s - 2m + 2}.$$

Since

$$\frac{m-1}{s} + \frac{1}{\lambda_0} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [5]) we know that there is a constant $B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}} \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have, for all $i = 1, \ldots., m$,

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n} \left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_0}\right)^{\frac{1}{\lambda_0}} \leq B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}} \|T\|. \tag{1.10}$$

Above, $\sum_{\widehat{j_i}=1}^{n}$ means the sum over all j_k for all $k \neq i$. If we choose

$$s = \frac{2mp}{mp + p - 2m},$$

we have
$$\lambda_0 < s \le 2.$$

The multiple exponent
$$(\lambda_0, s, s, ..., s)$$
can be obtained by interpolating the multiple exponents $(1, 2..., 2)$ and $\left(\frac{2m}{m+1}, ..., \frac{2m}{m+1}\right)$ with, respectively,
$$\theta_1 = 2\left(\frac{1}{\lambda_0} - \frac{1}{s}\right)$$
$$\theta_2 = m\left(\frac{2}{s} - 1\right),$$
in the sense of [5].

It is thus important to control the constants associated with the multiple exponents $(1, 2..., 2)$ and $\left(\frac{2m}{m+1}, ..., \frac{2m}{m+1}\right)$. The exponent $\left(\frac{2m}{m+1}, ..., \frac{2m}{m+1}\right)$ is the classical exponent of the Bohnenblust–Hille inequality and the estimate of the constant associated with $(1, 2..., 2)$ is well-known (we present the details for the sake of completeness). In fact, in general, for the exponent $\left(\frac{2k}{k+1}, ..., \frac{2k}{k+1}, 2, ..., 2\right)$ (with $\frac{2k}{k+1}$ repeated k times and 2 repeated $m-k$ times), using the multiple Khinchine inequality (1), we have, for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$,

$$\begin{aligned}
&\left(\sum_{j_1,...,j_k=1}^{n}\left(\sum_{j_{k+1},...,j_m=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^2\right)^{\frac{1}{2}\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \\
&\le \left(\sum_{j_1,...,j_k=1}^{n}\left(A_{\frac{2k}{k+1}}^{-(m-k)}\left(\int_{[0,1]^{m-k}}\left|\sum_{j_{k+1},...,j_m=1}^{n} r_{j_{k+1}}(t_{k+1})\cdots r_{j_m}(t_m)\right.\right.\right.\right. \\
&\qquad \left.\left.\left.\left.\times T\left(e_{j_1},...,e_{j_m}\right)\right|^{\frac{2k}{k+1}} dt_{k+1}\cdots dt_m\right)^{\frac{k+1}{2k}}\right)^{\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \\
&= A_{\frac{2k}{k+1}}^{-(m-k)}\left(\sum_{j_1,...,j_k=1}^{n}\int_{[0,1]^{m-k}}\left(T\left(e_{j_1},...,e_{j_k},\sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1})e_{j_{k+1}},...,\right.\right.\right. \\
&\qquad \left.\left.\left.\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right\|^{\frac{2k}{k+1}} dt_{k+1}\cdots dt_m\right)^{\frac{k+1}{2k}} \\
&= A_{\frac{2k}{k+1}}^{-(m-k)}\left(\int_{[0,1]^{m-k}}\sum_{j_1,...,j_k=1}^{n}\left(T\left(e_{j_1},...,e_{j_k},\sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1})e_{j_{k+1}},...,\right.\right.\right. \\
&\qquad \left.\left.\left.\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right|^{\frac{2k}{k+1}} dt_{k+1}\cdots dt_m\right)^{\frac{k+1}{2k}} \\
&\le A_{\frac{2k}{k+1}}^{-(m-k)} \sup_{t_{k+1},...,t_m\in[0,1]} B_{\mathbb{K},k}^{\mathrm{mult}}\left\|T\left(\,\cdot\,,...,\,\cdot\,,\sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1})e_{j_{k+1}},...,\sum_{j_m=1}^{n} r_{j_m}(t_m)e_{j_m}\right)\right\| \\
&= A_{\frac{2k}{k+1}}^{-(m-k)} B_{\mathbb{K},k}^{\mathrm{mult}}\left\|T\right\|.
\end{aligned}$$

So, choosing $k = 1$, since $A_1 = \left(\sqrt{2}\right)^{-1}$ and $B_{\mathbb{K},1}^{\mathrm{mult}} = 1$ we conclude that the constant associated with the multiple exponent $(1, 2, ..., 2)$ is $\left(\sqrt{2}\right)^{m-1}$.

Therefore, the optimal constant associated with the multiple exponent

$$(\lambda_0, s, ..., s)$$

is (for real scalars) less than or equal to

$$\left(\left(\sqrt{2}\right)^{m-1}\right)^{2\left(\frac{1}{\lambda_0}-\frac{1}{s}\right)}\left(B_{\mathbb{R},m}^{\text{mult}}\right)^{m\left(\frac{2}{s}-1\right)}$$

i.e.,

$$B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}} \leq \left(\sqrt{2}\right)^{\frac{2m(m-1)}{p}}\left(B_{\mathbb{R},m}^{\text{mult}}\right)^{\frac{p-2m}{p}}. \tag{1.11}$$

More precisely, (1.10) is valid with $B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}}$ as above. For complex scalars we can use the Khinchine inequality for Steinhaus variables and replace $\sqrt{2}$ by $\frac{2}{\sqrt{\pi}}$ as in [103].

Let

$$\lambda_j = \frac{\lambda_0 p}{p-\lambda_0 j}$$

for all $j = 1, ..., m$. Note that

$$\lambda_m = s$$

and that

$$\left(\frac{p}{\lambda_j}\right)^* = \frac{\lambda_{j+1}}{\lambda_j}$$

for all $j = 0, ..., m-1$.

Let us suppose that $1 \leq k \leq m$ and that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}\left|T(e_{j_1}, ..., e_{j_m})\right|^s\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \leq B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}}\|T\|$$

is true for all continuous m-linear forms $T : \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$. Let us prove that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}\left|T(e_{j_1}, ..., e_{j_m})\right|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}} \leq B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}}\|T\|$$

for all continuous m-linear forms $T : \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$.

The initial case (the case $k = 0$) is precisely (1.10) with $B_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}^{\text{mult}}$ as in (1.11).

Consider

$$T \in \mathcal{L}(\underbrace{\ell_p^n, ..., \ell_p^n}_{k \text{ times}}, \ell_\infty^n, ..., \ell_\infty^n; \mathbb{K})$$

and for each $x \in B_{\ell_p^n}$ define

$$\begin{array}{rcl} T^{(x)} & : & \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n \rightarrow \mathbb{K} \\ & & (z^{(1)}, ..., z^{(m)}) \qquad \mapsto T(z^{(1)}, ..., z^{(k-1)}, xz^{(k)}, z^{(k+1)}, ..., z^{(m)}), \end{array}$$

with $xz^{(k)} = (x_j z_j^{(k)})_{j=1}^n$. Observe that

$$\|T\| = \sup\{\|T^{(x)}\| : x \in B_{\ell_p^n}\}.$$

By applying the induction hypothesis to $T^{(x)}$, we obtain

$$\begin{aligned} & \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n \left| T\left(e_{j_1}, ..., e_{j_m}\right)\right|^s \left|x_{j_k}\right|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ & = \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n \left| T\left(e_{j_1}, ..., e_{j_{k-1}}, xe_{j_k}, e_{j_{k+1}}, ..., e_{j_m}\right)\right|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ & = \left(\sum_{j_i=1}^n \left(\sum_{\widehat{j}_i=1}^n \left| T^{(x)}\left(e_{j_1}, ..., e_{j_m}\right)\right|^s \right)^{\frac{1}{s}\lambda_{k-1}} \right)^{\frac{1}{\lambda_{k-1}}} \\ & \leq B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)} \|T^{(x)}\| \\ & \leq B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)} \|T\| \end{aligned} \tag{1.12}$$

for all $i = 1, ..., m$.

We shall analyze two cases:

- $i = k$.

Since

$$\left(\frac{p}{\lambda_{j-1}}\right)^* = \frac{\lambda_j}{\lambda_{j-1}}$$

for all $j = 1, ..., m$, we conclude that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}} \\
&=\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1}\left(\frac{p}{\lambda_{k-1}}\right)^{*}}\right)^{\frac{1}{\lambda_{k-1}}\frac{1}{\left(\frac{p}{\lambda_{k-1}}\right)^{*}}} \\
&=\left\|\left(\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)_{j_k=1}^{n}\right\|_{\left(\frac{p}{\lambda_{k-1}}\right)^{*}}^{\frac{1}{\lambda_{k-1}}} \\
&=\left(\sup_{y\in B_{\ell^{n}_{\frac{p}{\lambda_{k-1}}}}}\sum_{j_k=1}^{n}\left|y_{j_k}\right|\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&=\left(\sup_{x\in B_{\ell^{n}_{p}}}\sum_{j_k=1}^{n}\left|x_{j_k}\right|^{\lambda_{k-1}}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&=\sup_{x\in B_{\ell^{n}_{p}}}\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\left|x_{j_k}\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1}}\right)^{\frac{1}{\lambda_{k-1}}} \\
&\le B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|
\end{aligned}$$

where the last inequality holds by (1.12).

- $i \neq k$.

Let us first suppose that $k \in \{1, ..., m-1\}$. It is important to note that in this case $\lambda_{k-1} < \lambda_k < s$ for all $k \in \{1, ..., m-1\}$. Denoting, for $i = 1, ..., m$,

$$S_i = \left(\sum_{\widehat{j_i}=1}^{n}\left|T(e_{j_1}, ..., e_{j_m})\right|^{s}\right)^{\frac{1}{s}}$$

we get

$$
\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k} = \sum_{j_i=1}^{n} S_i^{\lambda_k} = \sum_{j_i=1}^{n} S_i^{\lambda_k-s}S_i^s \\
&= \sum_{j_i=1}^{n}\sum_{\widehat{j}_i=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_k}} = \sum_{j_k=1}^{n}\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_k}} \\
&= \sum_{j_k=1}^{n}\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^{\frac{s(s-\lambda_k)}{s-\lambda_{k-1}}}}{S_i^{s-\lambda_k}}|T(e_{j_1},...,e_{j_m})|^{\frac{s(\lambda_k-\lambda_{k-1})}{s-\lambda_{k-1}}}.
\end{aligned}
$$

Therefore, using Hölder's inequality twice we obtain

$$
\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k} \\
&\leq \sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{s-\lambda_k}{s-\lambda_{k-1}}}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{\lambda_k-\lambda_{k-1}}{s-\lambda_{k-1}}} \\
&\leq \left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{\lambda_k}{\lambda_{k-1}}}\right)^{\frac{\lambda_{k-1}}{\lambda_k}\cdot\frac{s-\lambda_k}{s-\lambda_{k-1}}} \\
&\qquad\times\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}.
\end{aligned} \tag{1.13}
$$

We know from the case $i=k$ that

$$
\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k}\right)^{\frac{1}{\lambda_k}\cdot\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}} \leq \left(B^{\mathrm{mult}}_{\mathbb{K},m,(\lambda_0,s,...,s)}\|T\|\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}. \tag{1.14}
$$

Now we investigate the first factor in (1.13). From Hölder's inequality and (1.12) it follows

that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\ldots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}\right)^{\frac{\lambda_k}{\lambda_{k-1}}}\right)^{\frac{\lambda_{k-1}}{\lambda_k}}\\
&=\left\|\left(\sum_{\widehat{j_k}}\frac{|T(e_{j_1},\ldots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}\right)_{j_k=1}^{n}\right\|_{\left(\frac{p}{\lambda_{k-1}}\right)^*}\\
&=\sup_{y\in B_{\ell^n_{\frac{p}{\lambda_{k-1}}}}}\sum_{j_k=1}^{n}|y_{j_k}|\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\ldots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}\\
&=\sup_{x\in B_{\ell^n_p}}\sum_{j_k=1}^{n}\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\ldots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1}}}|x_{j_k}|^{\lambda_{k-1}} \qquad (1.15)\\
&=\sup_{x\in B_{\ell^n_p}}\sum_{j_i=1}^{n}\sum_{\widehat{j_i}=1}^{n}\frac{|T(e_{j_1},\ldots,e_{j_m})|^{s-\lambda_{k-1}}}{S_i^{s-\lambda_{k-1}}}|T(e_{j_1},\ldots,e_{j_m})|^{\lambda_{k-1}}|x_{j_k}|^{\lambda_{k-1}}\\
&\le\sup_{x\in B_{\ell^n_p}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\frac{|T(e_{j_1},\ldots,e_{j_m})|^s}{S_i^s}\right)^{\frac{s-\lambda_{k-1}}{s}}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\ldots,e_{j_m})|^s|x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1}}\\
&=\sup_{x\in B_{\ell^n_p}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\ldots,e_{j_m})|^s|x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1}}\le\left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}\|T\|\right)^{\lambda_{k-1}}.
\end{aligned}$$

Replacing (1.14) and (1.15) in (1.13) we finally conclude that

$$\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\ldots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_k}\\
&\le\left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}\|T\|\right)^{\lambda_{k-1}\frac{s-\lambda_k}{s-\lambda_{k-1}}}\left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}\|T\|\right)^{\frac{(\lambda_k-\lambda_{k-1})s}{s-\lambda_{k-1}}}\\
&=\left(B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}\|T\|\right)^{\lambda_k}.
\end{aligned}$$

It remains to consider $k=m$. In this case $\lambda_m=s$ and we have the more simple situation since

$$\begin{aligned}
\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\ldots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_m}\right)^{\frac{1}{\lambda_m}} &= \left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}|T(e_{j_1},\ldots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_m}\right)^{\frac{1}{s}}\\
&\le B^{\text{mult}}_{\mathbb{K},m,(\lambda_0,s,\ldots,s)}\|T\|,
\end{aligned}$$

where the inequality is due to the case $i=k$. This concludes the proof. $\square$

In order present a more concrete formula for the constants of the Hardy–Littlewood inequality and to show that for $p \geq m^2$ these constants have a subpolynomial growth, we need to recall the optimal constants of the Khinchin inequality (1) and the best known constants of the Bohnenblust–Hille inequality.

The best known upper estimates for $B_{\mathbb{R},m}^{\text{mult}}$ and $B_{\mathbb{C},m}^{\text{mult}}$ can be found in [23]:

$$B_{\mathbb{K},m}^{\text{mult}} \leq \prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}.$$

Combining these results we have, for $2m < p < \infty$,

$$C_{\mathbb{R},m,p}^{\text{mult}} \leq \left(2^{\frac{4m^2-pm-2m}{2p-4m}+\frac{446381}{55440}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}} \right)^{\frac{p-2m}{p}} \quad \text{for } m \geq 14,$$

$$C_{\mathbb{R},m,p}^{\text{mult}} \leq \left(\sqrt{2}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^{m} 2^{\frac{1}{2j-2}} \right)^{\frac{p-2m}{p}} \quad \text{for } 2 \leq m \leq 13$$

and

$$C_{\mathbb{C},m,p}^{\text{mult}} \leq \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{\frac{p-2m}{p}}. \tag{1.16}$$

From [23] we know that

$$\begin{aligned} B_{\mathbb{C},m}^{\text{mult}} &< m^{\frac{1-\gamma}{2}} < m^{0.21139}, \\ B_{\mathbb{R},m}^{\text{mult}} &< 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} < 1.3 \cdot m^{0.36482}, \end{aligned}$$

for all m's, where γ is the famous Euler–Mascheroni constant. We, thus, conclude that, if $p \geq m^2$ then $\left(C_{\mathbb{K},m,p}^{\text{mult}}\right)_{m=1}^{\infty}$ has a subpolynomial growth. Similarly, in general we can conclude that $\left(C_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}\right)_{m=1}^{\infty}$ has a subpolynomial growth if $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{m}$.

Now we will provide nontrivial lower bounds for $C_{\mathbb{R},m,p}^{\text{mult}}$. Nowadays the best lower bounds for the constants of the real case of the Hardy–Littlewood inequalities can be founded in [44] (see Remark 1.5), but the next result it was the first in this direction and we will present the proof for the sake of completeness.

Theorem 1.11. *The optimal constants of the Hardy–Littlewood inequalities satisfies*

$$C_{\mathbb{R},m,p}^{\text{mult}} \geq 2^{\frac{mp+2m-2m^2-p}{mp}} > 1 \quad \textit{for} \quad 2m < p \leq \infty,$$

and

$$C_{\mathbb{R},m,2m}^{\text{mult}} > 1.$$

Proof. Following the lines of [65], it is possible to prove that $C_{\mathbb{R},m,p}^{\text{mult}} \geq 2^{\frac{mp+2m-2m^2-p}{mp}} > 1$ for $2m < p \leq \infty$, but note that when $p = 2m$ we have $2^{\frac{mp+2m-2m^2-p}{mp}} = 1$ and thus we do not have nontrivial information.

All that it left to prove is the case $p = 2m$. This first step follows the lines of [65].

For $2m \le p \le \infty$, consider $T_{2,p} : \ell_p^2 \times \ell_p^2 \to \mathbb{R}$ given by $(x^{(1)}, x^{(2)}) \mapsto x_1^{(1)}x_1^{(2)} + x_1^{(1)}x_2^{(2)} + x_2^{(1)}x_1^{(2)} - x_2^{(1)}x_2^{(2)}$ and $T_{m,p} : \ell_p^{2^{m-1}} \times \cdots \times \ell_p^{2^{m-1}} \to \mathbb{R}$ given by $(x^{(1)}, ..., x^{(m)}) \mapsto (x_1^{(m)} + x_2^{(m)})T_{m-1,p}(x^{(1)}, ..., x^{(m)}) + (x_1^{(m)} - x_2^{(m)})T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), B^{2^{m-2}}(x^{(2)}), ..., B^2(x^{(m-1)}))$, where $x^{(k)} = (x_j^{(k)})_{j=1}^{2^{m-1}} \in \ell_p^{2^{m-1}}$, $1 \le k \le m$, and B is the backward shift operator in $\ell_p^{2^{m-1}}$. Observe that

$$\begin{aligned}
|T_{m,p}(x^{(1)}, ..., x^{(m)})| &\le |x_1^{(m)} + x_2^{(m)}||T_{m-1,p}(x^{(1)}, ..., x^{(m)})| \\
&\quad + |x_1^{(m)} - x_2^{(m)}||T_{m-1,p}(B^{2^{m-1}}(x^{(1)}), B^{2^{m-2}}(x^{(2)}), ..., B^2(x^{(m-1)}))| \\
&\le \|T_{m-1,p}\|(|x_1^{(m)} + x_2^{(m)}| + |x_1^{(m)} - x_2^{(m)}|) \\
&= \|T_{m-1,p}\| 2 \max\{|x_1^{(m)}|, |x_2^{(m)}|\} \\
&\le 2\|T_{m-1,p}\|\|x^{(m)}\|_p.
\end{aligned}$$

Therefore,

$$\|T_{m,p}\| \le 2^{m-2}\|T_{2,p}\|. \tag{1.17}$$

Note that $\|T_{2,p}\| = \sup\{\|T_{2,p}^{(x^{(1)})}\| : \|x^{(1)}\|_p = 1\}$, where $T_{2,p}^{(x^{(1)})} : \ell_p^2 \to \mathbb{R}$ is given by $x^{(2)} \mapsto T_{2,p}\left(x^{(1)}, x^{(2)}\right)$. Thus we have the operator $T_{2,p}^{(x^{(1)})}(x^{(2)}) = (x_1^{(1)} + x_2^{(1)})x_1^{(2)} + (x_1^{(1)} - x_2^{(1)})x_2^{(2)}$. Since $(\ell_p)^* = \ell_{p^*}$, we obtain $\|T_{2,p}^{(x^{(1)})}\| = \|(x_1^{(1)} + x_2^{(1)}, x_1^{(1)} - x_2^{(1)}, 0, 0, ...)\|_{p^*}$. Therefore $\|T_{2,p}\| = \sup\{(|x_1^{(1)} + x_2^{(1)}|^{p^*} + |x_1^{(1)} - x_2^{(1)}|^{p^*})^{\frac{1}{p^*}} : |x_1^{(1)}|^p + |x_2^{(1)}|^p = 1\}$. We can verify that it is enough to maximize the above expression when $x_1^{(1)}, x_2^{(1)} \ge 0$. Then

$$\begin{aligned}
\|T_{2,p}\| &= \sup\{((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + |x - (1 - x^p)^{\frac{1}{p}}|^{p^*})^{\frac{1}{p^*}} : x \in [0, 1]\} \\
&= \max\{\sup\{f_p(x) : x \in [0, 2^{-\frac{1}{p}}]\}, \sup\{g_p(x) : x \in [2^{-\frac{1}{p}}, 1]\}\}
\end{aligned}$$

where $f_p(x) := ((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + ((1 - x^p)^{\frac{1}{p}} - x)^{p^*})^{\frac{1}{p^*}}$ and $g_p(x) := ((x + (1 - x^p)^{\frac{1}{p}})^{p^*} + (x - (1 - x^p)^{\frac{1}{p}})^{p^*})^{\frac{1}{p^*}}$. Examining the maps f_p and g_p we easily conclude that

$$\|T_{2,p}\| < 2 \tag{1.18}$$

(for instance, the precise value of $\|T_{2,4}\|$ seems to be graphically $\sqrt{3}$ (see Figure 1.1)).

From (1.17) and (1.18) we would conclude that $\|T_{m,p}\| < 2^{m-1}$. On the other hand, from Theorem 1.3 we have

$$(4^{m-1})^{\frac{mp+p-2m}{2mp}} = \left(\sum_{j_1,...,j_m=1}^{2^{m-1}} |T_{m,p}(e_{j_1}, ..., e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} < C_{\mathbb{R},m,p}^{\text{mult}} 2^{m-1}.$$

and thus

$$C_{\mathbb{R},m,p}^{\text{mult}} > \frac{(4^{m-1})^{\frac{mp+p-2m}{2mp}}}{2^{m-1}} = 2^{\frac{mp+2m-2m^2-p}{mp}} = 1.$$

□

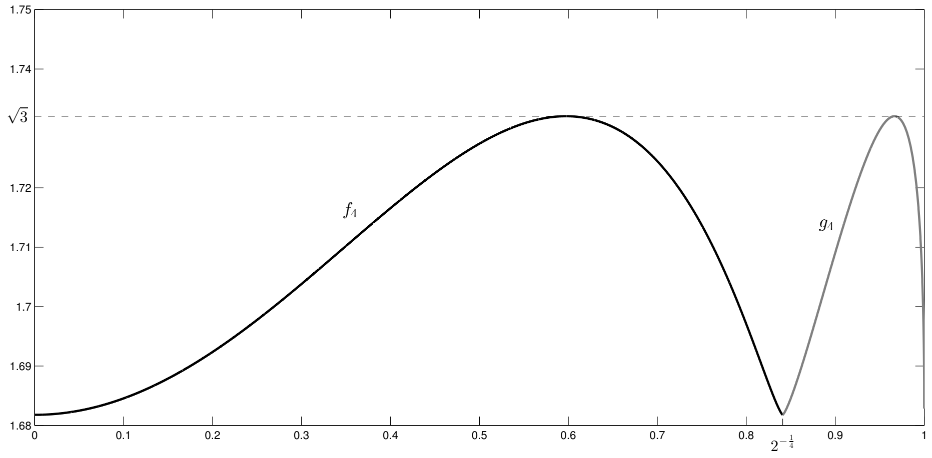

Figure 1.1: Graphs of the functions f_4 and g_4, respectively.

1.2 On the constants of the generalized Bohnenblust-Hille and Hardy–Littlewood inequalities

In this section, among other results, we show that for $p > 2m^3 - 4m^2 + 2m$ the constant $C^{\text{mult}}_{\mathbb{K},m,p}$ has the exactly same upper bounds that we have now for the Bohnenblust–Hille constants (1.2). More precisely we shall show that if $p > 2m^3 - 4m^2 + 2m$, then

$$
\begin{aligned}
C^{\text{mult}}_{\mathbb{C},m,p} &\leq \prod_{j=2}^{m} \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\
C^{\text{mult}}_{\mathbb{R},m,p} &\leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}}, \quad \text{for } m \geq 14, \\
C^{\text{mult}}_{\mathbb{R},m,p} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, \quad \text{for } 2 \leq m \leq 13.
\end{aligned}
\tag{1.19}
$$

It is not difficult to verify that (1.19) in fact improves (1.8). However the most interesting point is that in (1.19), contrary to (1.8), we have no dependence on p in the formulas and, besides, these new estimates are precisely the best known estimates for the constants of the Bohnenblust–Hille inequality (see (1.2)).

To prove these new estimates we also improve the best known estimates for the generalized Bohnenblust–Hille inequality (see Section 1.2.1). The importance of this result (generalized Bohnenblust–Hille inequality) trancends the intrinsic mathematical novelty since, as it was recently shown (see [23]), this new approach is fundamental to improve the estimates of the constants of the classical Bohnenblust–Hille inequality. In Section 1.2.2 we use these estimates to prove new estimates for the constants of the Hardy–Littlewood inequality. In the final section (Section 1.2.3) the estimates of the previous sections (sections 1.2.1 and 1.2.2) are used to obtain new constants for the generalized Hardy–Littlewood inequality.

1.2.1 Estimates for the constants of generalized Bohnenblust–Hille inequality

The best known estimates for the constants $B^{\text{mult}}_{\mathbb{K},m,(q_1,\dots,q_m)}$ are presented in [4]. More precisely, for complex scalars and $1 \leq q_1 \leq \cdots \leq q_m \leq 2$, from [4] we know that, for $\mathbf{q} = (q_1, ..., q_m)$,

$$B^{\text{mult}}_{\mathbb{C},m,\mathbf{q}} \leq \left(\prod_{j=1}^{m} \Gamma \left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2m\left(\frac{1}{q_m} - \frac{1}{2}\right)} \times \left(\prod_{k=1}^{m-1} \left(\Gamma \left(\tfrac{3k+1}{2k+2}\right)^{\left(\frac{-k-1}{2k}\right)(m-k)} \prod_{j=1}^{k} \Gamma \left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} \right)^{2k\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right)} \right). \tag{1.20}$$

In the present section we improve the above estimates for a certain family of $(q_1, ..., q_m)$. More precisely, if $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then

$$B^{\text{mult}}_{\mathbb{C},m,(q_1,\dots,q_m)} \leq \prod_{j=2}^{m} \Gamma \left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}.$$

A similar result holds for real scalars. These results have a crucial importance in the next sections.

Lemma 1.12. *Let $m \geq 2$ and $i \in \{1, ..., m\}$. If $q_i \in [\frac{2m-2}{m}, 2]$ and $q = \frac{2(m-1)q_i}{(m+1)q_i-2}$, then*

$$B^{\text{mult}}_{\mathbb{K},m,(q,\dots,q,q_i,q,\dots,q)} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}},$$

with q_i in the i-th position.

Proof. There is no loss of generality in supposing that $i = 1$. By [23, Proposition 3.1] we have, for each $k = 1, ..., m$,

$$\left(\sum_{\widehat{j_k}=1}^{n} \left(\sum_{j_k=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^2 \right)^{\frac{1}{2}\frac{2m-2}{m}} \right)^{\frac{m}{2m-2}} \leq A^{-1}_{\frac{2m-2}{m}} B^{\text{mult}}_{\mathbb{K},m-1} \|T\| \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}} \|T\|$$

(see [16, Section 2] for details).

We define $\mathbf{q}_k = (\mathbf{q}_k(1), ..., \mathbf{q}_k(m)) = ((2m-2)/m, ..., (2m-2)/m, 2, (2m-2)/m, ..., (2m-2)/m)$, where the 2 is in the k-th coordinate and take $\theta_1 = m - (2m-2)/q_1$ and $\theta_2 = \cdots = \theta_m = 2/q_1 - 1$. Recalling that $q_1 \geq \frac{2m-2}{m}$ we can see that $\theta_k \in [0, 1]$ for all $k = 1, ..., m$. It can be easily checked that

$$\frac{\theta_1}{\mathbf{q}_1(1)} + \cdots + \frac{\theta_m}{\mathbf{q}_m(1)} = \frac{1}{q_1} \quad \text{and} \quad \frac{\theta_1}{\mathbf{q}_1(j)} + \cdots + \frac{\theta_m}{\mathbf{q}_m(j)} = \frac{1}{q} \text{ for } j = 2, ..., m.$$

Then a straightforward application of the Minkowski inequality (using that $\frac{2m-2}{m} < 2$) and of the generalized Hölder inequality ([24, 68]) completes the proof. $\square$

Lemma 1.13. *Let $m \geq 2$ be a positive integer, let $2m < p \leq \infty$, let $q_1, ..., q_m \in \left[\frac{p}{p-m}, 2\right]$. If $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp+p-2m}{2p}$, then, for all $s \in (\max q_i, 2]$, the vector $\left(q_1^{-1}, ..., q_m^{-1}\right)$ belongs to the convex hull in $\mathbb{R}^m$ of*

$$\left\{ \sum_{k=1}^{m} a_{1k} e_k, ..., \sum_{k=1}^{m} a_{mk} e_k \right\},$$

where $a_{jk} = s^{-1}$ if $k \neq j$ and $a_{jk} = \lambda_{m,s}^{-1}$ if $k = j$, and $\lambda_{m,s} = \frac{2ps}{mps+ps+2p-2mp-2ms}$.

Proof. We want to prove that for $(q_1, ..., q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ and $s \in (\max q_i, 2]$ there are $0 < \theta_{j,s} < 1$, $j = 1, ..., m$, such that

$$\sum_{j=1}^{m} \theta_{j,s} = 1,$$

$$\frac{1}{q_1} = \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \cdots + \frac{\theta_{m,s}}{s},$$

$$\vdots$$

$$\frac{1}{q_m} = \frac{\theta_{1,s}}{s} + \cdots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}}.$$

Observe initially that from $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp+p-2m}{2p}$ we have $\max q_i \geq \frac{2mp}{mp+p-2m}$. Note also that for all $s \in \left[\frac{2mp-2p}{mp-2m}, 2\right]$ we have

$$mps + ps + 2p - 2mp - 2ms > 0 \qquad \text{and} \qquad \frac{p}{p-m} \leq \lambda_{m,s} \leq 2. \tag{1.21}$$

Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} > \frac{2mp-2p}{mp-2m}$ (the last inequality is strict because we are not considering the case $p = 2m$) it follows that $\lambda_{m,s}$ is well defined for all $s \in (\max q_i, 2]$. Furthermore, for all $s > \frac{2mp}{mp+p-2m}$ it is possible to prove that $\lambda_{m,s} < s$. In fact, $s > \frac{2mp}{mp+p-2m}$ implies $mps + ps - 2ms > 2mp$ and thus adding $2p$ in both sides of this inequality we can conclude that

$$\frac{2ps}{mps + ps + 2p - 2mp - 2ms} < \frac{2ps}{2p} = s,$$

i.e.,

$$\lambda_{m,s} < s. \tag{1.22}$$

For each $j = 1, ..., m$, consider $\theta_{j,s} = \frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})}$. Since $\sum_{j=1}^{m} \frac{1}{q_j} = \frac{mp+p-2m}{2p}$ we conclude that

$$\sum_{j=1}^{m} \theta_{j,s} = \sum_{j=1}^{m} \frac{\lambda_{m,s}\,(s - q_j)}{q_j\,(s - \lambda_{m,s})} = \frac{\lambda_{m,s}}{s - \lambda_{m,s}} \left(s \sum_{j=1}^{m} \frac{1}{q_j} - m \right) = 1.$$

Since by hypothesis $s > \max q_i \geq q_j$ for all $j = 1, ..., m$, it follows that $\theta_{j,s} > 0$ for all $j = 1, ..., m$ and thus $0 < \theta_{j,s} < \sum_{j=1}^{m} \theta_{j,s} = 1$.

Finally, note that

$$\frac{\theta_{j,s}}{\lambda_{m,s}} + \frac{1-\theta_{j,s}}{s} = \frac{\frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})}}{\lambda_{m,s}} + \frac{1-\frac{\lambda_{m,s}(s-q_j)}{q_j(s-\lambda_{m,s})}}{s} = \frac{1}{q_j}.$$

Therefore

$$\frac{1}{q_1} = \frac{\theta_{1,s}}{\lambda_{m,s}} + \frac{\theta_{2,s}}{s} + \cdots + \frac{\theta_{m,s}}{s},$$

$$\vdots$$

$$\frac{1}{q_m} = \frac{\theta_{1,s}}{s} + \cdots + \frac{\theta_{m-1,s}}{s} + \frac{\theta_{m,s}}{\lambda_{m,s}},$$

and the proof is done. □

Combining the two previous lemmas we have:

Theorem 1.14. *Let $m \geq 2$ be a positive integer and $q_1, ..., q_m \in [1,2]$. If $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{m+1}{2}$, and $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then*

$$B^{\text{mult}}_{\mathbb{K},m,(q_1,...,q_m)} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}},$$

where $A_{\frac{2j-2}{j}}$ are the respective constants of the Khinchine inequality.

Proof. Let $s = \frac{2m^2-4m+2}{m^2-m-1}$ and $q = \frac{2m-2}{m}$. Since $\frac{m-1}{s} + \frac{1}{q} = \frac{m+1}{2}$, from Lemma 1.12 the Bohnenblust–Hille exponents $(t_1, ..., t_m) = (s, ..., s, q), ..., (q, s, ..., s)$ are associated to

$$B^{\text{mult}}_{\mathbb{K},m,(t_1,...,t_m)} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

Since by hypothesis $\max q_i < \frac{2m^2-4m+2}{m^2-m-1} = s$, from the previous lemma (Lemma 1.13) with $p = \infty$, the exponent $(q_1, ..., q_m)$ is the interpolation of

$$\left(\frac{2s}{ms+s+2-2m}, s, ..., s\right), ..., \left(s, ..., s, \frac{2s}{ms+s+2-2m}\right).$$

But note that

$$\frac{2s}{ms+s+2-2m} = \frac{2m-2}{m}$$

and from Lemma 1.12 they are associated to the constants

$$B^{\text{mult}}_{\mathbb{K},m,(q_1,...,q_m)} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

□

Corollary 1.15. *Let $m \geq 2$ be a positive integer and $q_1, ..., q_m \in [1,2]$. If $\frac{1}{q_1} + \cdots + \frac{1}{q_m} =$*

$\frac{m+1}{2}$, *and* $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, *then*

$$\begin{aligned} B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)} &\leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ B^{\text{mult}}_{\mathbb{R},m,(q_1,\ldots,q_m)} &\leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}}, \quad \textit{for } m \geq 14, \\ B^{\text{mult}}_{\mathbb{R},m,(q_1,\ldots,q_m)} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}, \quad \textit{for } 2 \leq m \leq 13. \end{aligned}$$

The following table compares the estimate obtained for $B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)}$ in [4] (see (1.20)) and the new and better estimate obtained in Theorem 1.14.

$m \geq 2$	$1 \leq q_1 \leq \cdots \leq q_m \leq 2$; $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{m+1}{2}$ and $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$	$B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)}$: Estimates of [4] (see (1.20))	$B^{\text{mult}}_{\mathbb{C},m,(q_1,\ldots,q_m)}$: Estimates of Theorem 1.14
4	$q_1=\cdots=q_3=\frac{486}{305}, q_4=1.62$	< 1.28964	< 1.28890
5	$q_1=\cdots=q_4=\frac{668}{401}, q_5=1.67$	< 1.34783	< 1.34745
6	$q_1=\cdots=q_5=\frac{430}{251}, q_6=1.72$	< 1.39885	< 1.39783
7	$q_1=\cdots=q_6=\frac{1053}{603}, q_7=1.755$	< 1.44344	< 1.44224
8	$q_1=\cdots=q_7=\frac{1246}{701}, q_8=1.78$	< 1.48273	< 1.48207
9	$q_1=\cdots=q_8=\frac{14408}{8005}, q_9=1.801$	< 1.51863	< 1.51827
10	$q_1=\cdots=q_9=\frac{327618}{180211}, q_{10}=1.8201$	< 1.55231	< 1.55151
20	$q_1=\cdots=q_{19}=\frac{14478}{7601}, q_{20}=1.905$	< 1.79162	< 1.79137
50	$q_1=\cdots=q_{49}=\frac{240198}{122501}, q_{50}=1.9608$	< 2.170671	< 2.170620
100	$q_1=\cdots=q_{99}=\frac{1960398}{990001}, q_{100}=1.9802$	< 2.511775	< 2.511760
1000	$q_1=\cdots=q_{999}=\frac{665334666000666}{333000000333667}$, $q_{1000}=1.998002000002$	< 4.08463471	< 4.08463446

1.2.2 Application 1: Improving the constants of the Hardy-Littlewood inequality

The main result of this section shows that for $2m^3-4m^2+2m < p \leq \infty$ the optimal constants satisfying the Hardy–Littlewood inequality for m-linear forms in ℓ_p spaces are dominated by the best known estimates for the constants of the m-linear Bohnenblust–Hille inequality; this result improves (for $2m^3-4m^2+2m < p \leq \infty$) the best estimates we have thus far (see (1.8)), and may suggest a more subtle connection between the optimal constants of these inequalities.

Theorem 1.16. *Let $m \geq 2$ be a positive integer and $2m^3-4m^2+2m < p \leq \infty$. Then, for all continuous m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n, we have*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq \left(\prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}\right)\|T\|. \tag{1.23}$$

Proof. The case $p = \infty$ in (1.23) is precisely the Bohnenblust–Hille inequality, so we just need to consider $2m^3 - 4m^2 + 2m < p < \infty$. Let $\frac{2m-2}{m} \leq s \leq 2$ and

$$\lambda_{0,s} = \frac{2s}{ms + s + 2 - 2m}.$$

Note that

$$ms + s + 2 - 2m > 0 \qquad \text{and} \qquad 1 \leq \lambda_{0,s} \leq 2. \tag{1.24}$$

Since

$$\frac{m-1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [5]) we know that there is a constant $C_m \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have, for all $i = 1, ..., m$,

$$\left(\sum_{j_i=1}^{n} \left(\sum_{\widehat{j_i}=1}^{n} \left| T\left(e_{j_1}, ..., e_{j_m}\right) \right|^s \right)^{\frac{1}{s}\lambda_{0,s}} \right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \left\| T \right\|. \tag{1.25}$$

Above, $\sum\limits_{\widehat{j_i}=1}^{n}$ means the sum over all j_k for all $k \neq i$. If we choose $s = \frac{2mp}{mp+p-2m}$ (note that this s belongs to the interval $\left[\frac{2m-2}{m}, 2\right]$), we have $s > \frac{2m}{m+1}$ (this inequality is strict because we are considering the case $p < \infty$) and thus $\lambda_{0,s} < s$. In fact, $s > \frac{2m}{m+1}$ implies $ms + s > 2m$ and thus adding 2 in both sides of this inequality we can conclude that

$$\frac{2s}{ms + s + 2 - 2m} < \frac{2s}{2} = s,$$

i.e.,

$$\lambda_{0,s} < s. \tag{1.26}$$

Since $p > 2m^3 - 4m^2 + 2m$ we conclude that

$$s < \frac{2m^2 - 4m + 2}{m^2 - m - 1}.$$

Thus, from Theorem 1.14, the optimal constant associated to the multiple exponent

$$(\lambda_{0,s}, s, s, ..., s)$$

is less than or equal to

$$C_m = \prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}.$$

More precisely, (1.25) is valid with C_m as above. Now the proof follows the same lines, *mutatis mutandis*, of the proof of Theorem 1.9 (see [16, Theorem 1.1]), which has its roots in the work of Praciano-Pereira [118]. □

Remark 1.17. *Note that it is simple to verify that these new estimates are better than* *the old ones. In fact, for complex scalars the inequality*

$$\prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{2m(m-1)}{p}} \left(\prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}\right)^{\frac{p-2m}{p}}$$

is a straightforward consequence of

$$\prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1} < \left(\frac{2}{\sqrt{\pi}}\right)^{m-1},$$

which is true for $m \geq 3$. *The case of real scalars is analogous.*

The following table compares the estimates for $C_{\mathbb{C},m,p}^{\text{mult}}$ obtained in Theorem 1.9 (see (1.16) and [16]) and the estimate obtained in Theorem 1.16 for $2m^3-4m^2+2m < p \leq \infty$.

$m \geq 2$	$2m^3 - 4m^2 + 2m < p \leq \infty$	$C_{\mathbb{C},m,p}^{\text{mult}}$ Estimates (1.16) (see [16] and Theorem 1.9)	$C_{\mathbb{C},m,p}^{\text{mult}}$ Estimates of Theorem 1.16
4	$p = 73$	< 1.30433	< 1.28890
	$p = 500$	< 1.29114	
	$p = 1000$	< 1.29002	
10	$p = 1621$	< 1.56396	< 1.55151
	$p = 3000$	< 1.55822	
	$p = 5000$	< 1.55553	
50	$p = 240101$	< 2.175275	< 2.170620
	$p = 500000$	< 2.172854	
	$p = 1000000$	< 2.171737	
100	$p = 1960201$	< 2.514590	< 2.511760
	$p = 5000000$	< 2.512869	
	$p = 20000000$	< 2.512037	
1000	$p = 1996002001$	< 4.08512258	< 4.08463446
	$p = 6000000000$	< 4.08479684	
	$p = 50000000000$	< 4.08465395	

Recall that from the previous section that for $p \geq m^2$ the constants of the Hardy–Littlewood inequality have a subpolynomial growth. The graph 1.2 illustrates what we have thus far, combined with Theorem 1.16.

1.2.3 Application 2: Estimates for the constants of the generalized Hardy–Littlewood inequality

The best known estimates for the constants $C_{\mathbb{K},m,p,\mathbf{q}}^{\text{mult}}$ are $\left(\sqrt{2}\right)^{m-1}$ for real scalars and $\left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ for complex scalars (see [5]). In the Theorem 1.9 (see [16, Theorem 1.1]) and in the previous section (see (1.19)) better constants were obtained when $q_1 = ... = q_m =$

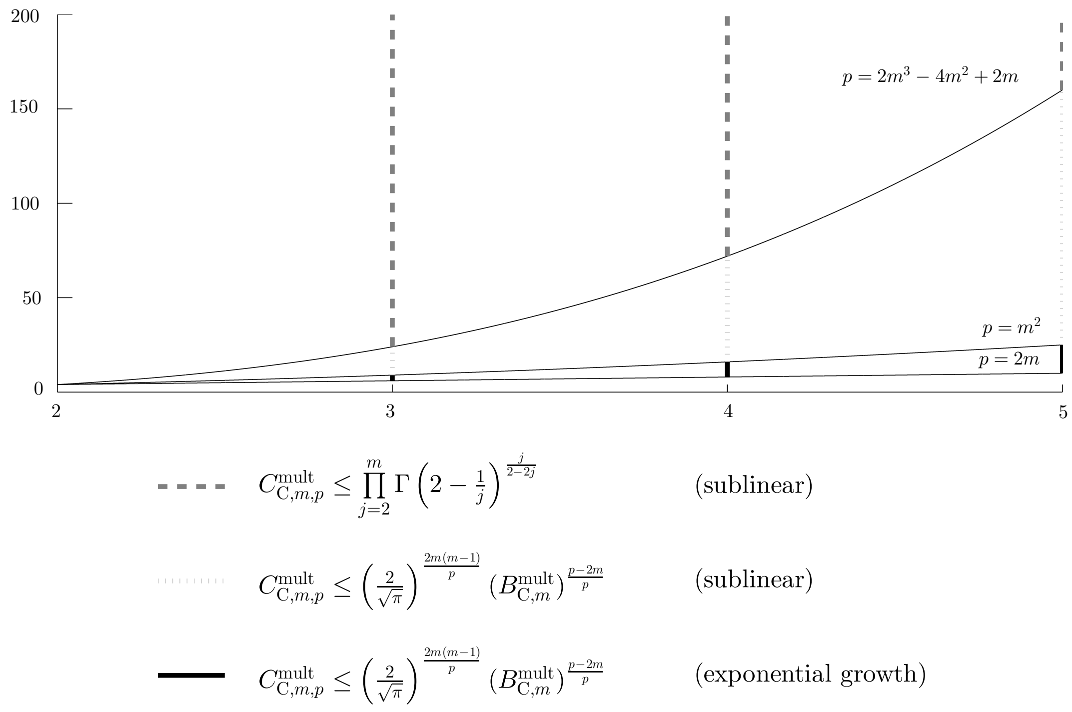

Figure 1.2: Behaviour of $C^{\mathrm{mult}}_{\mathbb{C},m,p}$.

$\frac{2mp}{mp+p-2m}$. Now we extend the results from [16] to general multiple exponents. Of course the interesting case is the border case, i.e., $\frac{1}{q_1} + ... + \frac{1}{q_m} = \frac{mp+p-2m}{2p}$. The proof is slightly more elaborated than the proof of Theorem 1.16 and also a bit more technical that the proof of the main result of [16].

Theorem 1.18. *Let $m \geq 2$ be a positive integer, let $2m < p \leq \infty$ and let $\mathbf{q} := (q_1, ..., q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ be such that $\frac{1}{q_1} + ... + \frac{1}{q_m} = \frac{mp+p-2m}{2p}$. If $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then*

$$C^{\mathrm{mult}}_{\mathbb{K},m,p,\mathbf{q}} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}.$$

Proof. The arguments follow the general lines of [16], but are slightly different and due the technicalities we present the details for the sake of clarity. Define for $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$,

$$\lambda_{m,s} = \frac{2ps}{mps + ps + 2p - 2mp - 2ms}. \tag{1.27}$$

Observe that $\lambda_{m,s}$ is well defined for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$. In fact, as we have in (1.21) note that for all $s \in \left[\frac{2mp-2p}{mp-2m}, 2\right]$ we have

$$mps + ps + 2p - 2mp - 2ms > 0 \qquad \text{and} \qquad \frac{p}{p-m} \leq \lambda_{m,s} \leq 2.$$

Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} > \frac{2mp-2p}{mp-2m}$ (the last inequality is strict because we are not considering the case $p = 2m$) and $\frac{2m^2-4m+2}{m^2-m-1} \leq 2$ it follows that $\lambda_{m,s}$ is well defined for all

$s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$.

Let us prove

$$C^{\text{mult}}_{\mathbb{K},m,p,(\lambda_{m,s},s,\ldots,s)} \leq \prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}} \tag{1.28}$$

for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$. In fact, for these values of s, consider

$$\lambda_{0,s} = \frac{2s}{ms+s+2-2m}.$$

Observe that if $p = \infty$ then $\lambda_{m,s} = \lambda_{0,s}$. Since

$$\frac{m-1}{s} + \frac{1}{\lambda_{0,s}} = \frac{m+1}{2},$$

from the generalized Bohnenblust–Hille inequality (see [5]) we know that there is a constant $C_m \geq 1$ such that for all m-linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ we have, for all $i = 1, ..., m$,

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n} |T(e_{j_1}, ..., e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{0,s}}\right)^{\frac{1}{\lambda_{0,s}}} \leq C_m \|T\|. \tag{1.29}$$

Since

$$\frac{2m}{m+1} \leq \frac{2mp}{mp+p-2m} \leq \max q_i < s < \frac{2m^2-4m+2}{m^2-m-1}$$

it is not to difficult to prove that (see (1.26))

$$\lambda_{0,s} < s < \frac{2m^2-4m+2}{m^2-m-1}.$$

Since $s < \frac{2m^2-4m+2}{m^2-m-1}$ we conclude by Theorem 1.14 that the optimal constant associated to the multiple exponent

$$(\lambda_{0,s}, s, s, ..., s)$$

is less than or equal to

$$\prod_{j=2}^{m} A^{-1}_{\frac{2j-2}{j}}. \tag{1.30}$$

More precisely, (1.29) is valid with C_m as above. Since $\lambda_{m,s} = \lambda_{0,s}$ if $p = \infty$, we have (1.28) for all for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$ and the proof is done for this case.

For $2m < p < \infty$, let

$$\lambda_{j,s} = \frac{\lambda_{0,s}p}{p-\lambda_{0,s}j}$$

for all $j = 1, ..., m$. Note that

$$\lambda_{m,s} = \frac{2ps}{mps+ps+2p-2mp-2ms}$$

and this notation is compatible with (1.27). Since $s > \max q_i \geq \frac{2mp}{mp+p-2m} \geq \frac{2mp}{mp+p-2j}$ for all $j = 1, ..., m$ we also observe that

$$\lambda_{j,s} < s \tag{1.31}$$

for all $j = 1, ..., m$. Moreover, observe that

$$\left(\frac{p}{\lambda_{j,s}}\right)^* = \frac{\lambda_{j+1,s}}{\lambda_{j,s}}$$

for all $j = 0, ..., m-1$. From now on, part of the proof of (i) follows the same steps as those of the proof of the main result in [16, 118], but we prefer to show here the details for the sake of completeness (note that the final part of the proof of (i) requires a more subtle argument than the one employed in [16]).

Let us suppose that $1 \leq k \leq m$ and that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}} \leq C_m\|T\|$$

is true for all continuous m-linear forms $T : \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$. Let us prove that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}}\right)^{\frac{1}{\lambda_{k,s}}} \leq C_m\|T\|$$

for all continuous m-linear forms $T : \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, ..., m$.

The initial case (the case in which all $p = \infty$) is precisely (1.29) with C_m as in (1.30). Consider

$$T \in \mathcal{L}(\underbrace{\ell_p^n, ..., \ell_p^n}_{k \text{ times}}, \ell_\infty^n, ..., \ell_\infty^n; \mathbb{R})$$

and for each $x \in B_{\ell_p^n}$ define

$$\begin{array}{rcll} T^{(x)} & : & \underbrace{\ell_p^n \times \cdots \times \ell_p^n}_{k-1 \text{ times}} \times \ell_\infty^n \times \cdots \times \ell_\infty^n & \to \mathbb{R} \\ & & (z^{(1)}, ..., z^{(m)}) & \mapsto T(z^{(1)}, ..., z^{(k-1)}, xz^{(k)}, z^{(k+1)}, ..., z^{(m)}), \end{array}$$

with $xz^{(k)} = (x_j z_j^{(k)})_{j=1}^n$. Observe that

$$\|T\| = \sup\{\|T^{(x)}\| : x \in B_{\ell_p^n}\}.$$

By applying the induction hypothesis to $T^{(x)}$, we obtain

$$\begin{aligned}
&\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^s\left|x_{j_k}\right|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&=\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}\left|T\left(e_{j_1},...,e_{j_{k-1}},xe_{j_k},e_{j_{k+1}},...,e_{j_m}\right)\right|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&=\left(\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}\left|T^{(x)}\left(e_{j_1},...,e_{j_m}\right)\right|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&\leq C_m\|T^{(x)}\|\\
&\leq C_m\|T\|
\end{aligned} \tag{1.32}$$

for all $i=1,...,m$.

We will analyze two cases:

- $i=k$.

Since

$$\left(\frac{p}{\lambda_{j-1,s}}\right)^* = \frac{\lambda_{j,s}}{\lambda_{j-1}}$$

for all $j = 1, ..., m$, we conclude that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k,s}}\right)^{\frac{1}{\lambda_{k,s}}}\\
&=\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1,s}\left(\frac{p}{\lambda_{k-1,s}}\right)^{*}}\right)^{\frac{1}{\lambda_{k-1,s}}\frac{1}{\left(\frac{p}{\lambda_{k-1,s}}\right)^{*}}}\\
&=\left\|\left(\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)_{j_k=1}^{n}\right\|_{\left(\frac{p}{\lambda_{k-1,s}}\right)^{*}}^{\frac{1}{\lambda_{k-1,s}}}\\
&=\left(\sup_{y\in B_{\ell^n_{\frac{p}{\lambda_{k-1,s}}}}}\sum_{j_k=1}^{n}\left|y_{j_k}\right|\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&=\left(\sup_{x\in B_{\ell^n_p}}\sum_{j_k=1}^{n}\left|x_{j_k}\right|^{\lambda_{k-1,s}}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&=\sup_{x\in B_{\ell^n_p}}\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\left|T\left(e_{j_1},...,e_{j_m}\right)\right|^{s}\left|x_{j_k}\right|^{s}\right)^{\frac{1}{s}\lambda_{k-1,s}}\right)^{\frac{1}{\lambda_{k-1,s}}}\\
&\leq C_m\|T\|.
\end{aligned}$$

where the last inequality holds by (1.32).

- $i \neq k$.

It is clear that $\lambda_{k-1,s} < \lambda_{k,s}$ for all $1 \leq k \leq m$. Since $\lambda_{k,s} < s$ for all $1 \leq k \leq m$ (see (1.31)) we get

$$\lambda_{k-1,s} < \lambda_{k,s} < s \text{ for all } 1 \leq k \leq m.$$

Denoting, for $i = 1, ..., m$,

$$S_i = \left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},...,e_{j_m})|^{s}\right)^{\frac{1}{s}}$$

we get

$$\begin{aligned}
\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}} &= \sum_{j_i=1}^{n} S_i^{\lambda_{k,s}} = \sum_{j_i=1}^{n} S_i^{\lambda_{k,s}-s} S_i^s \\
&= \sum_{j_i=1}^{n}\sum_{\widehat{j}_i=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k,s}}} = \sum_{j_k=1}^{n}\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k,s}}} \\
&= \sum_{j_k=1}^{n}\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^{\frac{s(s-\lambda_{k,s})}{s-\lambda_{k-1,s}}}}{S_i^{s-\lambda_{k,s}}}|T(e_{j_1},...,e_{j_m})|^{\frac{s(\lambda_{k,s}-\lambda_{k-1,s})}{s-\lambda_{k-1,s}}}.
\end{aligned}$$

Therefore, using Hölder's inequality (twice) we obtain

$$\begin{aligned}
&\sum_{j_i=1}^{n}\left(\sum_{\widehat{j}_i=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}} \\
&\leq \sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}\right)^{\frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{\lambda_{k,s}-\lambda_{k-1,s}}{s-\lambda_{k-1,s}}} \\
&\leq \left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}\frac{|T(e_{j_1},...,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}\right)^{\frac{\lambda_{k,s}}{\lambda_{k-1,s}}}\right)^{\frac{\lambda_{k-1,s}}{\lambda_{k,s}}\cdot\frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}} \\
&\quad\times\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}}\right)^{\frac{1}{\lambda_{k,s}}\cdot\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}}.
\end{aligned} \tag{1.33}$$

We know from the case $i=k$ that

$$\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j}_k=1}^{n}|T(e_{j_1},...,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}}\right)^{\frac{1}{\lambda_{k,s}}\cdot\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}} \leq (C_m\|T\|)^{\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}}. \tag{1.34}$$

Now we investigate the first factor in (1.33). From Hölder's inequality and (1.32) it follows

that

$$
\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\dots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}\right)^{\frac{\lambda_{k,s}}{\lambda_{k-1,s}}}\right)^{\frac{\lambda_{k-1,s}}{\lambda_{k,s}}}\\
&=\left\|\left(\sum_{\widehat{j_k}}\frac{|T(e_{j_1},\dots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}\right)_{j_k=1}^{n}\right\|_{\left(\frac{p}{\lambda_{k-1,s}}\right)^*}\\
&=\sup_{y\in B_{\ell^n_{\frac{p}{\lambda_{k-1,s}}}}}\sum_{j_k=1}^{n}|y_{j_k}|\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\dots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}\\
&=\sup_{x\in B_{\ell_p^n}}\sum_{j_k=1}^{n}\sum_{\widehat{j_k}=1}^{n}\frac{|T(e_{j_1},\dots,e_{j_m})|^s}{S_i^{s-\lambda_{k-1,s}}}|x_{j_k}|^{\lambda_{k-1,s}}\\
&=\sup_{x\in B_{\ell_p^n}}\sum_{j_i=1}^{n}\sum_{\widehat{j_i}=1}^{n}\frac{|T(e_{j_1},\dots,e_{j_m})|^{s-\lambda_{k-1,s}}}{S_i^{s-\lambda_{k-1,s}}}|T(e_{j_1},\dots,e_{j_m})|^{\lambda_{k-1,s}}|x_{j_k}|^{\lambda_{k-1,s}}\\
&\leq\sup_{x\in B_{\ell_p^n}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}\frac{|T(e_{j_1},\dots,e_{j_m})|^s}{S_i^{s}}\right)^{\frac{s-\lambda_{k-1,s}}{s}}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s|x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\\
&=\sup_{x\in B_{\ell_p^n}}\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s|x_{j_k}|^s\right)^{\frac{1}{s}\lambda_{k-1,s}}\leq\left(C_m\|T\|\right)^{\lambda_{k-1,s}}.
\end{aligned}
\tag{1.35}
$$

Replacing (1.34) and (1.35) in (1.33) we conclude that

$$
\begin{aligned}
\sum_{j_i=1}^{n}\left(\sum_{\widehat{j_i}=1}^{n}|T(e_{j_1},\dots,e_{j_m})|^s\right)^{\frac{1}{s}\lambda_{k,s}}&\leq\left(C_m\|T\|\right)^{\lambda_{k-1,s}\frac{s-\lambda_{k,s}}{s-\lambda_{k-1,s}}}\left(C_m\|T\|\right)^{\frac{(\lambda_{k,s}-\lambda_{k-1,s})s}{s-\lambda_{k-1,s}}}\\
&=\left(C_m\|T\|\right)^{\lambda_{k,s}}
\end{aligned}
$$

and finally the proof of (1.28) is done for all $s\in\left(\max q_i,\frac{2m^2-4m+2}{m^2-m-1}\right)$.

Now the proof uses a different argument from those from [16], since a new interpolation procedure is now needed. From (1.31) we know that $\lambda_{m,s}<s$ for all $s\in\left(\max q_i,\frac{2m^2-4m+2}{m^2-m-1}\right)$. Therefore, using the Minkowski inequality as in [5], it is possible to obtain from (1.28) that, for all fixed $i\in\{1,\dots,m\}$,

$$
C^{\text{mult}}_{\mathbb{K},m,p,(s,\dots,s,\lambda_{m,s},s,\dots,s)}\leq\prod_{j=2}^{m}A^{-1}_{\frac{2j-2}{j}}\tag{1.36}
$$

for all $s\in\left(\max q_i,\frac{2m^2-4m+2}{m^2-m-1}\right)$ with $\lambda_{m,s}$ in the i-th position. Finally, from Lemma 1.13

we know that $\left(q_1^{-1}, ..., q_m^{-1}\right)$ belongs to the convex hull of

$$\left\{\left(\lambda_{m,s}^{-1}, s^{-1}, ..., s^{-1}\right), ..., \left(s^{-1}, ..., s^{-1}, \lambda_{m,s}^{-1}\right)\right\}$$

for all $s \in \left(\max q_i, \frac{2m^2-4m+2}{m^2-m-1}\right)$ with certain constants $\theta_{1,s}, ..., \theta_{m,s}$ and thus, from the interpolative technique from [5], we get

$$\begin{aligned} C_{\mathbb{K},m,p,\mathbf{q}}^{\mathrm{mult}} &\leq \left(C_{\mathbb{K},m,p,(\lambda_{m,s},s,...,s)}^{\mathrm{mult}}\right)^{\theta_{1,s}} \cdots \left(C_{\mathbb{K},m,p,(s,...,s,\lambda_{m,s})}^{\mathrm{mult}}\right)^{\theta_{m,s}} \\ &\leq \left(\prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}\right)^{\theta_{1,s}+\cdots+\theta_{m,s}} = \prod_{j=2}^{m} A_{\frac{2j-2}{j}}^{-1}. \end{aligned}$$

□

Corollary 1.19. *Let $m \geq 2$ be a positive integer and $2m < p \leq \infty$. Let also $\mathbf{q} := (q_1, ..., q_m) \in \left[\frac{p}{p-m}, 2\right]^m$ be such that $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{mp+p-2m}{2p}$. If $\max q_i < \frac{2m^2-4m+2}{m^2-m-1}$, then*

$$\begin{aligned} C_{\mathbb{C},m,p,\mathbf{q}}^{\mathrm{mult}} &\leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ C_{\mathbb{R},m,p,\mathbf{q}}^{\mathrm{mult}} &\leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} && \textit{if } 2 \leq m \leq 13, \\ C_{\mathbb{R},m,p,\mathbf{q}}^{\mathrm{mult}} &\leq 2^{\frac{446381}{55440}-\frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}} && \textit{if } m \geq 14. \end{aligned}$$

Chapter 2

Optimal Hardy–Littlewood type inequalities for m-linear forms on ℓ_p spaces with $1 \leq p \leq m$

In this chapter we present results from the following:

[14] G. Araújo, and D. Pellegrino, *Optimal Hardy–Littlewood type inequalities for m-linear forms on* ℓ_p *spaces with* $1 \leq p \leq m$, Arch. Math. **105** (2015), 285–295.

In [37, Corollary 5.20] it is shown that in ℓ_2^n the Hardy–Littlewood multilinear inequalities has an extra power of n in its right hand side. Therefore, a natural question is:

- For $1 \leq p \leq m$, what power of n (depending on r, m, p) will appear in the right hand side of the Hardy–Littlewood multilinear inequalities if we replace the optimal exponents $2mp/(mp + p - 2m)$ and $p/(p - m)$ by a smaller value r?

This case ($1 \leq p \leq m$) was only explored for the case of Hilbert spaces ($p = 2$, see [37, Corollary 5.20] and [51]) and the case $p = \infty$ was explored in [46]. The results of this chapter answer the remains cases of the above question (see Theorem 2.1) and extends [37, Corollary 5.20] to $1 \leq p \leq m$ (see Theorem 2.1(a) and Proposition 2.4).

The main result of this chapter is the following:

Theorem 2.1. *Let $m \geq 2$ be a positive integer.*

(a) If $(r, p) \in ([1, 2] \times [2, 2m)) \cup ([1, \infty) \times [2m, \infty])$, then there is a constant $H^{\text{mult}}_{\mathbb{K},m,p,r} > 0$ (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq H^{\text{mult}}_{\mathbb{K},m,p,r} n^{\max\left\{ \frac{2mr+2mp-mpr-pr}{2pr}, 0 \right\}} \|T\|$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n. Moreover, the exponent $\max\{(2mr + 2mp - mpr - pr)/2pr, 0\}$ is optimal.

(b) If $(r,p) \in [2,\infty) \times (m, 2m]$, *then there is a constant* $H_{\mathbb{K},m,r,p}^{\text{mult}} > 0$ *(not depending on* n*) such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^r \right)^{\frac{1}{r}} \leq H_{\mathbb{K},m,r,p}^{\text{mult}} n^{\max\left\{ \frac{p+mr-rp}{pr}, 0 \right\}} \|T\|$$

for all m*-linear forms* $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ *and all positive integers* n*. Moreover, the exponent* $\max\{(p+mr-rp)/pr, 0\}$ *is optimal.*

Remark 2.2. *The first item of the above theorem recovers [37, Corollary 5.20(i)] (just make* $p = 2$*) and [46, Proposition 5.1].*

Proof of Theorem 2.1. Let $1 \leq q \leq r \leq \infty$ and E be a Banach space. We say that an m-linear form $S : E \times \cdots \times E \to \mathbb{K}$ is multiple $(r;q)$-summing if there is a constant $C > 0$ such that

$$\left\| \left(S(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)}) \right)_{j_1,\ldots,j_m=1}^{n} \right\|_{\ell_r} \leq C \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^{n} |\varphi(x_j^{(1)})|^q \right)^{\frac{1}{q}} \cdots \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^{n} |\varphi(x_j^{(m)})|^q \right)^{\frac{1}{q}}$$

for all positive integers n.

(a) Let us consider first $(r,p) \in [1,2] \times [2,2m)$. From now on $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ is an m-linear form. Since

$$\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} |\varphi(e_j)| = n n^{-\frac{1}{p^*}} = n^{\frac{1}{p}}$$

and since T is multiple $(2m/(m+1);1)$-summing (we will see in the next chapter that from the Bohnenblust–Hille inequality it is possible to prove that all continuous m-linear forms are multiple $(2m/(m+1);1)$-summing with constant $B_{\mathbb{K},m}^{\text{mult}}$), we conclude that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\text{mult}} \|T\| \, n^{\frac{m}{p}}. \tag{2.1}$$

Therefore, if $1 \leq r < 2m/(m+1)$, using the Hölder inequality and (2.1), we have

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^r \right)^{\frac{1}{r}} \\
&\leq \left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \left(\sum_{j_1,\ldots,j_m=1}^{n} |1|^{\frac{2mr}{2m-rm-r}} \right)^{\frac{2m-rm-r}{2mr}} \\
&= \left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1},\ldots,e_{j_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} (n^m)^{\frac{2m-rm-r}{2mr}} \\
&\leq B_{\mathbb{K},m}^{\text{mult}} \|T\| \, n^{\frac{m}{p}} n^{\frac{2m-rm-r}{2r}} \\
&= B_{\mathbb{K},m}^{\text{mult}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\|.
\end{aligned}$$

Now we consider the case $2m/(m+1) \leq r \leq 2$. From the proof of [15, Theorem 3.2(i)] we know that, for all $2m/(m+1) \leq r \leq 2$ and all Banach spaces E, every continuous m-linear form $S : E \times \cdots \times E \to \mathbb{K}$ is multiple $(r; 2mr/(mr+2m-r))$-summing with constant $C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}}$. Therefore

$$\begin{aligned}&\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^r\right)^{\frac{1}{r}} \\ &\leq C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} \|T\| \left[\left(\sup_{\varphi\in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} |\varphi(e_j)|^{\frac{2mr}{mr+2m-r}}\right)^{\frac{mr+2m-r}{2mr}}\right]^m.\end{aligned} \tag{2.2}$$

Since $1 \leq 2mr/(mr+2m-r) \leq 2m/(2m-1) = (2m)^* < p^*$, we have

$$\begin{aligned}\left(\sup_{\varphi\in B_{(\ell_p^n)^*}} \sum_{j=1}^{n} |\varphi(e_j)|^{\frac{2mr}{mr-r+2m}}\right)^{\frac{mr-r+2m}{2mr}} &= \left(n(n^{-\frac{1}{p^*}})^{\frac{2mr}{mr-r+2m}}\right)^{\frac{mr-r+2m}{2mr}} \\ &= n^{\frac{2mr+2mp-mpr-pr}{2mpr}}\end{aligned} \tag{2.3}$$

and finally, from (2.2) and (2.3), we obtain

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^r\right)^{\frac{1}{r}} \leq C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\|.$$

Now we prove the optimality of the exponents. Suppose that the theorem is valid for an exponent s, i.e.,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^r\right)^{\frac{1}{r}} \leq H^{\text{mult}}_{\mathbb{K},m,p,r} n^s \|T\|.$$

Since $p \geq 2$, from the generalized Kahane–Salem–Zygmund inequality (2) we have

$$n^{\frac{m}{r}} \leq C_m H^{\text{mult}}_{\mathbb{K},m,p,r} n^s n^{\frac{m+1}{2}-\frac{m}{p}}$$

and thus, making $n \to \infty$, we obtain

$$s \geq \frac{2mr+2mp-mpr-pr}{2pr}.$$

The case $(r,p) \in [1, 2mp/(mp+p-2m)] \times [2m, \infty]$ is analogous. In fact, from the Hardy–Littlewood/Praciano-Pereira inequality we know that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq C^{\text{mult}}_{\mathbb{K},m,p} \|T\|. \tag{2.4}$$

Therefore, from Hölder's inequality and (2.4), we have

$$\begin{aligned}
&\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^r\right)^{\frac{1}{r}} \\
&\leq \left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \left(\sum_{j_1,\dots,j_m=1}^{n} |1|^{\frac{2mpr}{2mp+2mr-mpr-pr}}\right)^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\
&\leq C^{\text{mult}}_{\mathbb{K},m,p} \|T\| \left(n^m\right)^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\
&= C^{\text{mult}}_{\mathbb{K},m,p} n^{\frac{2mp+2mr-mpr-pr}{2pr}} \|T\| .
\end{aligned} \tag{2.5}$$

Since $p \geq 2m$, the optimality of the exponent is obtained *ipsis litteris* as in the previous case.

If $(r,p) \in (2mp/(mp+p-2m), \infty) \times [2m, \infty]$ we have

$$\tfrac{2mr+2mp-mpr-pr}{2pr} < 0$$

and

$$\begin{aligned}
\left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{j_1,\dots,j_m=1}^{n} |T(e_{j_1},\dots,e_{j_m})|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \\
&\leq C^{\text{mult}}_{\mathbb{K},m,p} \|T\| \\
&= C^{\text{mult}}_{\mathbb{K},m,p} \|T\| \, n^{\max\left\{\frac{2mr+2mp-mpr-pr}{2pr},0\right\}}.
\end{aligned}$$

In this case the optimality of the exponent $\max\{(2mr+2mp-mpr-pr)/2pr, 0\}$ is immediate, since one can easily verify that no negative exponent of n is possible.

(b) Let us first consider $(r,p) \in [2, p/(p-m)] \times (m, 2m]$. Define

$$q = \tfrac{mr}{r-1}$$

and note that $q \leq 2m$ and $r = q/(q-m)$. Since $q/(q-m) = r \leq p/(p-m)$ we have $p \leq q$. Then $m < p \leq q \leq 2m$. Note that

$$q^* = \tfrac{mr}{mr+1-r}.$$

Since $m < q \leq 2m$, by the Hardy-Littlewood/Dimant-Sevilla-Peris inequality and using [63, Section 5] we know that every continuous m-linear form on any Banach space E is multiple $(q/(q-m); q^*)$-summing with constant $D^{\text{mult}}_{\mathbb{K},m,q}$, i.e., multiple $\left(r; \frac{mr}{mr+1-r}\right)$-

summing with constant $D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}}$. So for $T:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$ we have (since $q^*\le p^*$),

$$\begin{aligned}
&\left(\sum_{j_1,\dots,j_m=1}^{n}\left|T\left(e_{j_1},\dots,e_{j_m}\right)\right|^r\right)^{\frac{1}{r}}\\
&\le D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}}\left\|T\right\|\left[\left(\sup_{\varphi\in B_{(\ell_p^n)^*}}\sum_{j=1}^{n}\left|\varphi\left(e_j\right)\right|^{\frac{mr}{mr+1-r}}\right)^{\frac{mr+1-r}{mr}}\right]^m\\
&= D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}}\left\|T\right\|\left[\left(n(n^{-\frac{1}{p^*}})^{\frac{mr}{mr+1-r}}\right)^{\frac{mr+1-r}{mr}}\right]^m\\
&= D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}}\left\|T\right\|n^{\frac{p+mr-rp}{pr}}.
\end{aligned}$$

Note that if we have tried to use above an argument similar to (2.5), via Hölder's inequality, we would obtain worse exponents. Now we prove the optimality following the lines of [63]. Defining $R:\ell_p^n\times\cdots\times\ell_p^n\to\mathbb{K}$ by $R(x^{(1)},\dots,x^{(m)})=\sum_{j=1}^n x_j^{(1)}\cdots x_j^{(1)}$, from Hölder's inequality we can easily verify that

$$\|R\|\le n^{1-\frac{m}{p}}.$$

So if the theorem holds for n^s, plugging the m-linear form R into the inequality we have

$$n^{\frac{1}{r}}\le H^{\text{mult}}_{\mathbb{K},m,p,r}n^s n^{1-\frac{m}{p}}$$

and thus, by making $n\to\infty$, we obtain

$$s\ge\tfrac{p+mr-rp}{pr}.$$

If $(r,p)\in(p/(p-m),\infty)\times(m,2m]$ we have

$$\tfrac{p+mr-rp}{pr}<0$$

and

$$\begin{aligned}
\left(\sum_{j_1,\dots,j_m=1}^{n}\left|T(e_{j_1},\dots,e_{j_m})\right|^r\right)^{\frac{1}{r}} &\le \left(\sum_{j_1,\dots,j_m=1}^{n}\left|T(e_{j_1},\dots,e_{j_m})\right|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}}\\
&\le D^{\text{mult}}_{\mathbb{K},m,p}\left\|T\right\|\\
&= D^{\text{mult}}_{\mathbb{K},m,p}\left\|T\right\|n^{\max\left\{\frac{p+mr-rp}{pr},0\right\}}
\end{aligned}$$

In this case the optimality of the exponent $\max\left\{(p+mr-rp)/pr,0\right\}$ is immediate, since one can easily verify that no negative exponent of n is possible. □

Remark 2.3. *Observing the proof of Theorem 2.1 we conclude that the optimal constant $H^{\text{mult}}_{\mathbb{K},m,p,r}$ satisfies:*

$$H^{\text{mult}}_{\mathbb{K},m,p,r} \leq \begin{cases} B^{\text{mult}}_{\mathbb{K},m} & \textit{if } (r,p) \in \left[1, \frac{2m}{m+1}\right] \times [2,2m), \\ C^{\text{mult}}_{\mathbb{K},m,\frac{2mr}{r+mr-2m}} & \textit{if } (r,p) \in \left[\frac{2m}{m+1}, 2\right] \times [2,2m), \\ C^{\text{mult}}_{\mathbb{K},m,p} & \textit{if } (r,p) \in [1,\infty) \times [2m,\infty], \\ D^{\text{mult}}_{\mathbb{K},m,\frac{mr}{r-1}} & \textit{if } (r,p) \in \left[2, \frac{p}{p-m}\right] \times (m,2m], \\ D^{\text{mult}}_{\mathbb{K},m,p} & \textit{if } (r,p) \in \left(\frac{p}{p-m}, \infty\right) \times (m,2m]. \end{cases}$$

Using results of the previous chapters, we have the following estimates for the constants $H^{\text{mult}}_{\mathbb{K},m,p,r}$*:*

$$H^{\text{mult}}_{\mathbb{K},m,p,r} \leq \begin{cases} \eta_{\mathbb{K},m} & \textit{if } (r,p) \in \left[1, \frac{2m}{m+1}\right] \times [2,2m), \\ (\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}} (\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}} & \textit{if } (r,p) \in \left(\frac{2m}{m+1}, 2\right] \times [2,2m), \\ (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (\eta_{\mathbb{K},m})^{\frac{p-2m}{p}} & \textit{if } (r,p) \in [1,\infty) \times [2m, 2m^3-4m^2+2m], \\ \eta_{\mathbb{K},m} & \textit{if } (r,p) \in [1,\infty) \times (2m^3-4m^2+2m, \infty], \\ (\sqrt{2})^{m-1} & \textit{if } (r,p) \in [2,\infty) \times (m,2m], \end{cases}$$

where $\sigma_{\mathbb{R}} = \sqrt{2}$ *and* $\sigma_{\mathbb{C}} = 2/\sqrt{\pi}$ *and*

$$\begin{aligned} \eta_{\mathbb{C},m} &:= \prod_{j=2}^{m} \Gamma\left(2 - \tfrac{1}{j}\right)^{\frac{j}{2-2j}}, \\ \eta_{\mathbb{R},m} &:= \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} && \textit{for } m \leq 13, \\ \eta_{\mathbb{R},m} &:= 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}} && \textit{for } m \geq 14. \end{aligned}$$

Now we will obtain partial answers for the cases not covered by our main theorem, i.e., the cases $(r,p) \in [1,2] \times [1,2)$ and $(r,p) \in (2,\infty) \times [1,m]$.

Proposition 2.4. *Let $m \geq 2$ be a positive integer.*

(a) If $(r,p) \in [1,2] \times [1,2)$, then there is a constant $H^{\text{mult}}_{\mathbb{K},m,p,r} > 0$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^r \right)^{\frac{1}{r}} \leq H^{\text{mult}}_{\mathbb{K},m,p,r} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\| \tag{2.6}$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n. Moreover the optimal exponent of n is not smaller than $(2m-r)/2r$.

(b) If $(r,p) \in (2,\infty) \times [1,m]$, then there is a constant $H^{\text{mult}}_{\mathbb{K},m,p,r} > 0$ such that

$$\left(\sum_{j_1,.,\ldots,j_m=1}^{n} |T(e_{j_1},\ldots,e_{j_m})|^r \right)^{\frac{1}{r}} \leq \begin{cases} H^{\text{mult}}_{\mathbb{K},m,p,r} n^{\frac{2m-p+\varepsilon}{pr}} \|T\| & \textit{if } p > 2 \\ H^{\text{mult}}_{\mathbb{K},m,p,r} n^{\frac{2m-p}{pr}} \|T\| & \textit{if } p = 2 \end{cases}$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all positive integers n and all $\varepsilon > 0$. Moreover the optimal exponent of n is not smaller than $(2mr + 2mp - mpr - pr)/2pr$ and not smaller than $(2m - r)/2r$ if $2 \leq p \leq m$.

Proof. *(a)* The proof of (2.6) is the same of the proof of Theorem 2.1*(a)*. The estimate for the bound of the optimal exponent also uses the generalized Kahane–Salem–Zygmund inequality (2). Since $p \leq 2$ we have

$$n^{\frac{m}{r}} \leq C_m H_{\mathbb{K},m,p,r}^{\text{mult}} n^s n^{\frac{1}{2}}$$

and thus, by making $n \to \infty$,

$$s \geq \tfrac{2m-r}{2r}.$$

(b) Let $\delta = 0$ if $p = 2$ and $\delta > 0$ if $p > 2$. First note that every continuous m-linear form on ℓ_p spaces is obviously multiple $(\infty; p^* - \delta)$-summing and also multiple $(2; 2m/(2m-1))$-summing (this is a consequence of the Hardy–Littlewood inequality and [63, Section 5]). Using [37, Proposition 4.3] we conclude that every continuous m-linear form on ℓ_p spaces is multiple $(r; mpr/(2m + mpr - mr - p + \varepsilon))$-summing for all $\varepsilon > 0$ (and $\varepsilon = 0$ if $p = 2$). Therefore, there exist $H_{\mathbb{K},m,p,r}^{\text{mult}} > 0$ such that

$$\begin{aligned}
&\left(\sum_{j_1,\ldots,j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \\
&\leq H_{\mathbb{K},m,p,r}^{\text{mult}} \left[(n (n^{-\frac{1}{p^*}})^{\frac{mpr}{2m+mpr-mr-p+\varepsilon}})^{\frac{2m+mpr-mr-p+\varepsilon}{mpr}} \right]^m \|T\| \\
&= H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m+mpr-mr-p+\varepsilon}{pr}} (n^{\frac{1}{p}-1})^m \|T\| \\
&= H_{\mathbb{K},m,p,r}^{\text{mult}} n^{\frac{2m-p+\varepsilon}{pr}} \|T\|.
\end{aligned}$$

The bounds for the optimal exponents are obtained via the generalized Kahane–Salem–Zygmund inequality (2) as in the previous cases. □

Remark 2.5. *Item (b) of the Proposition 2.4 with $p = 2$ recovers [37, Corollary 5.20(ii)].*

Chapter 3

The polynomial Bohnenblust–Hille and Hardy–Littlewood inequalities

In this chapter we present results of the paper:

[11] G. Araújo, and D. Pellegrino, *Lower bounds for the complex polynomial Hardy-Littlewood inequality*, Linear Algebra Appl. **474** (2015), 184-191.

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, define $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$. The polynomial Bohnenblust–Hille inequality (see [5, 32] and the references therein) ensures that, given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_∞^n given by

$$P(x_1, ..., x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha,$$

then

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m}^{\mathrm{pol}} \|P\| \tag{3.1}$$

for some constant $B_{\mathbb{K},m}^{\mathrm{pol}} \geq 1$ which does not depend on n (the exponent $\frac{2m}{m+1}$ is optimal), where $\|P\| := \sup_{z \in B_{\ell_\infty^n}} |P(z)|$. The search of precise estimates of the growth of the constants $B_{\mathbb{K},m}^{\mathrm{pol}}$ is fundamental for different applications and remains an important open problem (see [23] and the references therein).

For real scalars it was shown in [45, Theorem 2.2] that

$$(1.17)^m \leq B_{\mathbb{R},m}^{\mathrm{pol}} \leq C(\varepsilon)\,(2+\varepsilon)^m,$$

where $C(\varepsilon)\,(2+\varepsilon)^m$ means that given $\varepsilon > 0$, there is a constant $C(\varepsilon) > 0$ such that $B_{\mathbb{C},m}^{\mathrm{pol}} \leq C(\varepsilon)\,(2+\varepsilon)^m$ for all m. In other words, this means that for real scalars the hypercontractivity of $B_{\mathbb{R},m}^{\mathrm{pol}}$ is optimal.

For complex scalars the behavior of $B_{\mathbb{K},m}^{\mathrm{pol}}$ is still unknown. The best information we have thus far about $B_{\mathbb{C},m}^{\mathrm{pol}}$ are due D. Núñez-Alarcón [101] (lower bounds) and F. Bayart,

D. Pellegrino and J.B. Seoane-Sepúlveda [23] (upper bounds)

$$B^{\mathrm{pol}}_{\mathbb{C},m} \geq \begin{cases} \left(1+\dfrac{1}{2^{m-1}}\right)^{\frac{1}{4}} & \text{for } m \text{ even;} \\ \left(1+\dfrac{1}{2^{m-1}}\right)^{\frac{m-1}{4m}} & \text{for } m \text{ odd;} \end{cases}$$
$$B^{\mathrm{pol}}_{\mathbb{C},m} \leq C(\varepsilon)\,(1+\varepsilon)^m .$$

The following diagram shows the evolution of the estimates of $B^{\mathrm{pol}}_{\mathbb{K},m}$ for complex scalars.

Authors	**Year**	**Estimate**
Bohnenblust and Hille	1931, [32] (*Ann.Math.*)	$B^{\mathrm{pol}}_{\mathbb{C},m} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1}$
Defant, Frerick, Ortega-Cerdá, Ounaïes, and Seip	2011, [55] (*Ann.Math.*)	$B^{\mathrm{pol}}_{\mathbb{C},m} \leq \left(1+\frac{1}{m-1}\right)^{m-1} \sqrt{m}\left(\sqrt{2}\right)^{m-1}$
Bayart, Pellegrino, and Seoane-Sepúlveda	2014, [23] (*Adv.Math.*)	$B^{\mathrm{pol}}_{\mathbb{C},m} \leq C(\varepsilon)\,(1+\varepsilon)^m$

When replacing ℓ_∞^n by ℓ_p^n the extension of the polynomial Bohnenblust–Hille inequality is called polynomial Hardy–Littlewood inequality and the optimal exponents are $\frac{2mp}{mp+p-2m}$ for $2m \leq p \leq \infty$. More precisely, given positive integers $m \geq 2$ and $n \geq 1$, as a consequence of the multilinear Hardy–Littlewood inequality (see [4, 63]), if P is a homogeneous polynomial of degree m on ℓ_p^n with $2m \leq p \leq \infty$ given by $P(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$, then there is a constant $C^{\mathrm{pol}}_{\mathbb{K},m,p} \geq 1$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}} \leq C^{\mathrm{pol}}_{\mathbb{K},m,p}\,\|P\|\,, \tag{3.2}$$

and $C^{\mathrm{pol}}_{\mathbb{K},m,p}$ does not depend on n, where $\|P\| := \sup_{z\in B_{\ell_p^n}} |P(z)|$. Using the generalized Kahane–Salem–Zygmund inequality (2) (see, for instance, [5]) we can verify that the exponents $\frac{2mp}{mp+p-2m}$ are optimal for $2m \leq p \leq \infty$. When $p=\infty$, since $\frac{2mp}{mp+p-2m} = \frac{2m}{m+1}$, we recover the polynomial Bohnenblust–Hille inequality.

As in the multilinear case, for $m < p < 2m$ there is also a version of the polynomial Hardy–Littlewood inequality (see [63]): given positive integers $m \geq 2$ and $n \geq 1$, if P is a homogeneous polynomial of degree m on ℓ_p^n with $m < p < 2m$ given by $P(x_1,\ldots,x_n) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha$, then there is a (optimal) constant $D^{\mathrm{pol}}_{\mathbb{K},m,p} \geq 1$ (not depending on n) such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{p}{p-m}}\right)^{\frac{p-m}{p}} \leq D^{\mathrm{pol}}_{\mathbb{K},m,p}\,\|P\|\,, \tag{3.3}$$

and the exponents $\frac{p}{p-m}$ are optimal.

In this chapter we look for upper and lower estimates for $C^{\mathrm{pol}}_{\mathbb{K},m,p}$ and $D^{\mathrm{pol}}_{\mathbb{K},m,p}$. Our main contributions regarding the constants of the polynomial Hardy–Littlewood inequality can be summarized in the following result (in this chapter we will only present the proof of the items *(1)(ii)* and *(3)*. For details of other results see [8]):

Theorem 3.1. *Let $m \geq 2$.*

(1) Let $2m \leq p \leq \infty$.

(i) If $\mathbb{K} = \mathbb{R}$, then $C_{\mathbb{R},m,p}^{\mathrm{pol}} \geq 2^{\frac{m^2p+10m-p-6m^2-4}{4mp}} \geq \left(\sqrt[16]{2}\right)^m$;

(ii) If $\mathbb{K} = \mathbb{C}$, then

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \begin{cases} 2^{\frac{m}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-1}{p}} & \text{for } m \text{ odd.} \end{cases}$$

(2) For $2m \leq p \leq \infty$,

$$C_{\mathbb{K},m,p}^{\mathrm{pol}} \leq C_{\mathbb{K},m,p}^{\mathrm{mult}} \frac{m^m}{(m!)^{\frac{mp+p-2m}{2mp}}},$$

(3) For $m < p < 2m$,

$$D_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \begin{cases} 2^{\frac{m}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-1}{p}} & \text{for } m \text{ odd.} \end{cases}$$

(4) For $m < p < 2m$,

$$D_{\mathbb{K},m,p}^{\mathrm{pol}} \leq D_{\mathbb{K},m,p}^{\mathrm{mult}} \frac{m^m}{(m!)^{\frac{p-m}{p}}},$$

Remark 3.2. *Trying to find a certain pattern in the behavior of the constants of the Bohnenblust–Hille and Hady–Littlewood inequalities, we define $B_{\mathbb{K},m}^{\mathrm{pol}}(n)$, $C_{\mathbb{K},m,p}^{\mathrm{pol}}(n)$ and $D_{\mathbb{K},m,p}^{\mathrm{pol}}(n)$ as the best (meaning* smallest*) value of the constants appearing in* (3.1), (3.2) *and* (3.3)*, respectively, for $n \in \mathbb{N}$ fixed. A number of papers related to these particular cases are being produced and we can summarize the main findings of these papers (see [9, 46, 48, 49, 78]) as follows:*

- $B_{\mathbb{C},2}^{\mathrm{pol}}(2) = \sqrt[4]{\frac{3}{2}}$;

- $B_{\mathbb{R},2}^{\mathrm{pol}}(2) = (2t_0^{4/3} + (2\sqrt{t_0 - t_0^2})^{4/3})^{3/4}$, *with* $t_0 = \frac{2\sqrt[3]{107+9\sqrt{129}}+\sqrt[3]{856-72\sqrt{129}}+16}{36}$;

- $B_{\mathbb{R},3}^{\mathrm{pol}}(2) \geq 2.5525$, $B_{\mathbb{R},5}^{\mathrm{pol}}(2) \geq 6.83591$, $B_{\mathbb{R},6}^{\mathrm{pol}}(2) \geq 10.7809$, $B_{\mathbb{R},7}^{\mathrm{pol}}(2) \geq 19.96308$, $B_{\mathbb{R},8}^{\mathrm{pol}}(2) \geq 33.36323$, $B_{\mathbb{R},10}^{\mathrm{pol}}(2) \geq 90.35556$, $B_{\mathbb{R},600}^{\mathrm{pol}}(2) \geq (1.65171)^{600}$, $B_{\mathbb{R},602}^{\mathrm{pol}}(2) \geq (1.61725)^{602}$;

- *For $4 \leq p \leq \infty$,*

$$\begin{aligned} C_{\mathbb{R},2,p}^{\mathrm{pol}}(2) &= \max_{\alpha\in[0,1]} \left[2\left|\frac{2\alpha^p - 1}{\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}}\right|^{\frac{4p}{3p-4}} \right. \\ &\quad \left. + \left(2\alpha\,(1-\alpha^p)^{\frac{1}{p}}\,\frac{\alpha^{p-2} + (1-\alpha^p)^{\frac{p-2}{p}}}{\alpha^2 + (1-\alpha^p)^{\frac{2}{p}}}\right)^{\frac{4p}{3p-4}} \right]^{\frac{3p-4}{4p}} ; \end{aligned}$$

- $C^{\mathrm{pol}}_{\mathbb{R},2,4}(2) = D^{\mathrm{pol}}_{\mathbb{R},2,4}(2) = \sqrt{2}$, $C^{\mathrm{pol}}_{\mathbb{R},3,6}(2) = D^{\mathrm{pol}}_{\mathbb{R},3,6}(2) \geq 2.236067$, $C^{\mathrm{pol}}_{\mathbb{R},5,10}(2) = D^{\mathrm{pol}}_{\mathbb{R},5,10}(2) \geq 6.236014$, $C^{\mathrm{pol}}_{\mathbb{R},6,12}(2) = D^{\mathrm{pol}}_{\mathbb{R},6,12}(2) \geq 10.636287$, $C^{\mathrm{pol}}_{\mathbb{R},7,14}(2) = D^{\mathrm{pol}}_{\mathbb{R},7,14}(2) \geq 18.095148$, $C^{\mathrm{pol}}_{\mathbb{R},8,16}(2) = D^{\mathrm{pol}}_{\mathbb{R},8,16}(2) \geq 31.727174$, $C^{\mathrm{pol}}_{\mathbb{R},10,20}(2) = D^{\mathrm{pol}}_{\mathbb{R},10,20}(2) \geq 91.640152$.

- *For* $2 < p \leq 4$, $D^{\mathrm{pol}}_{\mathbb{R},2,p}(2) = 2^{2/p}$.

3.1 Lower bounds for the complex polynomial Hardy–Littlewood inequality

In this section we provide nontrivial lower bounds for the constants of the complex case of the polynomial Hardy–Littlewood inequality. More precisely we prove that, for $m \geq 2$ and $2m \leq p < \infty$,

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq 2^{\frac{m}{p}}$$

for m even, and

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq 2^{\frac{m-1}{p}}$$

for m odd. For instance,

$$\sqrt{2} \leq C^{\mathrm{pol}}_{\mathbb{C},2,4} \leq 3.1915.$$

Let $m \geq 2$ be an even positive integer and let $p \geq 2m$. Consider the 2-homogeneous polynomials $Q_2 : \ell_p^2 \to \mathbb{C}$ and $\widetilde{Q_2} : \ell_\infty^2 \to \mathbb{C}$ both given by $(z_1, z_2) \mapsto z_1^2 - z_2^2 + cz_1z_2$. We know from [19, 45] that

$$\|\widetilde{Q_2}\| = \left(4 + c^2\right)^{\frac{1}{2}}.$$

If we follow the lines of [101] and we define the m-homogeneous polynomial $Q_m : \ell_p^m \to \mathbb{C}$ by $Q_m(z_1, ..., z_m) = z_3 \ldots z_m Q_2(z_1, z_2)$ we obtain

$$\|Q_m\| \leq 2^{-\frac{m-2}{p}} \|Q_2\| \leq 2^{-\frac{m-2}{p}} \|\widetilde{Q_2}\| = 2^{-\frac{m-2}{p}} \left(4 + c^2\right)^{\frac{1}{2}},$$

where we use the obvious inequality

$$\|Q_2\| \leq \|\widetilde{Q_2}\|.$$

Therefore, for $m \geq 2$ even and $c \in \mathbb{R}$, from the polynomial Hardy–Littlewood inequality it follows that

$$C^{\mathrm{pol}}_{\mathbb{C},m,p} \geq \frac{\left(2 + |c|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}} \left(4 + c^2\right)^{\frac{1}{2}}}.$$

If

$$c > \left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1 - 2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}},$$

it is not too difficult to prove that

$$2^{-\frac{m-2}{p}} \left(4 + c^2\right)^{\frac{1}{2}} < \left(\left(2^{\frac{mp+p-2m}{2mp}}\right)^2 + c^2\right)^{\frac{1}{2}},$$

i.e.,

$$2^{-\frac{m-2}{p}}\left(4+c^2\right)^{\frac{1}{2}} < \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_2.$$

Since $\frac{2mp}{mp+p-2m} \leq 2$, we know that $\ell_{\frac{2mp}{mp+p-2m}} \subset \ell_2$ and $\|\cdot\|_2 \leq \|\cdot\|_{\frac{2mp}{mp+p-2m}}$. Therefore, for all

$$c > \left(\frac{2^{\frac{2p+4-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned} 2^{-\frac{m-2}{p}}\left(4+c^2\right)^{\frac{1}{2}} &< \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_2 \\ &\leq \left\|\left(2^{\frac{mp+p-2m}{2mp}}, c\right)\right\|_{\frac{2mp}{mp+p-2m}} \\ &= \left(2+c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}, \end{aligned}$$

from which we conclude that

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left(2+c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}}\left(4+c^2\right)^{\frac{1}{2}}} > 1.$$

If $m \geq 3$ is odd, since $\|Q_m\| \leq \|Q_{m-1}\|$, then we have $\|Q_m\| \leq 2^{-\frac{m-3}{p}}\left(4+c^2\right)^{\frac{1}{2}}$ and thus we can now proceed analogously to the even case and finally conclude that for

$$c > \left(\frac{2^{\frac{2p+6-2m}{p}} - 2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}$$

we have

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left(2+c^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}}\left(4+c^2\right)^{\frac{1}{2}}} > 1.$$

So we have:

Proposition 3.3. *Let $m \geq 2$ be a positive integer and let $p \geq 2m$. Then, for every $\epsilon > 0$,*

$$C_{\mathbb{C},m,p}^{\text{pol}} \geq \frac{\left(2+\left(\left(\frac{2^{\frac{2p+4-2m}{p}}-2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-2}{p}}\left(4+\left(\left(\frac{2^{\frac{2p+4-2m}{p}}-2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-4}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^2\right)^{\frac{1}{2}}} > 1 \qquad \text{if } m \text{ is even}$$

and

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \frac{\left(2+\left(\left(\frac{2^{\frac{2p+6^{-}2m}{p}}-2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{2^{-\frac{m-3}{p}}\left(4+\left(\left(\frac{2^{\frac{2p+6^{-}2m}{p}}-2^{\frac{mp+p-2m}{mp}}}{1-2^{-\frac{2m-6}{p}}}\right)^{\frac{1}{2}}+\epsilon\right)^{2}\right)^{\frac{1}{2}}} > 1 \quad \text{if } m \text{ is odd.}$$

However, we have another approach to the problem, which is surprisingly simpler than the above approach and still seems to give best (bigger) lower bounds for the constants of the polynomial Hardy–Littlewood inequality (even for the case $m < p < 2m$).

Theorem 3.4. *Let $m \geq 2$ be a positive integer and let $m < p \leq \infty$. Then $2^{\frac{m}{p}}$ if m is even and $2^{\frac{m-1}{p}}$ if m is odd are lower bounds for the constants of the complex polynomial Hardy–Littlewood inequality.*

Proof. Let $m \geq 2$ be a positive integer and let $p \geq 2m$. Consider $P_2 : \ell_p^2 \to \mathbb{C}$ the 2-homogeneous polynomial given by $\mathbf{z} \mapsto z_1 z_2$. Observe that

$$\|P_2\| = \sup_{|z_1|^p+|z_2|^p=1} |z_1 z_2| = \sup_{|z|\leq 1} |z| \left(1-|z|^p\right)^{\frac{1}{p}} = 2^{-\frac{2}{p}}.$$

More generally, if $m \geq 2$ is even and P_m is the m-homogeneous polynomial given by $\mathbf{z} \mapsto z_1 \cdots z_m$, then

$$\|P_m\| \leq 2^{-\frac{m}{p}}.$$

Therefore, from the polynomial Hardy–Littlewood inequality we know that

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \frac{\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}}\right)^{\frac{mp+p-2m}{2mp}}}{\|P_m\|} \geq \frac{1}{2^{-\frac{m}{p}}} = 2^{\frac{m}{p}}.$$

If $m \geq 3$ is odd, we define again the m-homogeneous polynomial P_m given by $\mathbf{z} \mapsto z_1 \cdots z_m$ and since $\|P_m\| \leq \|P_{m-1}\|$, then we have $\|P_m\| \leq 2^{-\frac{m-1}{p}}$ and thus

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \frac{1}{2^{-\frac{m-1}{p}}} = 2^{\frac{m-1}{p}}.$$

With the same arguments used for the case $2m \leq p \leq \infty$, we obtain the similar estimate *(3)* of Theorem 3.1 for the case $m < p < 2m$. $\square$

The estimates of Theorem 3.3 seems to become better when ϵ grows (this seems to be a clear sign that we should avoid the terms z_1^2 and z_2^2 in our approach). Making $\epsilon \to \infty$ in Theorem 3.3 we obtain

$$C_{\mathbb{C},m,p}^{\mathrm{pol}} \geq \begin{cases} 2^{\frac{m-2}{p}} & \text{for } m \text{ even;} \\ 2^{\frac{m-3}{p}} & \text{for } m \text{ odd,} \end{cases}$$

which are slightly worse than the estimates from Theorem 3.4.

3.2 The complex polynomial Hardy-Littlewood inequality: Upper estimates

In this section, let us use the following notation: $S_{\ell_p^n}$ denotes the unit sphere on ℓ_p^n if $p<\infty$, and $S_{\ell_\infty^n}$ denotes the n-dimensional torus. More precisely: for $p\in(0,\infty)$

$$S_{\ell_p^n} := \left\{ \mathbf{z}=(z_1,...,z_n)\in\mathbb{C}^n : \|\mathbf{z}\|_{\ell_p^n}=1 \right\},$$

and

$$S_{\ell_\infty^n} := \mathbb{T}^n = \left\{ \mathbf{z}=(z_1,...,z_n)\in\mathbb{C}^n : |z_i|=1 \right\}.$$

Let μ^n the normalized Lebesgue measure on the respective set. The following lemma is a particular instance ($1\le p=s\le 2$ and $q=2$) of the Khinchin-Steinhaus polynomial inequalities (for polynomials homogeneous or not) and $p\le q$.

Lemma 3.5. *Let $1\le s\le 2$. For every m-homogeneous polynomial $P(\mathbf{z})=\sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$ on $\mathbb{C}^n$ with values in $\mathbb{C}$, we have*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^2\right)^{\frac{1}{2}} \le \left(\frac{2}{s}\right)^{\frac{m}{2}} \left(\int_{\mathbb{T}^n} |P(\mathbf{z})|^s \, d\mu^n(\mathbf{z})\right)^{\frac{1}{s}}.$$

When $n=1$ a result due to F.B. Weissler (see [126]) asserts that the optimal constant for the general case is $\sqrt{2/s}$. In the n-dimensional case the best constant for m-homogeneous polynomials is $(\sqrt{2/s})^m$ (see also [22]).

For $m\in[2,\infty]$ let us define $p_0(m)$ as the infimum of the values of $p\in[2m,\infty]$ such that for all $1\le s\le\frac{2p}{p-2}$ there is a $K_{s,p}>0$ such that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2p}{p-2}}\right)^{\frac{p-2}{2p}} \le K_{s,p}^m \left(\int_{S_{\ell_p^n}} |P(\mathbf{z})|^s \, d\mu^n(\mathbf{z})\right)^{\frac{1}{s}} \tag{3.4}$$

for all positive integers n and all m-homogeneous polynomials $P:\mathbb{C}^n\to\mathbb{C}$. From Lemma 3.5 we know that this definition makes sense, since from this lemma we know that (3.4) is valid for $p=\infty$. We conjecture that $p_0(m)\le m^2$. If it is true that $p_0(m)<\infty$, it is possible to prove the following new estimate for $C_{\mathbb{C},m,p}^{\mathrm{pol}}$ (see [8]).

Theorem 3.6. *Let $m\in[2,\infty]$ and $1\le k\le m-1$. If $p_0(m-k)<p\le\infty$ (and $p=\infty$ if $p_0(m-k)=\infty$) then, for every m-homogeneous polynomial $P:\ell_p^n\to\mathbb{C}$, defined by*

$P(\mathbf{z}) = \sum_{|\alpha|=m} a_\alpha \mathbf{z}^\alpha$, *we have*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}}$$
$$\leq K^{m-k}_{\frac{2kp}{kp+p-2k},p} \cdot \frac{m^m}{(m-k)^{m-k}} \cdot \left(\frac{(m-k)!}{m!} \right)^{\frac{p-2}{2p}} \left(\frac{2}{\sqrt{\pi}} \right)^{\frac{2k(k-1)}{p}} \cdot \left(B^{\mathrm{mult}}_{\mathbb{C},k} \right)^{\frac{p-2k}{p}} \|P\|,$$

where $B^{\mathrm{mult}}_{\mathbb{C},k}$ *is the optimal constant of the multilinear Bohnenblust–Hille inequality associated with* k*-linear forms.*

Part II

Summability of multilinear operators

Chapter 4

Maximal spaceability and optimal estimates for summing multilinear operators

In this chapter we present results of the paper:

[15] G. Araújo, and D. Pellegrino, *Optimal estimates for summing multilinear operators*, arXiv:1403.6064v2 [math.FA].

If $1 \leq p \leq q < \infty$, we say that a continuous linear operator $T : E \to F$ is (q,p)-summing if $(T(x_j))_{j=1}^{\infty} \in \ell_q(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. The class of (q,p)-summing linear operators from E to F will be represented by $\Pi_{(q;p)}(E,F)$. An equivalent formulation asserts that $T : E \to F$ is (q,p)-summing if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{\infty} \|T(x_j)\|^q\right)^{1/q} \leq C \left\|(x_j)_{j=1}^{\infty}\right\|_{w,p}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$. The above inequality can also be replaced by: there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{n} \|T(x_j)\|^q\right)^{1/q} \leq C \left\|(x_j)_{j=1}^{n}\right\|_{w,p}$$

for all $x_1, \ldots, x_n \in E$ and all positive integers n. The infimum of all C that satisfy the above inequalities defines a norm, denoted by $\pi_{(q;p)}(T)$, and $\left(\Pi_{(q;p)}\left(E,F\right), \pi_{(q;p)}\left(\cdot\right)\right)$ is a Banach space.

More generally, we can define:

Definition 4.1. *For $1 \leq p_1, ..., p_m < \infty$ and $\frac{1}{q} \leq \sum_{j=1}^{m} \frac{1}{p_j}$ recall that a continuous m-linear operator $T : E_1 \times \cdots \times E_m \to F$ is absolutely $(q; p_1, ..., p_m)$-summing if there is a $C > 0$ such that*

$$\left(\sum_{j=1}^{n} \left\|T(x_j^{(1)}, ..., x_j^{(m)})\right\|^q\right)^{\frac{1}{q}} \leq C \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

for all positive integers n and all $(x_j^{(k)})_{j=1}^{n} \in E_k$, $k = 1, ..., m$.

- We represent the class of all absolutely $(q;p_1,...,p_m)$-summing multiple operators from $E_1,....,E_m$ to F by $\Pi^m_{\mathrm{as}(q;p_1,...,p_m)}(E_1,...,E_m;F)$;
- When $p_1=\cdots=p_m=p$, we denote $\Pi^m_{\mathrm{as}(q;p,...,p)}(E_1,...,E_m;F)$ by

$$\Pi^m_{\mathrm{as}(q;p)}(E_1,...,E_m;F).$$

The infimum over all C as above defines a norm on $\Pi^m_{\mathrm{as}(q;p_1,...,p_m)}(E_1,...,E_m;F)$, which we denote by $\pi_{\mathrm{as}(q;p_1,...,p_m)}(T)$ (or $\pi_{\mathrm{as}(q;p)}(T)$ if $p_1=\cdots=p_m=p$).

In 2003 Matos [86] and, independently, Bombal, Pérez-García and Villanueva [34] introduced the notion of multiple summing multilinear operators.

Definition 4.2 (Multiple summing operators [34, 86]). *Let $1\leq p_1,\ldots,p_m\leq q<\infty$. A bounded m-linear operator $T:E_1\times\cdots\times E_m\to F$ is multiple $(q;p_1,\ldots,p_m)$-summing if there exists $C_m>0$ such that*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left\|T(x^{(1)}_{j_1},\ldots,x^{(m)}_{j_m})\right\|^q\right)^{\frac{1}{q}}\leq C_m\prod_{k=1}^{m}\left\|(x^{(k)}_j)_{j=1}^{\infty}\right\|_{w,p_k}\tag{4.1}$$

for every $(x^{(k)}_j)_{j=1}^{\infty}\in\ell^w_{p_k}(E_k)$, $k=1,\ldots,m$.

- The class of all multiple $(q;p_1,\ldots,p_m)$-summing operators from $E_1\times\cdots\times E_m$ to F will be denoted by $\Pi^m_{\mathrm{mult}(q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F)$.
- When $p_1=\cdots=p_m=p$ we write $\Pi^m_{\mathrm{mult}(q;p)}(E_1,...,E_m;F)$ instead of

$$\Pi^m_{\mathrm{mult}(q;p,...,p)}(E_1,...,E_m;F).$$

The infimum over all C_m satisfying (4.1) defines a norm in $\Pi^m_{\mathrm{mult}(q;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F)$ and is denoted by $\pi_{\mathrm{mult}(q;p_1,\ldots,p_m)}(T)$ (or $\pi_{\mathrm{mult}(q;p)}(T)$ if $p_1=\cdots=p_m=p$).

Using that $\mathcal{L}(c_0;E)$ is isometrically isomorphic to $\ell^w_1(E)$ (see [62]), Bohnenblust–Hille's inequality can be re-written as:

Theorem 4.3 (Bohnenblust–Hille *re-written* [115] (see also [37])). *If $m\geq 2$ is a positive integer and $T\in\mathcal{L}(E_1,\ldots,E_m;\mathbb{K})$, then*

$$\left(\sum_{j_1,\ldots,j_m=1}^{\infty}\left|T(x^{(1)}_{j_1},\ldots,x^{(m)}_{j_m})\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\leq B^{\mathrm{mult}}_{\mathbb{K},m}\|T\|\prod_{k=1}^{m}\left\|(x^{(k)}_j)_{j=1}^{\infty}\right\|_{w,1}\tag{4.2}$$

for every $(x^{(k)}_j)_{j=1}^{\infty}\in\ell^w_1(E_k)$, $k=1,\ldots,m$ and $j=1,\ldots,N$, where $B^{\mathrm{mult}}_{\mathbb{K},m}$ is the optimal constant of the classical Bohnenblust–Hille inequality.

Proof. Let $T\in\mathcal{L}(E_1,\ldots,E_m;\mathbb{K})$ and let $(x^{(k)}_j)_{j=1}^{\infty}\in\ell^w_1(E_k)$, $k=1,\ldots,m$. From [62, Prop. 2.2.] we have the boundedness of the linear operator $u_k:c_0\to E_k$ such that $u_k(e_j)=x^{(k)}_j$ and

$$\|u_k\|=\left\|(x^{(k)}_j)_{j=1}^{\infty}\right\|_{w,1},$$

for each $k = 1, \ldots, m$. Thus, $S : c_0 \times \cdots \times c_0 \to \mathbb{K}$ defined by

$$S(y_1, \ldots, y_m) = T\left(u_1\left(y_1\right), \ldots, u_m\left(y_m\right)\right)$$

is a bounded m-linear operator and $\|S\| \leq \|T\| \|u_1\| \cdots \|u_m\|$. Therefore,

$$\begin{aligned}\left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left|T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} &= \left(\sum_{j_1,\ldots,j_m=1}^{\infty} \left|S\left(e_{j_1}, \ldots, e_{j_m}\right)\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \\ \leq B_{\mathbb{K},m}^{\text{mult}} \|S\| \leq B_{\mathbb{K},m}^{\text{mult}} \|T\| \prod_{k=1}^{m} \|u_k\| &= B_{\mathbb{K},m}^{\text{mult}} \|T\| \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,1}.\end{aligned}$$

□

In this sense the Bohnenblust–Hille theorem, (1.1) can be seen as the beginning of the notion of multiple summing operators, that is, in the modern terminology, the classical Bohnenblust–Hille inequality [32] ensures that, for all $m \geq 2$ and all Banach spaces $E_1, ..., E_m$,

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\text{mult}\left(\frac{2m}{m+1};1\right)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

4.1 Maximal spaceability and multiple summability

In this section we are interested in estimating the size of the set of non multiple summing (and non absolutely summing) multilinear operators. For this task we use the notion of spaceability.

For a given Banach space E, a subset $A \subset E$ is *spaceable* if $A \cup \{0\}$ contains a closed infinite-dimensional subspace V of E (for details on spaceability and the related notion of lineability we refer to [18, 26, 27, 47] and the references therein). When $\dim V = \dim E$, A is called *maximal spaceable*. From now on $\mathfrak{c}$ denotes the cardinality of the continuum.

Proposition 4.4. *Let $E_1, ..., E_m$ be separable Banach spaces. Then,*

$$\dim \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \mathfrak{c}.$$

Proof. From [36, Remark 2.5] we know that $\dim \mathcal{L}(E_1, ..., E_m; \mathbb{K}) \geq \mathfrak{c}$.

Since $E_1, ..., E_m$ are separable, let $\omega_j \subseteq E_j$, $j = 1, ..., m$, be a countable, dense subset of E_j and let γ be a basis of $\mathcal{L}(E_1, ..., E_m; \mathbb{K})$. Define

$$\begin{array}{cccc} g & : & \gamma & \to & \mathbb{K}^{\omega_1 \times \cdots \times \omega_m} \\ & & T & \mapsto & T|_{\omega_1 \times \cdots \times \omega_m}, \end{array}$$

with $\mathbb{K}^{\omega_1 \times \cdots \times \omega_m}$ the set of all functions from $\omega_1 \times \cdots \times \omega_m$ to $\mathbb{K}$. Observe that g injective. Indeed, let $S, T \in \gamma$ such that $g(S) = g(T)$, i.e.,

$$S|_{\omega_1 \times \cdots \times \omega_m} = T|_{\omega_1 \times \cdots \times \omega_m}.$$

Given $x \in E_1 \times \cdots \times E_m$, since $\omega_1 \times \cdots \times \omega_m$ is dense on $E_1 \times \cdots \times E_m$, there exist $(x_n)_{n=1}^{\infty} \subset \omega_1 \times \cdots \times \omega_m$ with $\lim_{n\to\infty} x_n = x$. Since S and T are continuous, it follows

that
$$S(x) = S(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x).$$
Thus $S = T$ and hence g is injective, as required.

Therefore,
$$\dim \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \operatorname{card}(\gamma) \le \operatorname{card}(\mathbb{K}^{\omega_1\times\cdots\times\omega_m}) = \operatorname{card}(\mathbb{K}^{\mathbb{N}}) = \mathfrak{c},$$
where $\mathbb{K}^{\mathbb{N}}$ is the set of all functions from $\mathbb{N}$ to $\mathbb{K}$. □

Corollary 4.5. $\dim(\mathcal{L}({}^m\ell_p; \mathbb{K})) = \mathfrak{c}$.

Before we introduce the next result, it is important to note that:

Remark 4.6. *Let $1 \le s \le r < \infty$ and let $E_1, ..., E_m, F$ be Banach spaces with $\dim E_j < \infty$ for all $j = 1, ..., m$. Then*
$$\mathcal{L}(E_1, ..., E_m; F) = \Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; F).$$

In fact, since $s \le r$ we have $\ell_s \subseteq \ell_r$ and $\|\cdot\|_r \le \|\cdot\|_s$. Since E_j has finite dimension for all $j = 1, ..., m$, it follows that $\ell_s^w(E_j) = \ell_s(E_j)$ for all $j = 1, ..., m$. Thus, consider $T \in \mathcal{L}(E_1, ..., E_m; F)$, $n \in \mathbb{N}$ and $(x^{(k)}_{j_k})_{j_k=1}^n \in \ell_s^w(E_k)$, $k = 1, ..., m$, and observe that
$$\begin{aligned}
\left(\sum_{j_1,\dots,j_m=1}^{n} \left\|T\left(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m}\right)\right\|^r\right)^{\frac{1}{r}} &= \left\|\left(\left\|T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m})\right\|\right)^n_{j_1,\dots,j_m=1}\right\|_r \\
\le \left\|\left(\left\|T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m})\right\|\right)^n_{j_1,\dots,j_m=1}\right\|_s &= \left(\sum_{j_1,\dots,j_m=1}^{n} \left\|T(x^{(1)}_{j_1}, ..., x^{(m)}_{j_m})\right\|^s\right)^{\frac{1}{s}} \\
\le \|T\| \left(\sum_{j_1,\dots,j_m=1}^{n} \left\|x^{(1)}_{j_1}\right\|^s \cdots \left\|x^{(m)}_{j_m}\right\|^s\right)^{\frac{1}{s}} &= \|T\| \left(\sum_{j_1=1}^{n} \left\|x^{(1)}_{j_1}\right\|^s \cdots \sum_{j_m=1}^{n} \left\|x^{(m)}_{j_m}\right\|^s\right)^{\frac{1}{s}} \\
= \|T\| \prod_{k=1}^{m} \left\|(x^{(k)}_{j_k})^n_{j_k=1}\right\|_s &= \|T\| \prod_{k=1}^{m} \left\|(x^{(k)}_{j_k})^n_{j_k=1}\right\|_{w,s},
\end{aligned}$$
i.e., $T \in \Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; F)$.

Theorem 4.7. *Let $m \ge 1$, $p \in [2, \infty)$. If $1 \le s < p^*$ and*
$$r < \frac{2ms}{s + 2m - ms}$$
then
$$\mathcal{L}({}^m\ell_p; \mathbb{K}) \setminus \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p; \mathbb{K})$$
is maximal spaceable.

Proof. We consider the case of complex scalars. The case of real scalars is obtained from the complex case via a standard complexification argument (see [37]). An extended version of the Kahane–Salem–Zygmund inequality (see (2) and [5, Lemma 6.1]) asserts that, if $m, n \geq 1$ and $p \in [2, \infty]$, there exists a m-linear map $A_n : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ of the form

$$A_n(z^{(1)}, \ldots, z^{(m)}) = \sum_{j_1,\ldots,j_m=1}^{n} \pm z_{j_1}^{(1)} \cdots z_{j_m}^{(m)} \tag{4.3}$$

such that

$$\|A_n\| \leq C_m n^{\frac{m+1}{2} - \frac{m}{p}}$$

for certain constant $C_m > 0$.

Let

$$\beta := \frac{p + s - ps}{ps}.$$

Observe that $s < p^*$ implies $\beta > 0$. From the previous remark (Remark 4.6) we have

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| A_n \left(\frac{e_{j_1}}{j_1^{\beta}}, \ldots, \frac{e_{j_m}}{j_m^{\beta}} \right) \right|^r \right)^{\frac{1}{r}} \leq \pi_{\mathrm{mult}(r;s)} \left(A_n \right) \left\| \left(\frac{e_j}{j^{\beta}} \right)_{j=1}^{n} \right\|_{w,s}^{m}$$

i.e.,

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| \frac{1}{j_1^{\beta} \ldots j_m^{\beta}} \right|^r \right)^{\frac{1}{r}} \leq \pi_{\mathrm{mult}(r;s)} \left(A_n \right) \left\| \left(\frac{e_j}{j^{\beta}} \right)_{j=1}^{n} \right\|_{w,s}^{m}. \tag{4.4}$$

Let us investigate separately the both sides of (4.4). On the one hand,

$$\begin{aligned} \left(\sum_{j_1,\ldots,j_m=1}^{n} \left| \frac{1}{j_1^{\beta} \ldots j_m^{\beta}} \right|^r \right)^{\frac{1}{r}} &= \left(\sum_{j_1=1}^{n} \cdots \sum_{j_m=1}^{n} \left| \frac{1}{j_1^{\beta} \ldots j_m^{\beta}} \right|^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{j_1=1}^{n} \frac{1}{j_1^{r\beta}} \cdots \sum_{j_m=1}^{n} \frac{1}{j_m^{r\beta}} \right)^{\frac{1}{r}} = \left(\sum_{j=1}^{n} \frac{1}{j^{r\beta}} \right)^{\frac{m}{r}}. \end{aligned} \tag{4.5}$$

On the other hand, for $n \geq 2$, since $\dfrac{1}{\frac{1}{\beta s}} + \dfrac{1}{\frac{p^*}{s}} = 1$, we obtain

$$\begin{aligned} \left\| \left(\tfrac{e_j}{j^{\beta}} \right)_{j=1}^{n} \right\|_{w,s} &= \sup_{\varphi \in B_{(\ell_p)'}} \left(\sum_{j=1}^{n} \left| \varphi \left(\frac{e_j}{j^{\beta}} \right) \right|^s \right)^{\frac{1}{s}} = \sup_{\varphi \in B_{\ell_{p^*}}} \left(\sum_{j=1}^{n} |\varphi_j|^s \frac{1}{j^{\beta s}} \right)^{\frac{1}{s}} \\ &\leq \left(\left(\sum_{j=1}^{n} |\varphi_j|^{p^*} \right)^{\frac{s}{p^*}} \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{\beta s} \right)^{\frac{1}{s}} \leq \left(\sum_{j=1}^{n} \frac{1}{j} \right)^{\beta} \\ &= \left(1 + \sum_{j=2}^{n} \inf \left\{ \frac{1}{x} : x \in [j-1, j] \right\} \right)^{\beta} < \left(1 + \int_1^n \frac{1}{x} dx \right)^{\beta} \\ &= (1 + \log n)^{\beta}. \end{aligned} \tag{4.6}$$

Hence, replacing (4.5) and (4.6) in (4.4), we have

$$\left(\sum_{j=1}^{n}\frac{1}{j^{r\beta}}\right)^{\frac{m}{r}} < \pi_{\mathrm{mult}(r;s)}\left(A_n\right)\left(1+\log n\right)^{m\beta}$$

and consequently (since $\sum\limits_{j=1}^{n}\frac{1}{j^{r\beta}} \geq \sum\limits_{j=1}^{n}\frac{1}{n^{r\beta}} = n^{1-r\beta}$)

$$\left(n^{1-r\beta}\right)^{\frac{m}{r}} < \pi_{\mathrm{mult}(r;s)}\left(A_n\right)\left(1+\log n\right)^{m\beta}.$$

Since $\|A_n\| \leq C_m n^{\frac{m+1}{2}-\frac{m}{p}}$, we have

$$\frac{\pi_{\mathrm{mult}(r;s)}\left(A_n\right)}{\|A_n\|} > \frac{n^{\frac{m}{r}-\left(\frac{p+s-ps}{ps}\right)m}}{\left(1+\log n\right)^{m\beta} C_m n^{\frac{m+1}{2}-\frac{m}{p}}} = \frac{n^{\frac{m}{r}+\frac{m}{2}-\frac{m}{s}-\frac{1}{2}}}{C_m\left(1+\log n\right)^{r\beta}}.$$

Using that $r < \frac{2ms}{s+2m-ms}$ we get $\frac{m}{r}+\frac{m}{2}-\frac{m}{s}-\frac{1}{2} > 0$. Therefore, by making $n \to \infty$, it follows that

$$\lim_{n\to\infty}\frac{\pi_{\mathrm{mult}(r;s)}\left(A_n\right)}{\|A_n\|} = \infty. \tag{4.7}$$

Using the above limit, let us prove that $\Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right)$ is not closed in $\mathcal{L}\left({}^m\ell_p;\mathbb{K}\right)$. In fact, suppose (contrary to our claim) that $\Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right)$ is closed in $\mathcal{L}\left({}^m\ell_p;\mathbb{K}\right)$. Then $\left(\Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right), \|\cdot\|\right)$ is Banach space and since $\|\cdot\| \leq \pi_{\mathrm{mult}(r;s)}(\cdot)$ (see Proposition 5.3) we conclude that

$$\mathrm{id}:\left(\Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right), \pi_{\mathrm{mult}(r;s)}(\cdot)\right) \to \left(\Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right), \|\cdot\|\right)$$

is continuous. Thus by Open Mapping Theorem (see [43, Corollary 2.7]) we conclude that id^{-1} is also continuous and thus there exists $C > 0$ such that $\pi_{\mathrm{mult}(r;s)}(\cdot) \leq C\|\cdot\|$, contrary to (4.7).

Therefore, from [67, Theorem 5.6 and its reformulation] (see also [80]) we conclude that $\mathcal{L}\left({}^m\ell_p;\mathbb{K}\right) \setminus \Pi^m_{\mathrm{mult}(r;s)}\left({}^m\ell_p;\mathbb{K}\right)$ is spaceable.

It remains to prove the maximal spaceability. From Corollary 4.5 we know that $\dim\left(\mathcal{L}({}^m\ell_p;\mathbb{K})\right) = \mathfrak{c}$. Thus, if

$$V \subseteq \left(\mathcal{L}({}^m\ell_p;\mathbb{K}) \setminus \Pi^m_{\mathrm{mult}(r;s)}({}^m\ell_p;\mathbb{K})\right) \cup \{0\}$$

is a closed infinite-dimensional subspace of $\mathcal{L}({}^m\ell_p;\mathbb{K})$, we have $\dim(V) \leq \mathfrak{c}$. Since V is a Banach space, we also have $\dim(V) \geq \mathfrak{c}$ (see [36, Remark 2.5]). Thus, by the Cantor-Bernstein-Schroeder Theorem, it follows that $\dim(V) = \mathfrak{c}$ and the proof is done. ☐

Remark 4.8. *Note that it was not necessary to suppose the Continuum Hypothesis. In fact, for instance the proof given in [36, Remark 2.5] of the fact that the dimension of every infinite-dimensional Banach space is, at least,* $\mathfrak{c}$ *does not depends on the Continuum Hypothesis.*

4.2 Some consequences

Here we show some consequences of the results of the previous section. For instance, we observe a new optimality component of the Bohnenblust–Hille inequality: the term 1 from the pair $\left(\frac{2m}{m+1}; 1\right)$ is also optimal.

The following result is a simple consequence of Theorem 4.7.

Corollary 4.9. *Let $m \geq 2$ and $r \in \left[\frac{2m}{m+1}, 2\right]$. Then*

$$\sup\left\{s : \mathcal{L}(^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p;\mathbb{K})\right\} \leq \frac{2mr}{mr+2m-r}$$

for all $2 \leq p < \frac{2mr}{r+mr-2m}$.

Proof. Since $\frac{2m}{m+1} \leq r \leq 2 < 2m$, it follows that $1 \leq \frac{2mr}{mr+2m-r}$ and $2 < \frac{2mr}{r+mr-2m}$. Note that

$$s > \frac{2mr}{mr+2m-r}$$

implies

$$r < \frac{2ms}{s+2m-ms}.$$

Therefore, for $2 \leq p < \frac{2mr}{r+mr-2m}$, from Theorem 4.7 we know that

$$\mathcal{L}\left(^m\ell_p;\mathbb{K}\right) \smallsetminus \Pi^m_{\mathrm{mult}(r;s)}\left(^m\ell_p;\mathbb{K}\right)$$

is spaceable for all $\frac{2mr}{mr+2m-r} < s < p^*$ (note that $p < \frac{2mr}{r+mr-2m}$ implies $p^* > \frac{2mr}{mr+2m-r}$). In particular, for $2 \leq p < \frac{2mr}{r+mr-2m}$,

$$\sup\left\{s : \mathcal{L}(^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p;\mathbb{K})\right\} \leq \frac{2mr}{mr+2m-r}.$$

$\square$

This corollary, together with our main result, ensure that, for $r \in \left[\frac{2m}{m+1}, 2\right]$ and $2 \leq p < \frac{2mr}{r+mr-2m}$,

$$\sup\left\{s : \mathcal{L}(^m\ell_p;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p;\mathbb{K})\right\} = \frac{2mr}{mr+2m-r}.$$

When $p = 2$ the expression above recovers the optimality of [37, Theorem 5.14] in the case of m-linear operators on $\ell_2 \times \cdots \times \ell_2$.

In 2010 G. Botelho, C. Michels and D. Pellegrino [37] have shown that for $m \geq 1$ and Banach spaces $E_1, ..., E_m$ of cotype 2,

$$\mathcal{L}\left(E_1, ..., E_m;\mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(2;\frac{2m}{2m-1}\right)}\left(E_1, ..., E_m;\mathbb{K}\right)$$

and for Banach spaces of cotype $k > 2$,

$$\mathcal{L}\left(E_1, ..., E_m;\mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(2;\frac{km}{km-1}-\epsilon\right)}\left(E_1, ..., E_m;\mathbb{K}\right)$$

for all sufficiently small $\epsilon > 0$.

We now remark that it is not necessary to make any assumptions on the Banach spaces $E_1, ..., E_m$ and $\frac{2m}{2m-1}$ holds in all cases. Given $k > 2$, in [110, page 194] it is said that it is not known if $s = \frac{km}{km-1}$ is attained or not in

$$\sup\{s : \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(2;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all } E_j \text{ of cotype } k\} \geq \frac{km}{km-1}.$$

The fact that $\frac{2m}{2m-1}$ can replace $\frac{km}{km-1}$ in all cases ensures that $s = \frac{km}{km-1}$ is not attained and thus improves the estimate of [110, Corollary 3.1], which can be improved to

$$\begin{aligned}\sup\{s : \mathcal{L}(E_1,..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(2;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all } E_j \text{ of cotype } k\} \\ \in \left[\frac{2m}{2m-1}, \frac{2km}{2km+k-2m}\right]\end{aligned}$$

if $k > 2$ and $m \geq k$ is a positive integer.

More precisely we prove the following more general result. Let us remark that part (i) of the theorem above can be also inferred from [4, 63], although it is not explicitly written in the aforementioned papers:

Theorem 4.10. *Let $m \geq 2$ and let $r \in \left[\frac{2m}{m+1}, \infty\right)$. Then the optimal s such that*

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

for all Banach spaces $E_1, ..., E_m$ is:

(i) $\frac{2mr}{mr+2m-r}$ *if* $r \in \left[\frac{2m}{m+1}, 2\right]$*;*

(ii) $\frac{mr}{mr+1-r}$ *if* $r \in (2, \infty)$.

Proof. *(i)* For $q \geq 1$, let $X_q = \ell_q$ and let us define $X_\infty = c_0$. Let

$$q := \frac{2mr}{r + mr - 2m}.$$

Since $r \in \left[\frac{2m}{m+1}, 2\right]$ we have that $q \in [2m, \infty]$. Since

$$\frac{m}{q} \leq \frac{1}{2} \qquad \text{and} \qquad r = \frac{2m}{m+1-\frac{2m}{q}},$$

from the multilinear Hardy–Littlewood inequality (see, for example, [5, 73, 118]) there is a constant $C \geq 1$ such that

$$\left(\sum_{j_1,...,j_m=1}^{\infty} \left|A\left(e_{j_1}, ..., e_{j_m}\right)\right|^r\right)^{\frac{1}{r}} \leq C \left\|A\right\|$$

for all continuous m-linear operators $A : X_q \times \cdots \times X_q \to \mathbb{K}$. Let $T \in \mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right)$ and $(x_j^{(k)})_{j=1}^{\infty} \in \ell^w_{q^*}(E_k)$, $k = 1, ..., m$. Now we use a standard argument (see [4]) to lift the result from X_q to arbitrary Banach spaces. From [62, Proposition 2.2] we know that exist a continuous linear operator $u_k : X_q \to E_k$ so that $u_k(e_{j_k}) = x_{j_k}^{(k)}$ and

$$\|u_k\| = \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,q^*}$$

for all $k = 1, ..., m$. Therefore, $S : X_q \times \cdots \times X_q \to \mathbb{K}$ defined by $S(y_1, ..., y_m) = T\left(u_1(y_1), ..., u_m(y_m)\right)$ is m-linear, continuous and

$$\|S\| \leq \|T\| \prod_{k=1}^{m} \|u_k\| = \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,q^*}.$$

Hence

$$\left(\sum_{j_1,...,j_m=1}^{\infty} \left| T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)} \right) \right|^r \right)^{\frac{1}{r}} \leq C\|T\| \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,q^*},$$

and, since $q^* = \frac{2mr}{mr+2m-r}$, the last inequality proves that, for all $m \geq 2$ and $r \in \left[\frac{2m}{m+1}, 2\right]$,

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{mult}\left(r; \frac{2mr}{mr+2m-r}\right)} \left(E_1, ..., E_m; \mathbb{K}\right).$$

Now let us prove the optimality. From what we have just proved, for $r \in \left[\frac{2m}{m+1}, 2\right]$, we have

$$\begin{aligned} &U_{m,r} \\ &:= \sup\left\{ s : \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; \mathbb{K}) \text{ for all Banach spaces } E_j \right\} \\ &\geq \frac{2mr}{mr+2m-r}. \end{aligned}$$

From Corollary 4.9 we have, for $2 \leq p < \frac{2mr}{r+mr-2m}$,

$$\sup\left\{ s : \mathcal{L}(^m\ell_p; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr+2m-r}.$$

Therefore,

$$U_{m,r} \leq \sup\left\{ s : \mathcal{L}(^m\ell_p; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(^m\ell_p; \mathbb{K}) \right\} \leq \frac{2mr}{mr+2m-r}.$$

and we conclude that $U_{m,r} = \frac{2mr}{mr+2m-r}$.

(ii) Given $r > 2$ consider $m < p < 2m$ such that $r = \frac{p}{p-m}$. In this case, $p = \frac{mr}{r-1}$ and $p^* = \frac{mr}{mr+1-r}$. From [63] we know that

$$\Pi^m_{\mathrm{mult}\left(\frac{p}{p-m}; p^*\right)}(E_1, ..., E_m; \mathbb{K}) = \mathcal{L}(E_1, ..., E_m; \mathbb{K}) \tag{4.8}$$

for all Banach spaces $E_1, ..., E_m$, i.e.,

$$\Pi^m_{\mathrm{mult}\left(r; \frac{mr}{mr+1-r}\right)}(E_1, ..., E_m; \mathbb{K}) = \mathcal{L}(E_1, ..., E_m; \mathbb{K})$$

for all Banach spaces $E_1, ..., E_m$. Also, for $E_j = \ell_p$ for all j we know that

$$\Pi^m_{\mathrm{mult}(\frac{p}{p-m};p^*)}(\ell_p, ..., \ell_p; \mathbb{K}) = \mathcal{L}(\ell_p, ..., \ell_p; \mathbb{K}) \tag{4.9}$$

is optimal, i.e., $\frac{p}{p-m}$ cannot be improved. If $s > p^*$ let $q^* = s$ and then $q < p$ (we can always suppose s close to p^* and thus $m < q < 2m$). From (4.8) we have

$$\Pi^m_{\mathrm{mult}(\frac{q}{q-m};q^*)}(E_1, ..., E_m; \mathbb{K}) = \mathcal{L}(E_1, ..., E_m; \mathbb{K})$$

and from (4.9) in the case of ℓ_q instead of ℓ_p, we have

$$\Pi^m_{\mathrm{mult}(\frac{q}{q-m};q^*)}(\ell_q, ..., \ell_q; \mathbb{K}) = \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K})$$

and $\frac{q}{q-m}$ is optimal. Since $\frac{q}{q-m} > \frac{p}{p-m}$ we conclude that

$$\Pi^m_{\mathrm{mult}(\frac{p}{p-m};q^*)}(\ell_q, ..., \ell_q; \mathbb{K}) \neq \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K}),$$

i.e.,

$$\Pi^m_{\mathrm{mult}(\frac{p}{p-m};s)}(\ell_q, ..., \ell_q; \mathbb{K}) \neq \mathcal{L}(\ell_q, ..., \ell_q; \mathbb{K}).$$

□

The following graph (Figure 4.1) illustrates for which $(r, s) \in [1, \infty) \times [1, r]$ we have

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; \mathbb{K}).$$

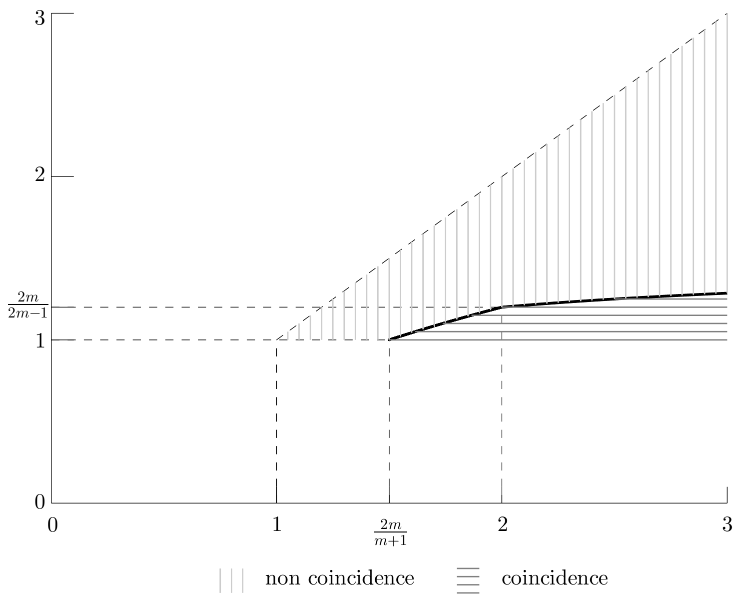

Figure 4.1: Areas of coincidence for $\Pi^m_{\mathrm{mult}(r;s)}(E_1, ..., E_m; \mathbb{K})$, $(r, s) \in [1, \infty) \times [1, r]$.

The table below details the results of coincidence and non coincidence in the "boundaries" of Figure 4.1. We can clearly see that the only case that remains open is the case $(r; s)$ with $r > 2$ and $\frac{2m}{2m-1} < s \leq \frac{mr}{mr+1-r}$.

$r \geq 1$	$s = r$	non coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non coincidence
$\frac{2m}{m+1} \leq r \leq 2$	$s = \frac{2mr}{mr+2m-r}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$r > 2$	$s = \frac{mr}{mr+1-r}$	coincidence

4.3 Multiple $(r;s)$-summing forms in c_0 and ℓ_∞ spaces

From standard localization procedures, coincidence results for c_0 and ℓ_∞ are the same; so we will restrict our attention to c_0. It is well known that $\Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right) = \mathcal{L}\left({}^m c_0;\mathbb{K}\right)$ whenever $r \geq s \geq 2$ (see [37]). When $s = 1$, as a consequence of the Bohnenblust–Hille inequality, we also know that the equality holds if and only if $s \geq \frac{2m}{m+1}$. The next result encompasses essentially all possible cases:

Proposition 4.11. *If $s \in [1,\infty)$ then*

$$\inf\left\{r : \Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right) = \mathcal{L}\left({}^m c_0;\mathbb{K}\right)\right\} = \begin{cases} \dfrac{2m}{m+1} & \text{if } \ 1 \leq s \leq \dfrac{2m}{m+1}, \\ s & \text{if } \ s \geq \dfrac{2m}{m+1}. \end{cases}$$

Proof. The case $r \geq s \geq 2$ is straight forward (see [37, Corollary 4.10]). The Bohnenblust–Hille inequality assures that when $s = 1$ the best choice for r is $\frac{2m}{m+1}$. So, it is obvious that for $1 \leq s \leq \frac{2m}{m+1}$ the best value for r is not smaller than $\frac{2m}{m+1}$. More precisely,

$$\Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right) \neq \mathcal{L}\left({}^m c_0;\mathbb{K}\right)$$

whenever $(r,s) \in \left[1,\frac{2m}{m+1}\right) \times \left[1,\frac{2m}{m+1}\right]$ and $r \geq s$. For linear operators a deep result due to Maurey and Pisier (see [62]) alerts us that the notions of absolutely $(r;1)$-summing operators and $(r;s)$-summing operators coincide when $s < r$. An adaptation of this result to multiple summing operators (see [116, Theorem 3.16] or [37, Lemma 5.2]) combined with the coincidence result for $(r;s) = \left(\frac{2m}{m+1};1\right)$ tells us that we also have coincidences for $\left(\frac{2m}{m+1};s\right)$ for all $1 < s < \frac{2m}{m+1}$. The remaining case $(r;s)$ with $\frac{2m}{m+1} < s < 2$ follows from an interpolation procedure in the lines of [37]. More precisely, given $\frac{2m}{m+1} < r < 2$ and $0 < \delta < \frac{r(2-\theta)-2}{2-\theta}$, where $\theta = \frac{mr+r-2m}{r}$, consider $\epsilon = \frac{2m}{m+1} - \frac{2(1-\theta)(r-\delta)}{2-\theta(r-\delta)}$. Since $1 < \frac{2m}{m+1} - \epsilon < \frac{2m}{m+1}$, we know that $\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi^m_{\mathrm{mult}\left(\frac{2m}{m+1};\frac{2m}{m+1}-\epsilon\right)}({}^m c_0;\mathbb{K})$. Since $\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi^m_{\mathrm{mult}(2;2)}({}^m c_0;\mathbb{K})$ and

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{\frac{2m}{m+1}} \qquad \text{and} \qquad \frac{1}{r-\delta} = \frac{\theta}{2} + \frac{1-\theta}{\frac{2m}{m+1}-\epsilon},$$

by interpolation we conclude $\mathcal{L}({}^m c_0;\mathbb{K}) = \Pi^m_{\mathrm{mult}(r;r-\delta)}({}^m c_0;\mathbb{K})$. $\square$

The following graph (Figure 4.2) illustrates for which $(r,s) \in [1,\infty) \times [1,r]$ we have

$$\mathcal{L}\left({}^m c_0;\mathbb{K}\right) = \Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0;\mathbb{K}\right).$$

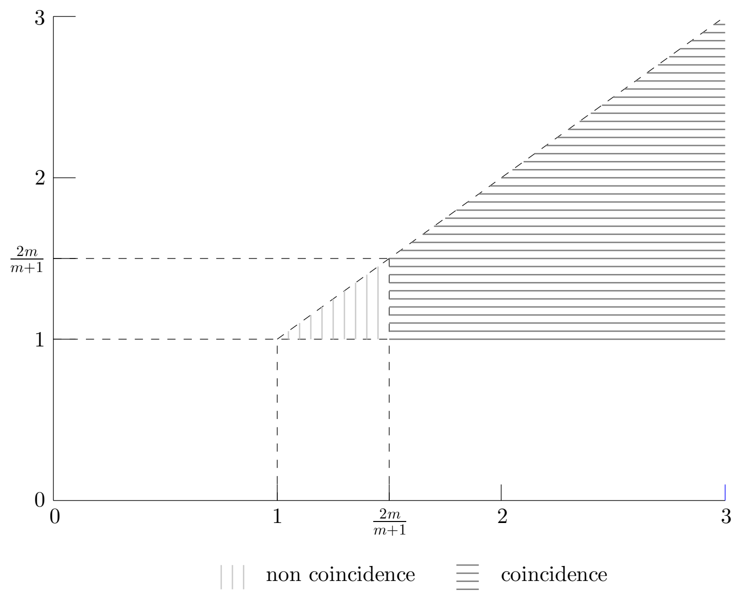

Figure 4.2: Areas of coincidence for $\Pi^m_{\mathrm{mult}(r;s)}\left({}^m c_0; \mathbb{K}\right)$, $(r,s) \in [1,\infty) \times [1,r]$.

The table below details the results of coincidence and non coincidence in the "boundaries" of Figure 4.2.

$1 \leq r < \frac{2m}{m+1}$	$s = 1$	non coincidence
$r = \frac{2m}{m+1}$	$1 \leq s < \frac{2m}{m+1}$	coincidence
$r \geq \frac{2m}{m+1}$	$s = 1$	coincidence
$1 \leq r < \frac{2m}{m+1}$	$s = r$	non coincidence
$\frac{2m}{m+1} \leq r < 2$	$s = r$	unknown
$r \geq 2$	$s = r$	coincidence

We can see that the only case that remains open is the case $(r;s)$ with $\frac{2m}{m+1} \leq r < 2$ and $s = r$.

4.4 Absolutely summing multilinear operators

In this section we investigate the optimality of coincidence results within the framework of absolutely summing multilinear operators and, as consequence, we observe that the Defant–Voigt theorem (first stated and proved in [7, Theorem 3.10]; see also, e.g., [20, Theorem 3], for complex scalars, or [39, Corollary 3.2]) is optimal.

Theorem 4.12 (Defant–Voigt)**.** *For all Banach spaces $E_1, ..., E_m$,*

$$\Pi^m_{\mathrm{as}(1;1)}(E_1, ..., E_m; \mathbb{K}) = \mathcal{L}(E_1, ..., E_m; \mathbb{K})$$

Combining the Defant–Voigt Theorem and a canonical inclusion theorem (see [42, 87]) we conclude that, for $r, s \geq 1$ and $s \leq \frac{mr}{mr+1-r}$, we have

$$\Pi^m_{\mathrm{as}(r;s)}(E_1, ..., E_m; \mathbb{K}) = \mathcal{L}(E_1, ..., E_m; \mathbb{K})$$

for all $E_1, ..., E_m$.

From [128, Proposition 1] it is possible to prove that for $r > 1$ and $\frac{r}{mr+1-r} \leq t < r$,

$$\Pi^m_{\mathrm{as}(t;\frac{mr}{mr+1-r})}(E_1, ..., E_m; \mathbb{K}) \neq \mathcal{L}(E_1, ..., E_m; \mathbb{K})$$

for some choices of $E_1, ..., E_m$. In fact (repeating an argument used in the proof of Theorem 4.10), given $r > 1$, consider $p > m$ such that $\frac{p}{p-m} = r$ and observe that in this case $\frac{mr}{mr+1-r} = p^*$ and thus we just need to prove that for all $\frac{p^*}{m} \leq t < \frac{p}{p-m}$,

$$\Pi^m_{\mathrm{as}(t;p^*)}(E_1, ..., E_m; \mathbb{K}) \neq \mathcal{L}(E_1, ..., E_m; \mathbb{K}).$$

From [128, Proposition 1] we know that if $p > m$ and $\frac{p^*}{m} \leq t < \frac{p}{p-m}$, then there is a continuous m-linear form ϕ such that

$$\phi \notin \Pi^m_{\mathrm{as}(t;p^*)}(E_1, ..., E_m; \mathbb{K}),$$

i.e.,

$$\Pi^m_{\mathrm{as}(t;p^*)}(E_1, ..., E_m; \mathbb{K}) \neq \mathcal{L}(E_1, ..., E_m; \mathbb{K}).$$

All these pieces of information provide Figure 4.3, which illustrates for which $(r, s) \in [1, \infty) \times [1, mr]$ we have

$$\mathcal{L}\left(E_1, ..., E_m; \mathbb{K}\right) = \Pi^m_{\mathrm{as}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right).$$

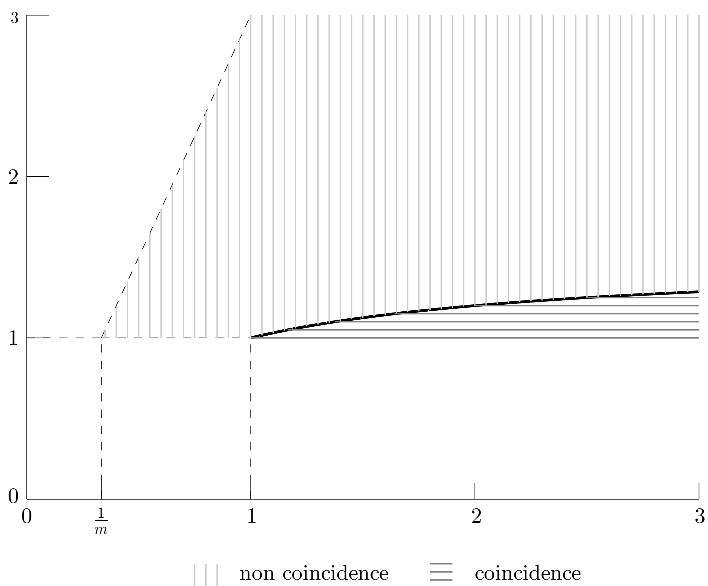

Figure 4.3: Areas of coincidence for $\Pi^m_{\mathrm{as}(r;s)}\left(E_1, ..., E_m; \mathbb{K}\right)$, $(r, s) \in [1, \infty) \times [1, mr]$.

The table below details the results of coincidence and non coincidence in the "boundaries" of Figure 4.3. The only possible open situation is the case $(r; s)$ with $s = 1$ and $r < 1$, which we answer in the next lines.

$\frac{1}{m} \leq r < 1$	$s = 1$	not known
$r > \frac{1}{m}$	$s = mr$	non coincidence
$r \geq 1$	$s = 1$	coincidence
$r \geq 1$	$s = \frac{mr}{mr+1-r}$	coincidence

Proposition 4.13. *The Defant–Voigt Theorem is optimal. More precisely if $m \geq 1$ is a positive integer,*

$$\inf \left\{ r : \begin{array}{l} \mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi^m_{\mathrm{as}(r;1)}(E_1, ..., E_m; \mathbb{K}) \text{ for all} \\ \text{infinite-dimentional Banach spaces } E_j \end{array} \right\} = 1.$$

Proof. The equality holds for $r = 1$; this is the so called Defant–Voigt theorem. It remains to prove that the equality does not hold for $r < 1$. This is simple; we just need to choose $E_j = c_0$ for all j and suppose that

$$\mathcal{L}(E_1, ..., E_m; \mathbb{K}) = \Pi_{\mathrm{as}(r;1)}(E_1, ..., E_m; \mathbb{K}). \tag{4.10}$$

For all positive integers n, consider the m-linear forms

$$T_n : c_0 \times \cdots \times c_0 \to \mathbb{K}$$

defined by

$$T_n(x^{(1)}, ..., x^{(m)}) = \sum_{j=1}^{n} x_j^{(1)} \cdots x_j^{(m)}.$$

Then it is plain that $\|T_n\| = n$, and from (4.10) there is a $C \geq 1$ such that

$$\left(\sum_{j=1}^{n} |T_n(e_j, ..., e_j)|^r \right)^{\frac{1}{r}} \leq C \|T_n\| \prod_{k=1}^{m} \sup_{\varphi \in B_{E_k^*}} \sum_{j=1}^{n} |\varphi(e_j)| = Cn,$$

i.e., $n^{1/r} \leq Cn$ and thus $r \geq 1$. ☐

This simple proposition ensures that the zone defined by $r < 1$ and $s = 1$ in the Figure 4.3 is a non coincidence zone,i.e., the Defant–Voigt theorem is optimal. Therefore, we can make a new table for the results of coincidence and non coincidence in the “boundaries” of Figure 4.3:

$\frac{1}{m} \leq r < 1$	$s = 1$	non coincidence
$r \geq \frac{1}{m}$	$s = mr$	non coincidence
$r \geq 1$	$s = 1$	coincidence
$r \geq 1$	$s = \frac{mr}{mr+1-r}$	coincidence

Chapter 5

A unified theory and consequences

In this chapter we present results of the paper:

[2] N. Albuquerque, G. Araújo, D. Núñez-Alarcón, D. Pellegrino, and P. Rueda, *Summability of multilinear operators: a unified theory and consequences*, arXiv:1409.6769 [math.FA].

5.1 Multiple summing operators with multiple exponents

For $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty)^m$, we shall consider the space

$$\ell_{\mathbf{p}}(E) := \ell_{p_1}\left(\ell_{p_2}\left(\ldots\left(\ell_{p_m}(E)\right)\ldots\right)\right),$$

namely, a vector matrix $(x_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{\infty} \in \ell_{\mathbf{p}}(E)$ if, and only if,

$$\left\|(x_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{\infty}\right\|_{\ell_{\mathbf{p}}(E)} := \left(\sum_{i_1=1}^{\infty}\left(\ldots\left(\sum_{i_m=1}^{\infty}\|x_{i_1\ldots i_m}\|_E^{p_m}\right)^{\frac{p_{m-1}}{p_m}}\ldots\right)^{\frac{p_1}{p_1}}\right)^{\frac{1}{p_1}} < +\infty.$$

When $E = \mathbb{K}$, we simply write $\ell_{\mathbf{p}}$. The following definition seems natural:

Definition 5.1. *Let* $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. *A multilinear operator* $T : E_1 \times \cdots \times E_m \to F$ *is* multiple $(q_1, \ldots, q_m; p_1, \ldots, p_m)$-summing *if there exist a constant* $C > 0$ *such that*

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left\|T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right)\right\|_F^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,p_k}$$

for all $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^{w}(E_k)$. *We represent the class of all multiple* $(q_1, \ldots, q_m; p_1, \ldots, p_m)$-*summing operators by* $\Pi^m_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F)$.

Of course, when $q_1 = \cdots = q_m = q$, then

$$\Pi^m_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F) = \Pi^m_{\mathrm{mult}(q;p_1,\ldots,p_m)}(E_1, \ldots, E_m; F).$$

As it happens with absolutely and multiple summing operators, the following result characterizes the multiple $(q_1, \ldots, q_m; p_1, \ldots, p_m)$-summing operators.

Proposition 5.2. *Let $T : E_1 \times \cdots \times E_m \to F$ be a continuous multilinear operator and $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. The following are equivalent:*

(1) T is multiple $(q_1, \ldots, q_m; p_1, \ldots, p_m)$-summing;

(2) $\left(T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)})\right)_{j_1,\ldots,j_m=1}^{\infty} \in \ell_{\mathbf{q}}(F)$ whenever $(x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^{w}(E_k)$.

(3) There exist a constant $C > 0$ such that

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left\|T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right)\right\|_F^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{k=1}^{m}\left\|(x_j^{(k)})_{j=1}^{n}\right\|_{w,p_k}$$

for all positive integer n and all $(x_j^{(k)})_{j=1}^{n} \in \ell_{p_k}^{w}(E_k)$.

Proof. By definition, it follows that *(1) ⇒ (2)*. Let us prove now that *(2) ⇒ (1)*. Supposing *(2)*, we can define the m-linear operator

$$\begin{array}{rcccl} \widehat{T} & : & \ell_{p_1}^{w}(E_1) \times \cdots \times \ell_{p_m}^{w}(E_m) & \to & \ell_{\mathbf{q}}(F) \\ & & \left((x_j^{(1)})_{j=1}^{\infty}, \ldots, (x_j^{(m)})_{j=1}^{\infty}\right) & \mapsto & \left(T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)})\right)_{j_1,\ldots,j_m=1}^{\infty}. \end{array} \tag{5.1}$$

Observe that $\widehat{T}$ is a continuous m-linear operator. In fact, let $((x_{j,s}^{(k)})_{j=1}^{\infty})_{s=1}^{\infty} \subset \ell_{p_k}^{w}(E_k)$, $k = 1, \ldots, m$, such that

$$(x_{j,s}^{(k)})_{j=1}^{\infty} \to (x_j^{(k)})_{j=1}^{\infty} \text{ in } \ell_{p_k}^{w}(E_k) \tag{5.2}$$

and

$$\widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty}, \ldots, (x_{j_m,s}^{(m)})_{j_m=1}^{\infty}\right) \to (y_{j_1,\ldots,j_m})_{j_1,\ldots,j_m=1}^{\infty} \text{ in } \ell_{\mathbf{q}}(F). \tag{5.3}$$

From (5.2) we have that for every $k \in \{1, \ldots, m\}$, given $\epsilon > 0$, there exist $N \in \mathbb{N}$ which verify

$$s \geq N \Rightarrow \sup_{\varphi \in B_{E_k^*}}\left(\sum_{j=1}^{\infty}\left|\varphi(x_{j,s}^{(k)} - x_j^{(k)})\right|^{p_k}\right)^{\frac{1}{p_k}} < \epsilon.$$

So

$$s \geq N \Rightarrow \sum_{j=1}^{\infty}\left|\varphi(x_{j,s}^{(k)} - x_j^{(k)})\right|^{p_k} < \epsilon^{p_k} \text{ for all } \varphi \in B_{E_k^*} \text{ and all } k \in \{1, \ldots, m\}$$

and thus

$$\left|\varphi(x_{j,s}^{(k)} - x_j^{(k)})\right| < \epsilon \text{ for all } \varphi \in B_{E_k^*} \text{ and all } \{j, k\} \in \mathbb{N} \times \{1, \ldots, m\}.$$

Then, from the Hahn–Banach theorem we conclude that

$$s \geq N \Rightarrow \left\|x_{j,s}^{(k)} - x_j^{(k)}\right\|_{E_k} = \sup_{\varphi \in B_{E_k^*}}\left|\varphi(x_{j,s}^{(k)} - x_j^{(k)})\right| \leq \epsilon \text{ for all } \{j, k\} \in \mathbb{N} \times \{1, \ldots, m\},$$

i.e., $x_{j,s}^{(k)} \to x_j^{(k)}$ in E_k for all $j \in \mathbb{N}$ and all $k \in \{1, ..., m\}$. Since T is a continuous multilinear operator, it follows that

$$T\left(x_{j_1,s}^{(1)}, ..., x_{j_m,s}^{(m)}\right) \to T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right) \text{ in } F \text{ for all fixed } j_1, ..., j_k \in \mathbb{N}.$$

From (5.3), given $\epsilon > 0$, there exist $M \in \mathbb{N}$ such that

$$s \geq M \Rightarrow \left\| \widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m,s}^{(m)})_{j_m=1}^{\infty}\right) - (y_{j_1,...,j_m})_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} < \epsilon,$$

from which we can obtain that, for $s \geq M$,

$$\left\| T\left(x_{j_1,s}^{(1)}, ..., x_{j_m,s}^{(m)}\right) - y_{j_1,...,j_m} \right\|_F < \epsilon \text{ for all fixed } j_1, ..., j_k \in \mathbb{N}.$$

We deduce from the uniqueness of the limit that $T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right) = y_{j_1,...,j_m}$ for every $j_1, ..., j_k \in \mathbb{N}$. Thus

$$\widehat{T}\left((x_{j_1,s}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m,s}^{(m)})_{j_m=1}^{\infty}\right) = \left(T\left(x_{j_1,s}^{(1)}, ..., x_{j_m,s}^{(m)}\right)\right)_{j_1,...,j_m=1}^{\infty} = (y_{j_1,...,j_m})_{j_1,...,j_m=1}^{\infty},$$

and then, from the closed graph theorem, we obtain that $\widehat{T}$ is a continuous m-linear operator.

Thus, there is $C > 0$ such that

$$\begin{aligned}
&\left\| \left(T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right)\right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&\quad = \left\| \widehat{T}\left((x_{j_1}^{(1)})_{j_1=1}^{\infty}, ..., (x_{j_m}^{(m)})_{j_m=1}^{\infty}\right) \right\|_{\ell_{\mathbf{q}}(F)} \\
&\quad \leq C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m}.
\end{aligned}$$

(1) $\Rightarrow$ *(3)*. Fix $n \in \mathbb{N}$ and let $(x_j^{(1)})_{j=1}^{n} \in E_1, ..., (x_j^{(m)})_{j=1}^{n} \in E_m$. Then $(x_j^{(k)})_{j=1}^{\infty} = (x_1^{(k)}, x_2^{(k)}, ..., x_n^{(k)}, 0, 0, ...) \in \ell_{p_k}^{w}(E_k)$ for every $k \in \{1, ..., m\}$. Thus, using *(1)*, we get

$$\begin{aligned}
&\left\| \left(T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right)\right)_{j_1,...,j_m=1}^{n} \right\|_{\ell_{\mathbf{q}}(F} \\
&\quad = \left\| \left(T\left(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}\right)\right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F} \\
&\quad \leq C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m} \\
&\quad = C \left\| (x_j^{(1)})_{j=1}^{n} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{n} \right\|_{w,p_m}.
\end{aligned}$$

(3) ⇒ (1). Consider $(x_j^{(1)})_{j=1}^{\infty} \in \ell_{p_1}^w(E_1), ..., (x_j^{(m)})_{j=1}^{\infty} \in \ell_{p_m}^w(E_m)$. Therefore

$$\begin{aligned}
&\left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F} \\
&= \sup_n \left\| \left(T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{n} \right\|_{\ell_{\mathbf{q}}(F} \\
&\leq C \sup_n \left\| (x_j^{(1)})_{j=1}^{n} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{n} \right\|_{w,p_m} \\
&= C \left\| (x_j^{(1)})_{j=1}^{\infty} \right\|_{w,p_1} \cdots \left\| (x_j^{(m)})_{j=1}^{\infty} \right\|_{w,p_m}.
\end{aligned}$$

□

It is not to difficult to prove that $\Pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}^m(E_1, \ldots, E_m; F)$ is a subspace of $\mathcal{L}(E_1, \ldots, E_m; F)$ and the infimum of the constants satisfying the above definition (Definition 5.1), i.e.,

$$\inf \left\{ \begin{array}{l} C \geq 0 \ ; \ \left\| \left(T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}) \right)_{j_1,...,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \leq C \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^{\infty} \right\|_{w,p_k}, \\ \qquad \text{for all } (x_j^{(k)})_{j=1}^{\infty} \in \ell_{p_k}^w(E_k) \end{array} \right\}$$

defines a norm in $\Pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}^m(E_1, \ldots, E_m; F)$, which will be denoted by

$$\pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T).$$

Proposition 5.3. *Let $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. If $T \in \Pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}^m(E_1, \ldots, E_m; F)$, then*

$$\|T\|_{\mathcal{L}(E_1,...,E_m;F)} \leq \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T).$$

Proof. Consider $x_j \in B_{E_j}$, $j = 1, ..., m$, and define $(x_i^{(j)})_{i=1}^{\infty} := (x_j, 0, ...)$. It is clear that $(x_i^{(j)})_{i=1}^{\infty} \in \ell_{p_j}^w(E_j)$ for every $j = 1, ..., m$. Therefore, for $T \in \Pi_{(q_1,...,q_m;p_1,...,p_m)}^m(E_1, ..., E_m; F)$,

$$\begin{aligned}
&\|T(x_1, ..., x_m)\|_F \\
&= \left(\sum_{j_1=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \left\| T(x_{j_1}^{(1)}, ..., x_{j_m}^{(m)}) \right\|_F^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \\
&\leq \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T) \prod_{j=1}^{m} \left\| (x_i^{(j)})_{i=1}^{\infty} \right\|_{w,p_j} \\
&= \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T) \prod_{j=1}^{m} \sup_{\varphi \in B_{E_j^*}} \left(\sum_{i=1}^{\infty} \left| \varphi(x_i^{(j)}) \right|^{p_j} \right)^{\frac{1}{p_j}} \\
&= \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T) \prod_{j=1}^{m} \sup_{\varphi \in B_{E_j^*}} |\varphi(x_j)| \\
&= \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T) \prod_{j=1}^{m} \|x_j\|_{E_j} = \pi_{\text{mult}(q_1,...,q_m;p_1,...,p_m)}(T).
\end{aligned}$$

By means of taking the supremum, the result follows. □

Given $T \in \Pi^m_{(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F)$, we have defined in (5.1) the continuous m-linear operator $\widehat{T}$. Furthermore, let us prove now that

$$\|\widehat{T}\| = \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(T). \tag{5.4}$$

In fact, first note that

$$\begin{aligned}
&\left\| \left(T(x^{(1)}_{j_1}, \ldots, x^{(m)}_{j_m}) \right)_{j_1,\ldots,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&= \left\| \widehat{T}\left((x^{(1)}_j)_{j=1}^{\infty}, \ldots, (x^{(m)}_j)_{j=1}^{\infty} \right) \right\|_{\ell_{\mathbf{q}}(F)} \\
&\leq \|\widehat{T}\| \prod_{k=1}^{m} \left\| (x^{(k)}_j)_{j=1}^{\infty} \right\|_{w,p_k},
\end{aligned}$$

that is, $\pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(T) \leq \|\widehat{T}\|$. On the other hand, we have

$$\begin{aligned}
\|\widehat{T}\| &= \sup_{(x^{(k)}_j)_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \left\| \widehat{T}\left((x^{(1)}_j)_{j=1}^{\infty}, \ldots, (x^{(m)}_j)_{j=1}^{\infty} \right) \right\| \\
&= \sup_{(x^{(k)}_j)_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \left\| \left(T(x^{(1)}_{j_1}, \ldots, x^{(m)}_{j_m}) \right)_{j_1,\ldots,j_m=1}^{\infty} \right\|_{\ell_{\mathbf{q}}(F)} \\
&\leq \sup_{(x^{(k)}_j)_{j=1}^{\infty} \in B_{\ell^w_{p_k}(E_k)}} \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(T) \prod_{k=1}^{m} \left\| (x^{(k)}_j)_{j=1}^{\infty} \right\|_{w,p_k} \\
&= \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(T),
\end{aligned}$$

which proves (5.4).

We can naturally define the continuous operator

$$\begin{array}{rccc}
\widehat{\theta} : & \Pi^m_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F) & \to & \mathcal{L}\left(\ell^w_{p_1}(E_1), \ldots, \ell^w_{p_m}(E_m); \ell_{\mathbf{q}}(F) \right) \\
& T & \mapsto & \widehat{T},
\end{array}$$

which, due to equation (5.4), is an isometry. These facts allow us to prove the following:

Theorem 5.4. *Let* $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. *Then*

$$\left(\Pi^m_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F), \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(\cdot) \right)$$

is a Banach space.

Proof. Let $(T_j)_{j=1}^{\infty}$ be a Cauchy sequence in $\Pi^m_{(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1,\ldots,E_m;F)$. Since $\|\cdot\| \leq \pi_{(q_1,\ldots,q_m;p_1,\ldots,p_m)}(\cdot)$ (Proposition 5.3), it follows that $(T_j)_{j=1}^{\infty}$ is also a Cauchy sequence in $\mathcal{L}(E_1,\ldots,E_m;F)$. Thus, consider $T \in \mathcal{L}(E_1,\ldots,E_m;F)$ such that

$$T_j \to T \text{ in } \mathcal{L}(E_1,\ldots,E_m;F).$$

Let us prove that $T \in \Pi^m_{(q_1,...,q_m;p_1,...,p_m)}(E_1,...,E_m;F)$. In fact, let $(x_j^{(k)})_{j=1}^\infty \subset \ell^w_{p_k}(E_k)$, $k = 1,...,m$. It is enough to prove that $(T(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty \in \ell_{\mathbf{q}}(F)$. Since $\widehat{\theta}$ is an isometry, $(\widehat{T}_j)_{j=1}^\infty$ is a Cauchy sequence in $\mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F))$, which is a Banach space because $\ell_{\mathbf{q}}(F)$ is a Banach space. Thus, there exist $S \in \mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F))$ such that

$$\widehat{T}_j \to S \text{ in } \mathcal{L}(\ell^w_{p_1}(E_1),...,\ell^w_{p_m}(E_m);\ell_{\mathbf{q}}(F)).$$

Therefere, if we consider $P_{k_1,...,k_m} : \ell_{\mathbf{q}}(F) \to F$ the continuous linear operator given by

$$(y_{j_1\cdots j_m})_{j_1,...,j_m=1}^\infty \mapsto y_{k_1\cdots k_m},$$

and $\epsilon > 0$ a positive real number, there exist a positive integer N such that

$$\begin{aligned}
&\left\| P_{k_1,...,k_m}\left(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right) - T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F \\
&\quad\le \left\| P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right. \\
&\qquad \left. - P_{k_1,...,k_m}\left(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right\|_F \\
&\qquad + \left\| P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right) - T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F \\
&\quad= \left\| P_{k_1,...,k_m}\left(\widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right.\right. \\
&\qquad \left.\left. - S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right)\right\|_F \\
&\qquad + \left\| P_{k_1,...,k_m}\left((T_j(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty\right) - T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F \\
&\quad\le \|P_{k_1,...,k_m}\| \left\| \widehat{T}_j((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right. \\
&\qquad \left. - S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)\right\|_{\ell_{\mathbf{q}}(F)} \\
&\qquad + \left\| T_j(x_{k_1}^{(1)},...,x_{k_m}^{(m)}) - T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})\right\|_F \\
&\quad\le C\|P_{k_1,...,k_m}\|\|\widehat{T}_j - S\| + C\|T_j - T\| \\
&\quad< \epsilon
\end{aligned}$$

for every $j \ge N$. Then

$$P_{k_1,...,k_m}(S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty)) = T(x_{k_1}^{(1)},...,x_{k_m}^{(m)})$$

for all $k_1,...,k_m \in \mathbb{N}$, and consequently

$$S((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty) = (T(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty. \tag{5.5}$$

This proves that $(T(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty \in \ell_{\mathbf{q}}(F)$, as required.

By definition we have

$$\widehat{T}((x_{j_1}^{(1)})_{j_1=1}^\infty,...,(x_{j_m}^{(m)})_{j_m=1}^\infty) = (T(x_{j_1}^{(1)},...,x_{j_m}^{(m)}))_{j_1,...,j_m=1}^\infty.$$

Replacing the above expression in (5.5) we conclude that $\widehat{T} = S$. Thus, given $\epsilon > 0$, it follows from (5.4) that for sufficiently large j

$$\pi_{\mathrm{mult}(q_1,\dots,q_m;p_1,\dots,p_m)}(T_j - T) = \|\widehat{T_j - T}\| = \|\widehat{T}_j - \widehat{T}\| = \|\widehat{T}_j - S\| < \epsilon,$$

that is,

$$T_j \to T \text{ in } \Pi^m_{(q_1,\dots,q_m;p_1,\dots,p_m)}(E_1,\dots,E_m;F),$$

and this proves that $(\Pi^m_{(q_1,\dots,q_m;p_1,\dots,p_m)}(E_1,\dots,E_m;F), \pi_{\mathrm{mult}(q_1,\dots,q_m;p_1,\dots,p_m)}(\cdot))$ is a Banach space. □

Using that

$$\ell_q \setminus \ell_p \neq \emptyset \tag{5.6}$$

if $1 \le p < q \le \infty$, let us prove the following result:

Proposition 5.5. *If $q_j < p_j$ for some $j \in \{1,\dots,m\}$, then*

$$\Pi^m_{\mathrm{mult}(q_1,\dots,q_m;p_1,\dots,p_m)}(E_1,\dots,E_m;F) = \{0\}.$$

Proof. Since $q_j < p_j$, we know from (5.6) that there is a sequence $(\alpha_i)_{i=1}^\infty \in \ell_{p_j} \setminus \ell_{q_j}$. Let $x_j \in E_j \setminus \{0\}$. Then for all $\varphi \in E_j'$ we have

$$\sum_{i=1}^\infty |\varphi(\alpha_i x_j)|^{p_j} \le \sum_{i=1}^\infty \|\varphi\|^{p_j} |\alpha_i|^{p_j} \|x_j\|^{p_j} = \|\varphi\|^{p_j}\|x_j\|^{p_j} \sum_{i=1}^\infty |\alpha_i|^{p_j} < \infty,$$

i.e., $(\alpha_i x_j)_{i=1}^\infty \in \ell^w_{p_j}(E_j)$. Let us suppose, by contradiction, that there exist

$$T \in \Pi^m_{(q_1,\dots,q_m;p_1,\dots,p_m)}(E_1,\dots,E_m;F) \setminus \{0\}.$$

Then, we can take $x_k \in E_k \setminus \{0\}$, $k \in \{1,\dots,m\} \setminus \{j\}$, such that $T(x_1,\dots,x_m) \neq 0$. For each $k \in \{1,\dots,m\} \setminus \{j\}$ let us consider $(x_i^{(k)})_{i=1}^\infty = (x_k, 0, \dots)$. Since $(x_i^{(k)})_{i=1}^\infty \in \ell^w_{p_k}(E_k)$ for every $k \in \{1,\dots,m\} \setminus \{j\}$ and $(\alpha_i x_j)_{i=1}^\infty \in \ell^w_{p_j}(E_j)$, the Proposition 5.2 ensures that

$$\begin{aligned}
&\left\|\left(T(x^{(1)}_{i_1},\dots,x^{(j-1)}_{i_{j-1}},\alpha_{i_j}x_j,x^{(j+1)}_{i_{j+1}},\dots,x^{(m)}_{i_m})\right)_{i_1,\dots,i_m=1}^\infty\right\|_{\ell_{\mathbf{q}}(F)} \\
&\le C\left(\prod_{\substack{k=1\\k\neq j}}^m \left\|(x_i^{(k)})_{i=1}^\infty\right\|_{w,p_k}\right)\|(\alpha_i x_j)_{i=1}^\infty\|_{w,p_j}.
\end{aligned}$$

However,

$$\begin{aligned}
&\left\|\left(T(x^{(1)}_{i_1},\dots,x^{(j-1)}_{i_{j-1}},\alpha_{i_j}x_j,x^{(j+1)}_{i_{j+1}},\dots,x^{(m)}_{i_m})\right)_{i_1,\dots,i_m=1}^\infty\right\|_{\ell_{\mathbf{q}}(F)} \\
&= \left(\sum_{i_j=1}^\infty \left\|T\left(x_1,\dots,x_{j-1},\alpha_{i_j}x_j,x_{j+1},\dots,x_m\right)\right\|^{q_j}\right)^{\frac{1}{q_j}} \\
&= \|T(x_1,\dots,x_m)\|\left(\sum_{i=1}^\infty |\alpha_i|^{q_j}\right)^{\frac{1}{q_j}},
\end{aligned}$$

from where we can conclude

$$\|T(x_1,...,x_m)\| \left(\sum_{i=1}^{\infty} |\alpha_i|^{q_j}\right)^{\frac{1}{q_j}} \leq C \left(\prod_{\substack{k=1\\k\neq j}}^{m} \left\|(x_i^{(k)})_{i=1}^{\infty}\right\|_{w,p_k}\right) \|(\alpha_i x_j)_{i=1}^{\infty}\|_{w,p_j}.$$

Therefore, $\sum_{i=1}^{\infty} \left|\alpha_{i_j}\right|^{q_j} < \infty$, which is a contradiction since $(\alpha_i)_{i=1}^{\infty} \in \ell_{p_j} \setminus \ell_{q_j}$. □

Using the generalized Bohnenblust–Hille inequality (Theorem 1.1) together with the fact that $\mathcal{L}(c_0, E)$ and $\ell_1^w(E)$ are isometrically isomorphic (see [62, Proposition 2.2]), it is possible to prove the following result (recall the notation of the constants $B^{\text{mult}}_{\mathbb{K},m,(q_1,...,q_m)}$ in Theorem 1.1). The proof is similar to the proof of Theorem 4.3 and we omit it.

Proposition 5.6. *If $q_1, \ldots, q_m \in [1,2]$ are such that*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2},$$

then

$$\left(\sum_{j_1=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\left|T\left(x_{j_1}^{(1)},\ldots,x_{j_m}^{(m)}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}$$
$$\leq B^{\text{mult}}_{\mathbb{K},m,(q_1,...,q_m)} \|T\| \prod_{k=1}^{m} \left\|(x_j^{(k)})_{j=1}^{\infty}\right\|_{w,1},$$

for all m-linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$ and all sequences $(x_j^{(k)})_{j=1}^{\infty} \in \ell_1^w(E_k)$, $k = 1, \ldots, m$.

In other words, if $q_1, \ldots, q_m \in [1,2]$ are such that $\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2}$ we have the following coincidence result:

$$\Pi^m_{\text{mult}(q_1,...,q_m;1,...,1)}(E_1,\ldots,E_m;\mathbb{K}) = \mathcal{L}(E_1,\ldots,E_m;\mathbb{K}).$$

With the same idea than in the proof of Proposition 5.6 (but now using $\mathcal{L}(c_0,E) = \ell_1^w(E)$ and $\mathcal{L}(\ell_p,E) = \ell_{p^*}^w(E)$), we can re-write the Theorems 1.1 and 1.2 (recall the notation for the constants on each result):

Proposition 5.7. *Let $m \geq 1$, $\mathbf{p} := (p_1,\ldots,p_m) \in [1,\infty]^m$.*

(1) Let $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} := (q_1,\ldots,q_m) \in \left[\left(1-\left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^m$ such that

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|.$$

Then, for all continuous m-linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}\left|T\left(x_{i_1}^{(1)},\ldots,x_{i_m}^{(m)}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{\mathbb{K},m,\mathbf{p},\mathbf{q}}^{\text{mult}}\|T\|\prod_{k=1}^{m}\left\|(x_i^{(k)})_{i=1}^{\infty}\right\|_{w,p_k^*},$$

regardless of the sequences $(x_i^{(k)})_{i=1}^{\infty} \in \ell_{p_k^*}^{w}(E_k)$, $k = 1, \ldots, m$.

(2) If $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$, then, for all continuous m-linear forms $T : E_1 \times \cdots \times E_m \to \mathbb{K}$,

$$\left(\sum_{i_1,\ldots,i_m=1}^{N}\left|T\left(x_{i_1}^{(1)},\ldots,x_{i_m}^{(m)}\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{\mathbb{K},m,\mathbf{p}}^{\text{mult}}\|T\|\prod_{k=1}^{m}\left\|(x_i^{(k)})_{i=1}^{\infty}\right\|_{w,p_k^*}$$

regardless of the sequences $(x_i^{(k)})_{i=1}^{\infty} \in \ell_{p_k^*}^{w}(E_k)$, $k = 1, \ldots, m$.

In other words, the previous result says that if $\mathbf{p} := (p_1, \ldots, p_m) \in [1, \infty]^m$ and $(q_1, \ldots, q_m) \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^m$ are such that

$$0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left|\frac{1}{\mathbf{p}}\right|,$$

then

$$\Pi_{\text{mult}(q_1,\ldots,q_m;p_1^*,\ldots,p_m^*)}^{m}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

Also, if

$$\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1,$$

then

$$\Pi_{\text{mult}\left((1-|1/\mathbf{p}|)^{-1};p_1^*,\ldots,p_m^*\right)}^{m}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}).$$

The following proposition illustrates how, within this framework, coincidence results for m-linear forms can be extended to $m+1$-linear forms.

Proposition 5.8. *Let $\mathbf{p}, \mathbf{q} \in [1, +\infty)^m$. If*

$$\Pi_{\text{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}^{m}(E_1, \ldots, E_m; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m; \mathbb{K}),$$

then

$$\Pi_{\text{mult}(q_1,\ldots,q_m,2;p_1,\ldots,p_m,1)}^{m+1}(E_1, \ldots, E_m, E_{m+1}; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_m, E_{m+1}; \mathbb{K}).$$

Proof. Let us first prove that, for all continuous $(m+1)$-linear forms $T : E_1 \times \cdots \times E_m \times$

$c_0 \to \mathbb{K}$, there exist a constant $C > 0$ such that

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_{m+1}=1}^{n} \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^2 \right)^{\frac{q_m}{2}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}}$$
$$\leq C A_{q_m}^{-1} \|T\| \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^n \right\|_{w,p_k}, \tag{5.7}$$

where A_{q_m} is the constant of the Khintchine inequality (1). In fact, from Khintchine's inequality, we have

$$A_{q_m} \left(\sum_{j_{m+1}=1}^{n} \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^2 \right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^1 \left| \sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}} \right) \right|^{q_m} dt \right)^{\frac{1}{q_m}}$$
$$= \left(\int_0^1 \left| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, \sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}} \right) \right|^{q_m} dt \right)^{\frac{1}{q_m}}.$$

Thus,

$$\Big(\sum_{j_1=1}^{n} \Big(\cdots \Big(\sum_{j_{m+1}=1}^{n} \left| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}, e_{j_{m+1}}) \right|^2 \Big)^{\frac{q_m}{2}} \cdots \Big)^{\frac{q_1}{q_2}} \Big)^{\frac{1}{q_1}}$$
$$\leq A_{q_m}^{-1} \Big(\sum_{j_1=1}^{n} \Big(\cdots \Big(\sum_{j_m=1}^{n} \int_0^1 \Big| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)},$$
$$\sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}}) \Big|^{q_m} dt \Big)^{\frac{q_{m-1}}{q_m}} \cdots \Big)^{\frac{q_1}{q_2}} \Big)^{\frac{1}{q_1}}$$
$$= A_{q_m}^{-1} \Big(\sum_{j_1=1}^{n} \Big(\cdots \Big(\int_0^1 \sum_{j_m=1}^{n} \Big| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)},$$
$$\sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}}) \Big|^{q_m} dt \Big)^{\frac{q_{m-1}}{q_m}} \cdots \Big)^{\frac{q_1}{q_2}} \Big)^{\frac{1}{q_1}}$$
$$\leq A_{q_m}^{-1} \sup_{t \in [0,1]} \Big(\sum_{j_1=1}^{n} \Big(\cdots \Big(\sum_{j_m=1}^{n} \Big| T(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)},$$
$$\sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}}) \Big|^{q_m} \Big)^{\frac{q_{m-1}}{q_m}} \cdots \Big)^{\frac{q_1}{q_2}} \Big)^{\frac{1}{q_1}}$$
$$\leq A_{q_m}^{-1} \sup_{t \in [0,1]} \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)} \Big(T(\cdot, \ldots, \cdot, \sum_{j_{m+1}=1}^{n} r_{j_{m+1}}(t) e_{j_{m+1}}) \Big) \prod_{k=1}^{m} \left\| (x_j^{(k)})_{j=1}^n \right\|_{w,p_k}$$

Since $\|\cdot\| \leq \pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(\cdot)$ (see Proposition 5.3) and since, by hypothesis

$$\mathcal{L}(E_1,\ldots,E_m;\mathbb{K}) = \Pi^m_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(E_1,\ldots,E_m;\mathbb{K}),$$

the open mapping theorem ensures that the norms $\pi_{\mathrm{mult}(q_1,\ldots,q_m;p_1,\ldots,p_m)}(\cdot)$ and $\|\cdot\|$ are equivalents. Therefore, there exists a constant $C>0$ such that

$$\begin{aligned}
&\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m+1}=1}^{n}\left|T\left(x^{(1)}_{j_1},\ldots,x^{(m)}_{j_m},e_{j_{m+1}}\right)\right|^2\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\
&\leq CA_{q_m}^{-1}\sup_{t\in[0,1]}\left\|T\left(\cdot,\ldots,\cdot,\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}}\right)\right\|\prod_{k=1}^{m}\left\|(x^{(k)}_j)_{j=1}^n\right\|_{w,p_k}\\
&\leq CA_{q_m}^{-1}\|T\|\sup_{t\in[0,1]}\left\|\sum_{j_{m+1}=1}^{n}r_{j_{m+1}}(t)e_{j_{m+1}}\right\|\prod_{k=1}^{m}\left\|(x^{(k)}_j)_{j=1}^n\right\|_{w,p_k}\\
&\leq CA_{q_m}^{-1}\|T\|\prod_{k=1}^{m}\left\|(x^{(k)}_j)_{j=1}^n\right\|_{w,p_k}.
\end{aligned}$$

Let $T\in\mathcal{L}(E_1,\ldots,E_m,E_{m+1};\mathbb{K})$, $(x^{(k)}_j)_{j=1}^n\in\ell^w_{p_k}(E_k)$, $k=1,\ldots,m$, and $(x^{(m+1)}_j)_{j=1}^n\in\ell^w_1(E_{m+1})$. From [62, Proposition 2.2] we have the boundedness of the linear operator $u:c_0\to E_{m+1}$ such that $e_j\mapsto u(e_j)=x^{(m+1)}_j$ and $\|u\|=\left\|(x^{(m+1)}_j)_{j=1}^n\right\|_{1,w}$. Then, $S:E_1\times\cdots\times E_m\times c_0\to\mathbb{K}$ defined by $S(y_1,\ldots,y_{m+1})=T(y_1,\ldots,y_m,u(y_{m+1}))$ is a continuous $(m+1)$-linear form and $\|S\|\leq\|T\|\|u\|$. Therefore, from (5.7),

$$\begin{aligned}
&\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m+1}=1}^{n}\left|T\left(x^{(1)}_{j_1},\ldots,x^{(m)}_{j_m},x^{(m+1)}_{j_{m+1}}\right)\right|^2\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\
&=\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{jm+1=1}^{n}\left|S\left(x^{(1)}_{j_1},\ldots,x^{(m)}_{j_m},e_{j_{m+1}}\right)\right|^2\right)^{\frac{q_m}{2}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\
&\leq CA_{q_m}^{-1}\|T\|\,\|u\|\left\|(x^{(1)}_j)_{j=1}^n\right\|_{w,p_1}\cdots\left\|(x^{(m)}_j)_{j=1}^n\right\|_{w,p_m}\\
&=CA_{q_m}^{-1}\|T\|\left\|(x^{(1)}_j)_{j=1}^n\right\|_{w,p_1}\cdots\left\|(x^{(m)}_j)_{j=1}^n\right\|_{w,p_m}\left\|(x^{(m+1)}_j)_{j=1}^n\right\|_{w,1},
\end{aligned}$$

i.e., $T\in\Pi^{m+1}_{\mathrm{mult}(q_1,\ldots,q_m,2;p_1,\ldots,p_m,1)}(E_1,\ldots,E_{m+1};\mathbb{K})$. □

5.2 Partially multiple summing operators: the unifying concept

Let $1 \leq m \in \mathbb{N}$ and $1 \leq p_1, ..., p_m \leq \infty$. For $\mathbf{p} := (p_1, \ldots, p_m) \in [1, +\infty]^m$, recall that $\left|\frac{1}{\mathbf{p}}\right| := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $X_p := \ell_p$, for $1 \leq p < \infty$, and $X_\infty := c_0$. In addition to Bohnenblus–Hille and Hardy–Littlewood inequalities (see (1.1) and Theorems 1.1 and 1.2, respectively), the following results on summability of m-linear forms $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$ are well known.

- Zalduendo ([128], 1993): Let $\left|\frac{1}{\mathbf{p}}\right| < 1$. For every continuous m-linear form $T : X_{p_1} \times \cdots \times X_{p_m} \to \mathbb{K}$,

$$\left(\sum_{i=1}^{\infty} |T(e_i, ..., e_i)|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq \|T\|, \tag{5.8}$$

  and the exponent $1/(1 - |1/\mathbf{p}|)$ is optimal.

- Aron and Globevnik ([17], 1989): For every continuous m-linear form $T : c_0 \times \cdots \times c_0 \to \mathbb{K}$,

$$\sum_{i=1}^{\infty} |T(e_i, \ldots, e_i)| \leq \|T\|, \tag{5.9}$$

  and the exponent 1 is optimal.

The main purpose of [2] is to present a unified version of the Bohnenblust–Hille and the Hardy–Littlewood inequalities with partial sums (i.e., it was shown what happens when some of the indices of the sums $i_1, \ldots, i_m$ are repeated) which also encompasses Zalduendo's and Aron–Globevnik's inequalities. A tensorial perspective[1] was the key in this matter, establishing an intrinsic relationship between the exponents and constants involved and the number of indices taken on the sums.

From now on, if $n_1, \ldots, n_k \geq 1$ are such that $n_1 + \cdots + n_k = m$, then $\left(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}\right)$ will mean $(e_{i_1}, \overset{n_1 \text{ times}}{\ldots}, e_{i_1}, \ldots, e_{i_k}, \overset{n_k \text{ times}}{\ldots}, e_{i_k})$. The main result of [2] is:

Theorem 5.9. *Let $1 \leq k \leq m$ and $n_1, \ldots, n_k \geq 1$ be positive integers such that $n_1 + \cdots + n_k = m$ and assume that*

$$\mathbf{p} := \left(p_1^{(1)}, \overset{n_1 \text{ times}}{\ldots}, p_{n_1}^{(1)}, \ldots, p_1^{(k)}, \overset{n_k \text{ times}}{\ldots}, p_{n_k}^{(k)}\right) \in [1, \infty]^m$$

is such that $0 \leq \left|\frac{1}{\mathbf{p}}\right| < 1$. Let r_i given by $\frac{1}{r_i} = \frac{1}{p_1^{(i)}} + \cdots + \frac{1}{p_{n_i}^{(i)}}$, $i = 1, \ldots, k$.

(1) If $0 \leq \left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$ and $\mathbf{q} := (q_1, \ldots, q_k) \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^k$ then, for every continuous

[1]The idea of introducing in this context this tensorial perspective was of the Prof[a]. Maria Pilar Rueda Segado.

m–linear form $T : \left(\times_{1\leq i\leq n_1} X_{p_i^{(1)}}\right) \times \cdots \times \left(\times_{1\leq i\leq n_k} X_{p_i^{(k)}}\right) \to \mathbb{K}$,

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left|T\left(e_{i_1}^{n_1},\ldots,e_{i_k}^{n_k}\right)\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{\mathbb{K},k,(r_1,\ldots,r_k),\mathbf{q}}^{\text{mult}}\|T\| \quad (5.10)$$

if and only if $\left|\frac{1}{\mathbf{q}}\right| \leq \frac{k+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$. *In other words, the exponents are optimal.*

(2) If $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$ *then, for every continuous m–linear form* $T : \left(\times_{1\leq i\leq n_1} X_{p_i^{(1)}}\right) \times \cdots \times \left(\times_{1\leq i\leq n_k} X_{p_i^{(k)}}\right) \to \mathbb{K}$,

$$\left(\sum_{i_1,\ldots,i_k=1}^{\infty}\left|T\left(e_{i_1}^{n_1},\ldots,e_{i_k}^{n_k}\right)\right|^{\frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}}\right)^{1-\left|\frac{1}{\mathbf{p}}\right|} \leq D_{\mathbb{K},k,(r_1,\ldots,r_k)}^{\text{mult}}\|T\|. \quad (5.11)$$

Moreover, the exponent in (5.11) is optimal

Let us establish the following notation: for Banach spaces $E_1,\ldots,E_m$ and an element $x \in E_j$, for some $j \in \{1,\ldots,m\}$, the symbol $x_j \cdot e_j$ represents the vector $x_j \cdot e_j \in E_1 \times \cdots \times E_m$ such that its j-th coordinate is $x_j \in E_j$, and 0 otherwise. This theorem motivated us to give the following unifying notion of absolutely summing multilinear operators (the essence of the notion of partially multiple summing operators (below) was first sketched in [107, Definition 2.2.1] but it has not been explored since):

Definition 5.10. *Let $E_1,\ldots,E_m,F$ be Banach spaces, m,k be positive integers with $1 \leq k \leq m$, and $(\mathbf{p},\mathbf{q}) := (p_1,\ldots,p_m,q_1,\ldots,q_k) \in [1,\infty)^{m+k}$. Let also $\mathcal{I} = \{I_1,\ldots,I_k\}$ a family of non-void disjoints subsets of $\{1,\ldots,m\}$ such that $\cup_{i=1}^k I_i = \{1,\ldots,m\}$, that is, $\mathcal{I}$ is a partition of $\{1,\ldots,m\}$. A multilinear operator $T : E_1\times\cdots\times E_m \to F$ is $\mathcal{I}$-partially multiple $(\mathbf{q};\mathbf{p})$-summing if there exists a constant $C > 0$ such that*

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_k=1}^{\infty}\left\|T\left(\sum_{n=1}^{k}\sum_{j\in I_n} x_{i_n}^{(j)}\cdot e_j\right)\right\|_F^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\prod_{j=1}^{m}\left\|(x_i^{(j)})_{i=1}^{\infty}\right\|_{w,p_j}$$

for all $(x_i^{(j)})_{i=1}^{\infty} \in \ell_{p_j}^{w}(E_j)$, $j = 1,\ldots,m$. We represent the class of all $\mathcal{I}$-partially multiple $(\mathbf{q};\mathbf{p})$-summing operators by $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$. The infimum taken over all possible constants $C > 0$ satisfying the previous inequality defines a norm in $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$, which is denoted by $\pi_{(\mathbf{q};\mathbf{p})}$.

As usual, $\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1,\ldots,E_m;F)$ is a subspace of $\mathcal{L}(E_1,\ldots,E_m;F)$. Moreover, note that when

- $k = 1$, we recover the class of absolutely $(q;p_1,\ldots,p_m)$-summing operators, with $q := q_1$;

- $k = m$ and $q_1 = \cdots = q_m =: q$, we recover the class of multiple $(q; p_1, \ldots, p_m)$-summing operators;
- $k = m$, we recover the class of multiple $(q_1, \ldots, q_m; p_1, \ldots, p_m)$-summing operators, as we defined in the section 5.1.

The basis of this theory can be developed in the same lines as those from the previous section, as we will be presenting in what follows. From now on, m, k are positive integers with $1 \leq k \leq m$, $(\mathbf{p}, \mathbf{q}) := (p_1, \ldots, p_m, q_1, \ldots, q_k) \in [1, \infty)^{m+k}$ and $\mathcal{I} = \{I_1, \ldots, I_k\}$ is a partition of $\{1, \ldots, m\}$.

Proposition 5.11. *Let $T : E_1 \times \cdots \times E_m \to F$ be a continuous multilinear operator. The following assertions are equivalent:*
(1) T is $\mathcal{I}$-partially multiple $(\mathbf{q}; \mathbf{p})$-summing;
(2) $\left(T\left(\sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j\right)\right)_{i_1, \ldots, i_k = 1}^{\infty} \in \ell_{\mathbf{q}}(F)$ whenever $(x_i^{(j)})_{i=1}^{\infty} \in \ell_{p_j}^{w}(E_j)$, for $j = 1, \ldots, m$.

Proposition 5.12. *If $T \in \Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$, then*

$$\|T\|_{\mathcal{L}(E_1, \ldots, E_m; F)} \leq \pi_{(\mathbf{q};\mathbf{p})}(T).$$

Given $T \in \Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F)$ we may define the m-linear operator

$$\begin{array}{rcl} \widehat{T} : \ell_{p_1}^{w}(E_1) \times \cdots \times \ell_{p_m}^{w}(E_m) & \to & \ell_{\mathbf{q}}(F) \\ \left((x_i^{(1)})_{i=1}^{\infty}, \ldots, (x_i^{(m)})_{i=1}^{\infty}\right) & \mapsto & \left(T\left(\sum_{n=1}^{k} \sum_{j \in I_n} x_{i_n}^{(j)} \cdot e_j\right)\right)_{i_1, \ldots, i_k = 1}^{\infty}. \end{array} \tag{5.12}$$

By using both, the closed graph and the Hahn–Banach theorems, it is possible to prove that $\widehat{T}$ is a continuous m-linear operator. Furthermore, we can prove that

$$\|\widehat{T}\| = \pi_{(\mathbf{q};\mathbf{p})}(T), \tag{5.13}$$

therefore, naturally we define the isometric operator

$$\begin{array}{rcl} \widehat{\theta} : \Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F) & \to & \mathcal{L}\left(\ell_{p_1}^{w}(E_1), \ldots, \ell_{p_m}^{w}(E_m); \ell_{\mathbf{q}}(F)\right) \\ T & \mapsto & \widehat{T}. \end{array}$$

These facts lead us to the following result.

Theorem 5.13. $\left(\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F), \pi_{(\mathbf{q};\mathbf{p})}(\cdot)\right)$ *is a Banach space.*

Also,

Proposition 5.14. *If there exists $n \in \{1, \ldots, k\}$ such that $\frac{1}{q_n} > \sum_{j \in I_n} \frac{1}{p_j}$, then*

$$\Pi_{(\mathbf{q};\mathbf{p})}^{k,m,\mathcal{I}}(E_1, \ldots, E_m; F) = \{0\}.$$

As in Proposition 5.6, it is possible to prove the following result (now using the Bohnenblust–Hille inequality with partial sums, i.e., Theorem 5.9 with $\mathbf{p} = (\infty, ..., \infty)$):

If $\mathbf{q} = (q_1, \dots, q_k) \in [1,2]^k$ is such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} \le \frac{k+1}{2}$, then

$$\Pi^{k,m,\mathcal{I}}_{\left(\mathbf{q};1,\overset{m \text{ times}}{\cdots},1\right)}(E_1, \dots, E_m; F) = \mathcal{L}\left(E_1, \dots, E_m; \mathbb{K}\right).$$

With the same idea of Proposition 5.7, we can re-written Theorem 5.9 in general: Let $1 \le k \le m$ and $n_1, \dots, n_k \ge 1$ be positive integers such that $n_1 + \cdots + n_k = m$ and let $\mathbf{p} := (p_1, \dots, p_m) \in [1, \infty]^m$. Then,

(1) if $0 \le \left|\frac{1}{\mathbf{p}}\right| \le \frac{1}{2}$ and $\mathbf{q} := (q_1, \dots, q_k) \in \left[\left(1 - \left|\frac{1}{\mathbf{p}}\right|\right)^{-1}, 2\right]^k$ is such that $\frac{1}{q_1} + \cdots + \frac{1}{q_k} \le \frac{k+1}{2} - \left|\frac{1}{\mathbf{p}}\right|$, we have

$$\Pi^{k,m,\mathcal{I}}_{\left(\mathbf{q};p_1^*,\dots,p_m^*\right)}(E_1, \dots, E_m; F) = \mathcal{L}\left(E_1, \dots, E_m; \mathbb{K}\right);$$

(2) if $\frac{1}{2} \le \left|\frac{1}{\mathbf{p}}\right| < 1$, we have

$$\Pi^{k,m,\mathcal{I}}_{\left((1-|1/\mathbf{p}|)^{-1};p_1^*,\dots,p_m^*\right)}(E_1, \dots, E_m; F) = \mathcal{L}\left(E_1, \dots, E_m; \mathbb{K}\right).$$

5.3 Remark on Theorem 5.9

As a direct consequence of Theorem 5.9 yields the following particular case whenever $p_1 = \cdots = p_m = p$, which has a more friendly statement.

Corollary 5.15. *Let $m \ge k \ge 1$, $m < p \le \infty$ and let $n_1, \dots, n_k \ge 1$ be such that $n_1 + \cdots + n_k = m$. Then, for every continuous m–linear form $T : \ell_p \times \cdots \times \ell_p \to \mathbb{K}$, there is a constant $H(k,m,p,\rho,\mathbb{K}) \ge 1$ such that*

$$\left(\sum_{i_1,\dots,i_k=1}^{\infty} \left|T\left(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}\right)\right|^{\rho}\right)^{\frac{1}{\rho}} \le H(k,m,p,\rho,\mathbb{K}) \left\|T\right\|,$$

with

$$\rho = \frac{p}{p-m} \quad \text{for} \quad m < p \le 2m \quad \text{and} \quad H(k,m,p,\rho,\mathbb{K}) \le D^{\mathbb{K}}_{k,p}$$

and

$$\rho = \frac{2kp}{kp + p - 2m} \quad \text{for} \quad p \ge 2m \quad \text{and} \quad H(k,m,p,\rho,\mathbb{K}) \le C^{\mathbb{K}}_{k,p}. \tag{5.14}$$

Moreover, in both cases, the exponent ρ is optimal.

Remark 5.16. *It is very interesting to stress that the optimal exponent for the case $p > 2m$ is not the exponent of the k-linear case, as one may expect. It is a kind of surprising combination of the cases of k-linear and m-linear forms, as it can be seen in (5.14). In general the panorama is quite puzzling:*

- *If $m < p < 2m$ the optimal exponent depends only on m;*

- *If $p = 2m$, the optimal exponent does not depend on m or k.*
- *If $2m < p < \infty$, the optimal exponent depends on m and k;*
- *If $p = \infty$, the optimal exponent depends only on k.*

The following estimates summarizes the (optimal) exponents and respective best known constants that can be derived from Corollary 5.15 combined with estimates from [13, 63, 109] (below γ denotes the Euler-Mascheroni constant):

$$
\begin{aligned}
&H(k,m,p,\frac{p}{p-m},\mathbb{R}) \le (\sqrt{2})^{k-1};\\
&H(k,m,p,\frac{p}{p-m},\mathbb{C}) \le \left(\frac{2}{\sqrt{\pi}}\right)^{k-1};\\
&H(k,m,p,\frac{2kp}{kp+p-2m},\mathbb{R}) \le \begin{cases} 1.3k^{\left(\frac{2-\log 2-\gamma}{2}\right)(k-1)\left(\frac{2k-p+kp-2k^2}{k^2p-2kp}\right)} \left(\sqrt{2}\right)^{\frac{p-2k-kp+6k^2-6k^3+2k^4}{kp(k-2)}} \\ \qquad\qquad \text{if } 2m \le p \le 2m^3-4m^2+2m, \\ 1.3k^{\frac{2-\log 2-\gamma}{2}} \quad \text{if } 2m^3-4m^2+2m < p \le \infty; \end{cases}\\
&H(k,m,p,\frac{2kp}{kp+p-2m},\mathbb{C}) \le \begin{cases} k^{\left(\frac{1-\gamma}{2}\right)(k-1)\left(\frac{2k-p+kp-2k^2}{k^2p-2kp}\right)} \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{p-2k-kp+6k-6k^3+2k^4}{kp(k-2)}} \\ \qquad\qquad \text{if } 2m \le p \le 2m^3-4m^2+2m, \\ k^{\frac{1-\gamma}{2}} \quad \text{if } 2m^3-4m^2+2m < p \le \infty. \end{cases}
\end{aligned}
$$

Part III

Classical inequalities for polynomials on circle sectors

Chapter 6

Polynomial inequalities on the $\pi/4$-circle sector

In this chapter we present results of the paper:

[10] G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, and J.B. Seoane-Sepúlveda, *Polynomial inequalities on the* $\pi/4$*-circle sector*, arXiv:1503.06607 [math.FA].

The Krein–Milman Theorem ensures that every convex body (non-empty, compact set) in a Banach space is fully described by the set of its extreme points. We recall that if C is a convex body in a Banach space, a point $e \in C$ is said to be *extreme* if $x, y \in C$ and $\lambda x + (1\lambda)y = e$, for some $0 < \lambda < 1$, entails $x = y = e$. Equivalently, $e \in C$ is extreme if and only if $C \setminus \{e\}$ is convex. It is well known that a convex function (for instance, a polynomial norm) defined on a convex body attains its maximum at an extreme point of their domain. From now on we will refer to this method as the Krein–Milman approach.

In this chapter we apply this method in order to obtain sharp polynomial inequalities on the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ of 2-variable, real 2-homogeneous polynomials endowed with the supremum norm on the sector $D\left(\frac{\pi}{4}\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$.

Let us describe now the four inequalities that will be studied in this chapter. Namely, for a fixed $(x, y) \in D\left(\frac{\pi}{4}\right)$, we find the best (smallest) constant in the following inequalities:

- **Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.** For a fixed $(x, y) \in D\left(\frac{\pi}{4}\right)$, we find the smallest constant $\Phi(x, y)$ in the inequality
$$\|\nabla P(x, y)\|_2 \leq \Phi(x, y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$
for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $\|\cdot\|_2$ denotes the euclidean norm in $\mathbb{R}^2$

- **Markov global estimate on the gradient of polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.** For all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and all $(x, y) \in D\left(\frac{\pi}{4}\right)$, we find the smallest constant $M > 0$ in the inequality
$$\|\nabla P(x, y)\|_2 \leq M\|P\|_{D\left(\frac{\pi}{4}\right)}.$$

- **Polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.** We find the smallest constant $K > 0$ in the inequality
$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq K\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where P is an arbitrary polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and L is the polar of P.

- **Unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.** We find the smallest constant $C>0$ in the inequality

$$\||P|\|_{D\left(\frac{\pi}{4}\right)} \leq C\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$, where $|P|$ is the modulus of P, i.e., if $P(x,y)=ax^2+by^2+cxy$, then $|P|(x,y)=|a|x^2+|b|y^2+|c|xy$.

If $P(x,y)=ax^2+by^2+cxy$, we will often represent P as the point (a,b,c) in $\mathbb{R}^3$. Hence, the norm of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right) := \left\{e^{i\theta} : \theta \in \left[0, \frac{\pi}{4}\right]\right\}$ is in fact the norm in $\mathbb{R}^3$ given by

$$\|(a,b,c)\|_{D\left(\frac{\pi}{4}\right)} = \sup\left\{|ax^2+by^2+cxy| : (x,y) \in D\left(\frac{\pi}{4}\right)\right\}.$$

In Section 6.2, the notation $\mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ will be useful to represent the symmetric bilinear forms on $\mathbb{R}^2$ endowed with the supremum norm on $D\left(\frac{\pi}{4}\right)$.

An explicit description of the norm $\|\cdot\|_{D\left(\frac{\pi}{4}\right)}$ and the extreme points of the unit ball $B_{D\left(\frac{\pi}{4}\right)}$, denoted by $\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)$, will be required. Both are presented below (see [93, Theorem 3.1] and [93, Theorem 4.4], respectively):

Lemma 6.1. *If $P(x,y)=ax^2+by^2+cxy$, then*

$$\|P\|_{D\left(\frac{\pi}{4}\right)} = \begin{cases} \max\left\{|a|, \frac{1}{2}|a+b+c|, \frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\} & \text{if } c(a-b)\geq 0, \\ \max\{|a|, \frac{1}{2}|a+b+c|\} & \text{if } c(a-b)\leq 0, \end{cases}$$

Lemma 6.2. *The extreme points of the unit ball of $\mathcal{P}({}^2D(\frac{\pi}{4}))$ are given by*

$$\operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right) = \left\{\pm P_t, \pm Q_s, \pm(1,1,0) : \ -1\leq t\leq 1 \text{ and } 1\leq s\leq 5+4\sqrt{2}\right\},$$

where

$$\begin{aligned} P_t &:= (t, 4+t+4\sqrt{1+t}, -2-2t-4\sqrt{1+t}), \\ Q_s &:= (1, s, -2\sqrt{2(1+s)}). \end{aligned}$$

6.1 Bernstein and Markov-type inequalities for polynomials on sectors

In this section we provide sharp estimates on the Euclidean length of the gradient ∇P of a polynomial P in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Theorem 6.3. *For every $(x,y)\in D\left(\frac{\pi}{4}\right)$ and $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ we have*

$$\|\nabla P\|_2 \leq \Phi(x,y)\|P\|_{D\left(\frac{\pi}{4}\right)},$$

where

$$\Phi(x,y) = \begin{cases} 4\left[\left(13+8\sqrt{2}\right)x^2+\left(69+48\sqrt{2}\right)y^2-2\left(28+20\sqrt{2}\right)xy\right] \\ \qquad\qquad \text{if } 0\le y\le \frac{\sqrt{2}-1}{2}x \text{ or } \left(4\sqrt{2}-5\right)x\le y\le x, \\ \frac{x^4}{y^2}+4(x^2+y^2) \quad \text{if } \frac{\sqrt{2}-1}{2}x\le y\le\left(\sqrt{2}-1\right)x, \\ \frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2} \quad \text{if } \left(\sqrt{2}-1\right)x\le y\le\left(4\sqrt{2}-5\right)x. \end{cases}$$

Proof. In order to calculate $\Phi(x,y) := \sup\{\|\nabla P(x,y)\|_2 : \|P\|_{D(\frac{\pi}{4})} \le 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x,y)\|_2 : P\in \mathrm{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 6.2 with positive sign. Let us start first with $P_t(x,y) = tx^2+\left(4+t+4\sqrt{1+t}\right)y^2-2\left(1+t+2\sqrt{1+t}\right)xy$, $t\in[-1,1]$. Then,

$$\nabla P_t(x,y) = \left(2tx-2\left(1+t+2\sqrt{1+t}\right)y, 2\left(4+t+4\sqrt{1+t}\right)y-2\left(1+t+2\sqrt{1+t}\right)x\right),$$

so that

$$\begin{aligned}\|\nabla P_t(x,y)\|_2^2 =& 4t^2x^2+4\left(1+t+2\sqrt{1+t}\right)^2y^2-8t\left(1+t+2\sqrt{1+t}\right)xy \\ &+4\left(4+t+4\sqrt{1+t}\right)^2y^2+4\left(1+t+2\sqrt{1+t}\right)^2x^2 \\ &-8\left(4+t+4\sqrt{1+t}\right)\left(1+t+2\sqrt{1+t}\right)xy\end{aligned}$$

Make now the change $u=\sqrt{1+t}\in\left[0,\sqrt{2}\right]$, so that

$$\begin{aligned}\|\nabla P_u(x,y)\|_2^2 =& 8(x-y)^2u^4+16\left(x^2-4xy+3y^2\right)u^3 \\ &+8\left(x^2-10xy+13y^2\right)u^2+32\left(3y^2-xy\right)u+4\left(x^2+9y^2\right).\end{aligned}$$

Since

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 = 16\left(2\left(x-y\right)^2u^2+\left(x^2-8xy+7y^2\right)u+2y\left(3y-x\right)\right)(u+1),$$

it follows that the critical points of $\|DP_u(x,y)\|_2^2$ are $u=\frac{2y}{x-y}$, $u=\frac{3y-x}{2(x-y)}$ and $u=-1$ if $x\neq y$ and $u=4$ and $u=-1$ if $x=y$. Since we need to consider $0\le u\le\sqrt{2}$, we can directly omit the case $x=y$.

Therefore, we can write

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2 = 32(x-y)^2\left(u-\frac{2y}{x-y}\right)\left(u-\frac{3y-x}{2(x-y)}\right)(u+1).$$

Let $u_1 = \frac{2y}{x-y}$ and $u_2 = \frac{3y-x}{2(x-y)}$ (Again, since we need to consider $0 \le u \le \sqrt{2}$, we can omit the solution $u = -1$). Also, we have the extra conditions $u_1 \in [0, \sqrt{2}]$ whenever $0 \le y \le \left(\sqrt{2}-1\right) x$ and $u_2 \in [0, \sqrt{2}]$ whenever $\frac{1}{3}x \le y \le \left(4\sqrt{2}-5\right) x$. Considering all these facts, we need to compare the quantities

$$\begin{aligned} C_1(x,y) &:= \|\nabla P_{u_1}(x,y)\|_2^2 = \|\nabla P_{t_1}\|_2^2 \\ &= 4\frac{x^6 - 4x^5y + 7x^4y^2 - 8x^3y^3 + 7x^2y^4 - 4xy^5 + y^6}{(x-y)^4} = 4\left(x^2+y^2\right), \end{aligned}$$

for $0 \le y \le \left(\sqrt{2}-1\right) x$ and $t_1 = \frac{3y^2+2xy-x^2}{(x-y)^2}$,

$$\begin{aligned} C_2(x,y) &:= \|\nabla P_{u_2}(x,y)\|_2^2 = \|\nabla P_{t_2}\|_2^2 \\ &= \frac{9x^6 - 30x^5y + 55x^4y^2 - 68x^3y^3 + 55x^2y^4 - 30xy^5 + 9y^6}{2(x-y)^4} \\ &= \frac{\left(3x^2 - 2xy + 3y^2\right)^2}{2(x-y)^2}, \end{aligned}$$

for $\frac{1}{3}x \le y \le \left(4\sqrt{2}-5\right) x$ and $t_2 = \frac{5y^2+2xy-3x^2}{4(x-y)^2}$,

$$C_3(x,y) := \|\nabla P_{t_3=-1}\|_2^2 = 4\left(x^2+9y^2\right),$$

and

$$C_4(x,y) := \|\nabla P_{t_4=1}\|_2^2 = 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right].$$

Let us focus now on $Q_s = \left(1, s, -2\sqrt{2(1+s)}\right)$, $1 \le s \le 5+4\sqrt{2}$. Then, we have

$$\|\nabla Q_s(x,y)\|_2^2 = 4x^2 + 4s^2y^2 + 8(1+s)(x^2+y^2) - 8(1+s)\sqrt{2(1+s)}xy.$$

Making the change $v = \sqrt{2(1+s)} \in \left[2, 2+2\sqrt{2}\right]$, we need to study the function

$$\|\nabla Q_v(x,y)\|_2^2 = v^2\left(y^2v^2 - 4xyv + 4x^2\right) + 4\left(x^2+y^2\right).$$

If $x = y = 0$ we have $\|\nabla Q_v(x,y)\|_2^2 = 0$, so we will assume both $x \ne 0$ and $y \ne 0$. The critical points of $\|\nabla Q_v(x,y)\|_2^2$ are $v = \frac{x}{y}, v = \frac{2x}{y}$ and $v = 0$ (but $0 \notin [2, 2+2\sqrt{2}]$). Observe that $v_1 = \frac{x}{y} \in \left[2, 2+2\sqrt{2}\right]$ whenever $\frac{\sqrt{2}-1}{2}x \le y \le \frac{1}{2}x$ and $v_2 = \frac{2x}{y} \in \left[2, 2+2\sqrt{2}\right]$ whenever $y \ge \left(\sqrt{2}-1\right) x$. Thus, we also need to compare the quantities

$$C_5(x,y) := \|\nabla Q_{v_1}(x,y)\|_2^2 = \|\nabla Q_{s_1}(x,y)\|_2^2 = \frac{x^4}{y^2} + 4\left(x^2+y^2\right),$$

for $\frac{\sqrt{2}-1}{2}x \le y \le \frac{1}{2}x$ and $s_1 = \frac{x^2-2y^2}{2y^2}$,

$$C_6(x,y) := \|\nabla Q_{v_2}(x,y)\|_2^2 = \|\nabla Q_{s_2}(x,y)\|_2^2 = 4\left(x^2+y^2\right),$$

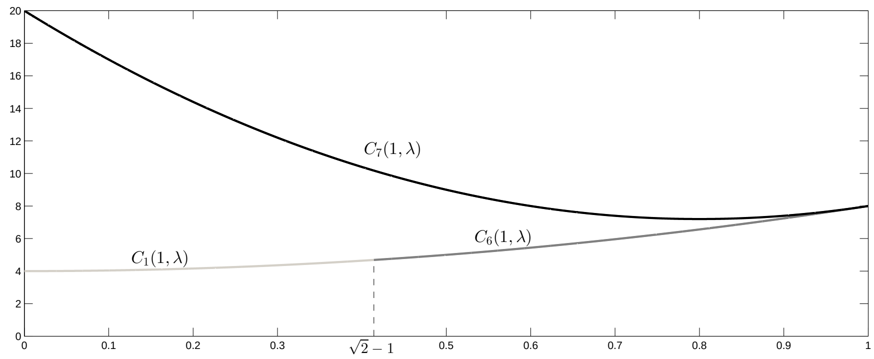

Figure 6.1: Graphs of the mappings $C_1(1,\lambda)$, $C_6(1,\lambda)$, $C_7(1,\lambda)$.

for $\left(\sqrt{2}-1\right)x \le y \le x$ and $s_2 = \frac{2x^2-y^2}{y^2}$, and also

$$C_7(x,y) := \|\nabla Q_{s_3=1}\|_2^2 = 4\left(x^2+y^2\right) + 16(x-y)^2,$$

and

$$\begin{aligned} C_8(x,y) &:= \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2 \\ &= \left(12+8\sqrt{2}\right)\left[4x^2 + \left(12+8\sqrt{2}\right)y^2 - \left(8+8\sqrt{2}\right)xy\right] + 4\left(x^2+y^2\right) \\ &= 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right]. \end{aligned}$$

Note that (the reader can take a look at Figures 6.1, 6.2 and 6.3)

$$\begin{aligned} &C_1(x,y), C_6(x,y) \le C_7(x,y) \le \begin{cases} C_4(x,y) & \text{if } 0 \le y \le \frac{2-\sqrt{2}}{2}x \text{ or } \frac{1}{2}x \le y \le x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \le y \le \frac{1}{2}x, \end{cases} \\ &C_3(x,y) \le \begin{cases} C_2(x,y) & \text{if } \frac{1}{3}x \le y \le \left(4\sqrt{2}-5\right)x, \\ C_4(x,y) & \text{if } 0 \le y \le \frac{1}{3}x \text{ or } (4\sqrt{2}-5)x \le y \le x, \end{cases} \\ &C_8(x,y) = C_4(x,y). \end{aligned}$$

Hence, for $(x,y) \in D\left(\frac{\pi}{4}\right)$,

$$\begin{aligned} \Phi(x,y) &= \sup\left\{\|\nabla P(x,y)\|_2 \,:\, P \in \operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\} \\ &= \begin{cases} C_4(x,y) & \text{if } 0 \le y \le \frac{\sqrt{2}-1}{2}x \text{ or } (4\sqrt{2}-5)x \le y \le x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \le y \le (\sqrt{2}-1)x, \\ C_2(x,y) & \text{if } (\sqrt{2}-1)x \le y \le (4\sqrt{2}-5)x. \end{cases} \end{aligned}$$

In order to illustrate the previous step, the reader can take a look at Figure 6.4. $\square$

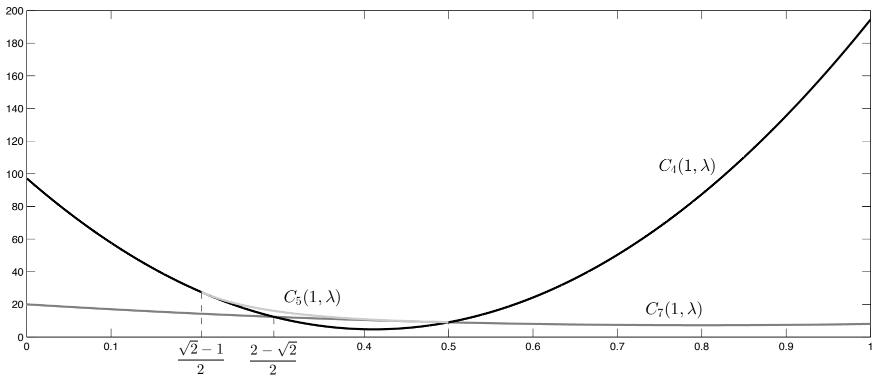

Figure 6.2: Graphs of the mappings $C_4(1, \lambda)$, $C_5(1, \lambda)$, $C_7(1, \lambda)$.

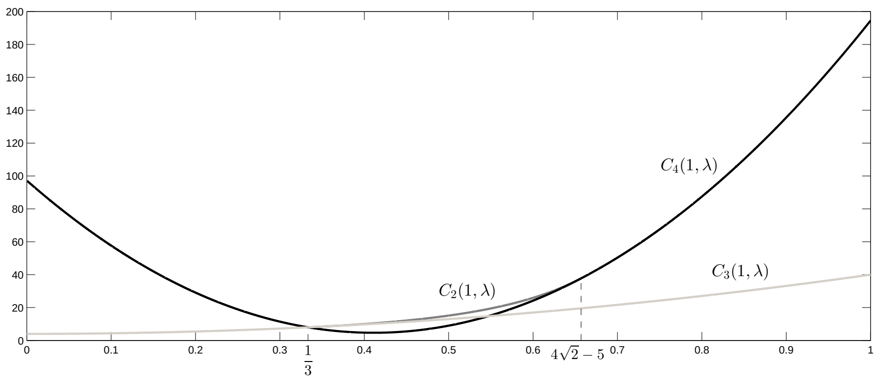

Figure 6.3: Graphs of the mappings $C_2(1, \lambda)$, $C_3(1, \lambda)$, $C_4(1, \lambda)$.

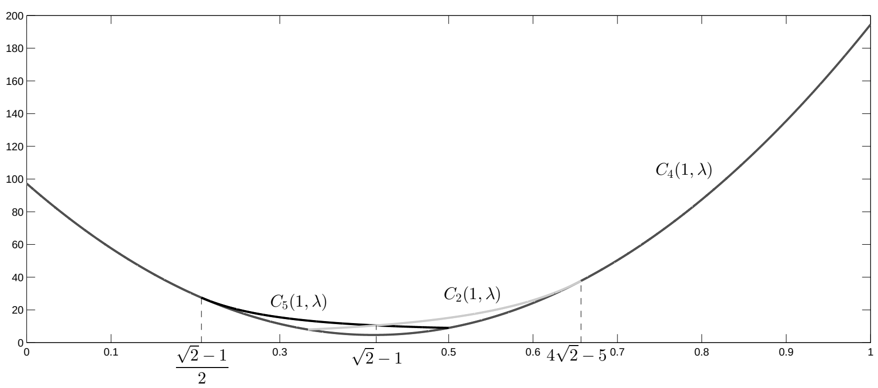

Figure 6.4: Graphs of the mappings $C_2(1, \lambda)$, $C_4(1, \lambda)$, $C_5(1, \lambda)$.

Corollary 6.4. *If $P \in \mathcal{P}\left(D\left(\frac{\pi}{4}\right)\right)$, then*

$$\sup\left\{\|\nabla P(x,y)\|_2 : (x,y) \in D\left(\frac{\pi}{4}\right)\right\} \leq 4(13+8\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

with equality for the polynomials $P_1(x,y) = \pm\left(x^2 + (5+4\sqrt{2})y^2 - 2(2+2\sqrt{2})xy\right)$.

6.2 Polarization constants for polynomials on sectors

In this section we find the exact value of the polarization constant of the space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. In order to do that, we prove a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Observe that if $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y) \in D\left(\frac{\pi}{4}\right)$ then the differential $DP(x,y)$ of P at (x,y) can be viewed as a linear form. What we shall do is to find the best estimate for $\|DP(x,y)\|_{D\left(\frac{\pi}{4}\right)}$ (the sup norm of $DP(x,y)$ over the sector $D\left(\frac{\pi}{4}\right)$) in terms of (x,y) and $\|P\|_{D\left(\frac{\pi}{4}\right)}$. First, we state a lemma that will be useful in the future:

Lemma 6.5. *Let $a, b \in \mathbb{R}$. Then,*

$$\begin{aligned}\sup_{\theta\in\left[0,\frac{\pi}{4}\right]} |a\cos\theta + b\sin\theta| &= \begin{cases} \max\left\{|a|, \frac{\sqrt{2}}{2}|a+b|\right\} & \text{if } \frac{b}{a} > 1 \text{ or } \frac{b}{a} < 0, \\ \sqrt{a^2+b^2} & \text{otherwise.}\end{cases} \\ &= \begin{cases} \sqrt{a^2+b^2} & \text{if } 0 < \frac{b}{a} < 1, \\ \frac{\sqrt{2}}{2}|a+b| & \text{if } \left(1-\sqrt{2}\right)b < a < b \text{ or } b < a < \left(1-\sqrt{2}\right)b, \\ |a| & \text{if } -\left(1+\sqrt{2}\right)a < b < 0 \text{ or } 0 < b < -\left(1+\sqrt{2}\right)a.\end{cases}\end{aligned}$$

Theorem 6.6. *For every $(x,y) \in D(\frac{\pi}{4})$ and $P \in \mathcal{P}({}^2D(\frac{\pi}{4}))$ we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{4})} \leq \Psi(x,y)\|P\|_{D(\frac{\pi}{4})}, \tag{6.1}$$

where

$$\Psi(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < (\sqrt{2}-1)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } (\sqrt{2}-1)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x\end{cases}$$

Moreover, inequality (6.1) is optimal for each $(x,y) \in D(\frac{\pi}{4})$.

Proof. In order to calculate $\Psi(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : \|P\|_{D(\frac{\pi}{4})}) \leq 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : P \in \text{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 6.2 with positive sign. Let us start first with

$$P_t(x,y) = tx^2 + \left(4+t+4\sqrt{1+t}\right)y^2 - \left(2+2t+4\sqrt{1+t}\right)xy.$$

So we may write

$$\begin{aligned}&\nabla P_t(x,y)\\&=\left(2tx-\left(2+2t+4\sqrt{1+t}\right)y,\ 2\left(4+t+4\sqrt{1+t}\right)y-\left(2+2t+4\sqrt{1+t}\right)x\right),\end{aligned}$$

from which

$$\begin{aligned}\|DP_t(x,y)\|_{D(\frac{\pi}{4})}&=\sup_{0\le\theta\le\frac{\pi}{4}}\Big|2\left[tx-\left(1+t+2\sqrt{1+t}\right)y\right]\cos\theta\\&\quad+2\left[\left(4+t+4\sqrt{1+t}\right)y-\left(1+t+2\sqrt{1+t}\right)x\right]\sin\theta\Big|\\&=2x\sup_{0\le\theta\le\frac{\pi}{4}}|f_\lambda(t,\theta)|,\end{aligned}$$

for

$$\begin{aligned}f_\lambda(t,\theta)=&\left[t-\left(1+t+2\sqrt{1+t}\right)\lambda\right]\cos\theta\\&+\left[\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)\right]\sin\theta,\end{aligned}$$

where $\lambda=\frac{y}{x}$, $x\neq 0$ (the case $x=0$ is trivial, since the only point in $D(\frac{\pi}{4})$ where $x=0$ is $(0,0)$, in which case $P_t(0,0)=\|DP_t(0,0)\|_{D(\frac{\pi}{4})}=0$).
We need to calculate

$$\sup_{-1\le t\le 1}\|DP_t(x,y)\|_{D(\frac{\pi}{4})}=2x\sup_{\substack{0\le\theta\le\frac{\pi}{4}\\-1\le t\le 1}}|f_\lambda(t,\theta)|.$$

Let us define $C_1=[-1,1]\times[0,\frac{\pi}{4}]$. We will analyze 5 cases.

(1) $(t,\theta)\in(-1,1)\times(0,\frac{\pi}{4})$.

We are interested just in critical points. Hence,

$$\begin{aligned}\frac{\partial f_\lambda}{\partial t}(t,\theta)=&\left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda-\left(1+\frac{1}{\sqrt{1+t}}\right)\right]\sin\theta\\&+\left[1-\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right]\cos\theta=0,\end{aligned}\tag{6.2}$$

$$\begin{aligned}\frac{\partial f_\lambda}{\partial\theta}(t,\theta)=&\left[\left(1+t+2\sqrt{1+t}\right)\lambda-t\right]\sin\theta\\&+\left[\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)\right]\cos\theta=0\end{aligned}\tag{6.3}$$

Equation (6.3) tells us that

$$\sin\theta=\frac{\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)}{t-\left(1+t+2\sqrt{1+t}\right)\lambda}\cos\theta.\tag{6.4}$$

If we now plug (6.4) in equation (6.2), we obtain

$$0 = \left\{ \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\right] \right.$$
$$\left. \times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t - \left(1+t+2\sqrt{1+t}\right)\lambda} \right\} \cos\theta.$$

Using that $0 < \theta < \frac{\pi}{4}$, we can conclude

$$0 = \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda\right] + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\right]$$
$$\times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t - \left(1+t+2\sqrt{1+t}\right)\lambda}$$

and thus

$$\begin{aligned}
0 &= \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda\right] \cdot \left[t - \left(1+t+2\sqrt{1+t}\right)\lambda\right] \\
&\quad + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\right] \cdot \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right] \\
&= t - \left(1+t+2\sqrt{1+t}\right)\lambda - t\lambda + \left(1+t+2\sqrt{1+t}\right)\lambda^2 - \frac{\lambda t}{\sqrt{1+t}} \\
&\quad + \frac{\lambda^2}{\sqrt{1+t}}\left(1+t+2\sqrt{1+t}\right) + \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda^2 \\
&\quad - \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda \\
&\quad + \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right) \\
&= t\left(1 - 2\lambda + 2\lambda^2 - 2\lambda + 1\right) + \left(-2\lambda + 2\lambda^2 + 4\lambda^2 - 2\lambda - 4\lambda + 2\right)\sqrt{1+t} \\
&\quad + \frac{t}{\sqrt{1+t}}\left(-\lambda + \lambda^2 + 2\lambda^2 - 2\lambda - \lambda + 1\right) + \frac{1}{\sqrt{1+t}}\left(\lambda^2 + 8\lambda^2 - 2\lambda - 4\lambda + 1\right) \\
&\quad + \left(-\lambda + \lambda^2 + 2\lambda^2 + 4\lambda^2 - \lambda - 4\lambda + 1 + 2 + 8\lambda^2 - 8\lambda\right) \\
&= 2t(\lambda-1)^2 + 6\sqrt{1+t}(\lambda-1)\left(\lambda - \frac{1}{3}\right) + 3\frac{t}{\sqrt{1+t}}(\lambda-1)\left(\lambda - \frac{1}{3}\right) \\
&\quad + \frac{1}{\sqrt{1+t}}(3\lambda-1)^2 + 15\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right).
\end{aligned}$$

Working with this last expression, we get

$$\begin{aligned}
0 &= 2t\sqrt{1+t}(\lambda-1)^2 + 6(1+t)(\lambda-1)\left(\lambda - \frac{1}{3}\right) + 3t(\lambda-1)\left(\lambda - \frac{1}{3}\right) \\
&\quad + (3\lambda-1)^2 + 15\sqrt{1+t}\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right)
\end{aligned}$$

and hence, rearranging terms,

$$\sqrt{1+t}\left[15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right)+2t(\lambda-1)^2\right]$$
$$=-9t(\lambda-1)\left(\lambda-\frac{1}{3}\right)-15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right). \tag{6.5}$$

If $\lambda=1$, we obtain

$$\sqrt{1+t}+1=0$$

and so, in particular, we have $\lambda\neq 1$. Equation (6.5) has two solutions,

$$t_1(\lambda)=\frac{-1+2\lambda+3\lambda^2}{(\lambda-1)^2}\quad\text{and}\quad t_2(\lambda)=\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2}.$$

Using equation (6.2), we may see

$$\tan\theta=\frac{\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda-1}{\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda-\left(1+\frac{1}{\sqrt{1+t}}\right)}.$$

In particular, evaluating in $t_1(\lambda)$ we obtain

$$\tan\theta_1=\frac{\left(1+\frac{1-\lambda}{2\lambda}\right)\lambda-1}{\left(1+\frac{1-\lambda}{\lambda}\right)\lambda-\left(1+\frac{1-\lambda}{2\lambda}\right)}=\lambda,$$

in which case we have

$$D_{1,1}(\lambda):=|f_\lambda(t_1,\theta_1)|=\left|-\sqrt{1+\lambda^2}\right|=\sqrt{1+\lambda^2}.$$

Regarding $t_2(\lambda)$, we obtain

$$\tan\theta_2=\frac{\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-1}{\left(1+2\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)}.$$

Since $\theta_2\in\left(0,\frac{\pi}{4}\right)$, we need to guarantee $0<\tan\theta_2<1$, and for this we need $0<\lambda<\frac{1}{5}$. Therefore

$$\tan\theta_2=\frac{5\lambda-1}{7\lambda-3}$$

and in this case,

$$\begin{aligned}
&D_{1,2}(\lambda) := |f_\lambda(t_2,\theta_2)| \\
&= \Big|\Big[\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2} - \Big(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{3\lambda-1}{\lambda-1}\Big)\lambda\Big]\frac{3-7\lambda}{\sqrt{74\lambda^2-52\lambda+10}} \\
&\quad + \Big[\Big(3+\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{6\lambda-2}{\lambda-1}\Big)\lambda - \Big(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2} + \frac{3\lambda-1}{\lambda-1}\Big)\Big] \\
&\qquad\qquad \times \frac{1-5\lambda}{\sqrt{74\lambda^2-52\lambda+10}}\Big| \\
&= \Big| - \frac{78\lambda^4-208\lambda^3+196\lambda^2-80\lambda+14}{4(\lambda-1)^2\sqrt{74\lambda^2-52\lambda+10}}\Big| \\
&= \Big| - \frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}\Big| \\
&= \frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}.
\end{aligned}$$

(2) $\theta = 0, -1 \le t \le 1$.

We have

$$f_\lambda(t,0) = t - \left(1+t+2\sqrt{1+t}\right)\lambda.$$

Then,

$$\begin{aligned}
f_\lambda(-1,0) &= -1, \\
f_\lambda(1,0) &= 1 - 2\left(1+\sqrt{2}\right)\lambda,
\end{aligned}$$

and hence

$$|f_\lambda(1,0)| = \begin{cases} 1-2(1+\sqrt{2})\lambda & \text{if } 0 \le \lambda < \frac{\sqrt{2}-1}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{\sqrt{2}-1}{2} \le \lambda \le 1. \end{cases}$$

Working now on $(-1,1)$, since

$$f'_\lambda(t,0) = 1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda,$$

the critical point of $f_\lambda(t,0)$ is

$$t = \frac{\lambda^2}{(1-\lambda)^2} - 1.$$

Recall that we need to make sure that $-1 < t < 1$. Therefore, in this case we also need to ask

$$\lambda < \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2}.$$

Plugging the critical point of $f_\lambda(t,0)$ into $f_\lambda(t,0)$, we obtain

$$f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2} - 1, 0\right) = \frac{\lambda^2}{(\lambda-1)^2} - 1 - \left[\frac{\lambda^2}{(\lambda-1)^2} + \frac{2\lambda}{1-\lambda}\right]\lambda = \frac{\lambda^2}{\lambda-1} - 1,$$

and hence

$$\left|f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2}-1,0\right)\right| = 1+\frac{\lambda^2}{1-\lambda}.$$

- Assume first $0 \le \lambda < \frac{\sqrt{2}-1}{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)| = \max\left\{1,\, 1-2\left(1+\sqrt{2}\right)\lambda,\, 1+\frac{\lambda^2}{1-\lambda}\right\} = 1+\frac{\lambda^2}{1-\lambda}.$$

- Assume now $\frac{\sqrt{2}-1}{2} \le \lambda < 2-\sqrt{2}$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)| = \max\left\{1,\, 2\left(1+\sqrt{2}\right)\lambda-1,\, 1+\frac{\lambda^2}{1-\lambda}\right\} = 1+\frac{\lambda^2}{1-\lambda}.$$

- Assume finally $2-\sqrt{2} \le \lambda \le 1$. Then,

$$\sup_{-1\le t\le 1}|f_\lambda(t,0)| = \max\left\{1,\, 2\left(1+\sqrt{2}\right)\lambda-1\right\} = 2\left(1+\sqrt{2}\right)\lambda-1.$$

Thus, in conclusion,

$$\begin{aligned}\sup_{-1\le t\le 1}|f_\lambda(t,0)| &= \begin{cases} 1+\frac{\lambda^2}{1-\lambda} & \text{if } 0\le\lambda<2-\sqrt{2},\\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\\ &=: \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2},\\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1.\end{cases}\end{aligned}$$

(3) $\theta = \frac{\pi}{4}$ and $-1 \le t \le 1$.

We have

$$\begin{aligned}f_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[t-\left(1+t+2\sqrt{1+t}\right)\lambda+\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)\right]\\ &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{1+t}\right)\lambda-\left(1+2\sqrt{1+t}\right)\right].\end{aligned}$$

Again, we have

$$\begin{aligned}f_\lambda\left(-1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left(3\lambda-1\right),\\ f_\lambda\left(1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right],\\ f'_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\frac{\lambda}{\sqrt{1+t}}-\frac{1}{\sqrt{1+t}}\right].\end{aligned}$$

and $f'_\lambda(t,\frac{\pi}{4}) = 0$ implies $\lambda = 1$ (in which case $f_\lambda(t,\frac{\pi}{4}) = \sqrt{2}$ for every t).

- Assume first $0 \leq \lambda < \frac{1}{3}$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda\left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max\left\{\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda,\ 1-3\lambda\right\}$$
$$= \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right]$$

- Assume now $\frac{1}{3} \leq \lambda < 4\sqrt{2}-5$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda\left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max\left\{\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda,\ 3\lambda-1\right\}$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } \frac{1}{3} \leq \lambda < \frac{2\sqrt{2}+1}{7}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{2\sqrt{2}+1}{7} \leq \lambda < 4\sqrt{2}-5. \end{cases}$$

- Assume finally $4\sqrt{2}-5 \leq \lambda \leq 1$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda\left(t, \frac{\pi}{4}\right)| = \frac{\sqrt{2}}{2} \max\left\{3\lambda-1,\ \left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right\} = \frac{\sqrt{2}}{2}(3\lambda-1).$$

Hence, we can say that

$$\begin{aligned} \sup_{-1\leq t\leq 1} |f_\lambda\left(t, \tfrac{\pi}{4}\right)| &= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2} - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}+1}{7} \\ \frac{\sqrt{2}}{2}\left(3\lambda-1\right) & \text{if } \frac{2\sqrt{2}+1}{7} \leq \lambda \leq 1. \end{cases} \\ &=: \begin{cases} D_{3,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}+1}{7} \\ D_{3,2}(\lambda) & \text{if } \frac{2\sqrt{2}+1}{7} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

(4) $t = -1$, $0 \leq \theta \leq \frac{\pi}{4}$.

Applying lemma 6.5, we obtain

$$\begin{aligned} \sup_{0\leq\theta\leq\frac{\pi}{4}} f_\lambda(-1,\theta) &= \begin{cases} 1 & \text{if } 0 \leq \lambda < \frac{1+\sqrt{2}}{3}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{1+\sqrt{2}}{3} \leq \lambda \leq 1. \end{cases} \\ &=: \begin{cases} D_{4,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{1+\sqrt{2}}{3}, \\ D_{4,2}(\lambda) & \text{if } \frac{1+\sqrt{2}}{3} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

(5) $t = 1$, $0 \leq \theta \leq \frac{\pi}{4}$.

We use again lemma 6.5, with $a = 1 - \left(2+2\sqrt{2}\right)\lambda$ and $b = \left(5+4\sqrt{2}\right)\lambda - \left(2+2\sqrt{2}\right)$. Through standard calculations, we see that $\frac{b}{a} < 0$ if and only if $\lambda \in \left[0, \frac{\sqrt{2}-1}{2}\right) \cup \left(\frac{6-2\sqrt{2}}{7}, 1\right]$

and $\frac{b}{a} > 1$ if and only if $\frac{\sqrt{2}-1}{2} < \lambda < \frac{3+4\sqrt{2}}{23}$. Therefore,

$$\begin{aligned}
&\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)| \\
&= \begin{cases} \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23}, \\ \sqrt{\left(1-\left(2+2\sqrt{2}\right)\lambda\right)^2+\left(\left(5+4\sqrt{2}\right)\lambda-\left(2+2\sqrt{2}\right)\right)^2} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7}, \\ \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}
\end{aligned}$$

Since $0 \le \lambda < \sqrt{2}-1$ implies $\left|1-\left(2+2\sqrt{2}\right)\lambda\right| < \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|$, it follows that

$$\begin{aligned}
&\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)| \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right| & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left|1-\left(2+2\sqrt{2}\right)\lambda\right| & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1 \end{cases} \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases} \\
&=: \begin{cases} D_{5,1}(\lambda) & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ D_{5,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ D_{5,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}
\end{aligned}$$

Since (see Figures 6.5 and 6.6)

$$\begin{aligned}
&D_{1,1}(\lambda) \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1, \end{cases} \\
&D_{1,2}(\lambda) \le D_{3,1}(\lambda) \text{ for } 0<\lambda<\tfrac{1}{5},
\end{aligned}$$

we can rule out case (1). Since

$$\begin{aligned}
D_{3,1}(\lambda) &= D_{5,1}(\lambda) \quad \text{for } 0\le\lambda\le\tfrac{3+4\sqrt{2}}{23}, \\
D_{3,2}(\lambda) &= D_{4,2}(\lambda) \quad \text{for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,
\end{aligned}$$

we can directly rule out case (3). Since (see Figures 6.5 and 6.7)

$$\begin{aligned}
&D_{4,1}(\lambda) = 1 \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda<\frac{1+\sqrt{2}}{3}, \end{cases} \\
&D_{4,2}(\lambda) \le D_{2,2} \text{ for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,
\end{aligned}$$

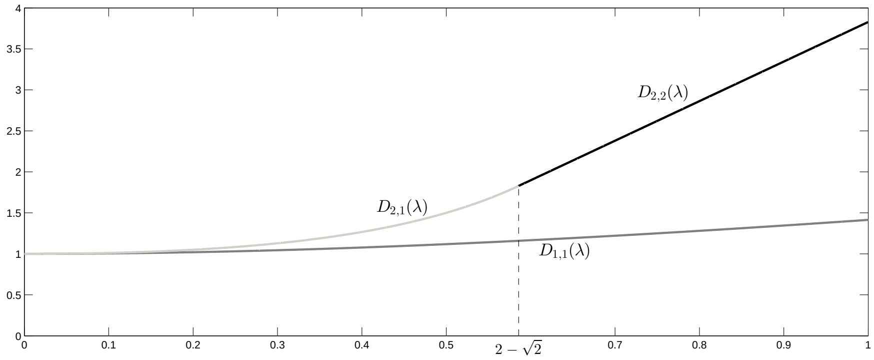

Figure 6.5: Graphs of the mappings $D_{1,1}(\lambda)$, $D_{2,1}(\lambda)$ and $D_{2,2}(\lambda)$.

we can rule out case (4). Finally, since (see Figure 6.8)

$$\begin{aligned} D_{5,2}(\lambda) &\leq D_{2,1}(\lambda) \quad \text{for } \tfrac{3+4\sqrt{2}}{23} \leq \lambda < \tfrac{6-2\sqrt{2}}{7}, \\ D_{5,3}(\lambda) &= D_{2,2}(\lambda) \quad \text{for } 2-\sqrt{2} \leq \lambda \leq 1, \end{aligned}$$

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of case (5).

Thus, putting all the above cases together, we may reach the conclusion

$$\begin{aligned} &\sup_{(t,\theta)\in C_1} |f_\lambda(t,\theta)| \\ &= \begin{cases} D_{5,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14} \leq \lambda < 2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2} \leq \lambda \leq 1, \end{cases} \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ 1+\frac{\lambda^2}{1-\lambda} & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14} \leq \lambda < 2-\sqrt{2}, \\ \left(2+2\sqrt{2}\right)\lambda - 1 & \text{if } 2-\sqrt{2} \leq \lambda \leq 1, \end{cases} \end{aligned}$$

and hence

$$\begin{aligned} &\sup_{-1\leq t\leq 1} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(t,\theta)\in C_1} |f_\lambda(t,\theta)| \\ &= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x, \end{cases} \end{aligned}$$

assuming in every moment $x \neq 0$ (in order to illustrate the previous step, the reader can take a look at Figure 6.9).

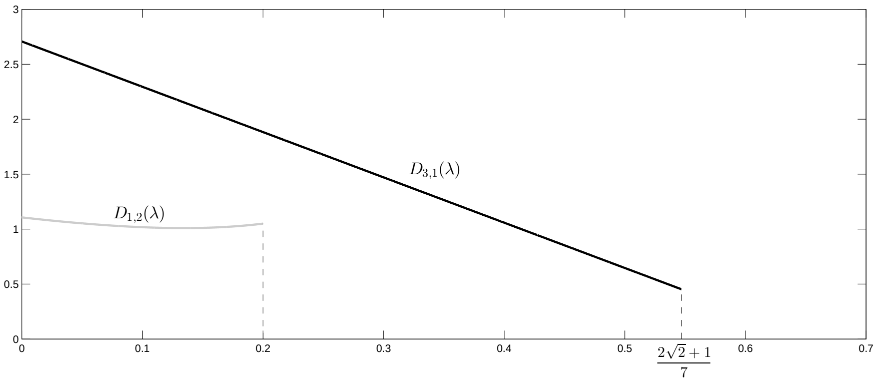

Figure 6.6: Graphs of the mappings $D_{1,2}(\lambda)$ and $D_{3,1}(\lambda)$.

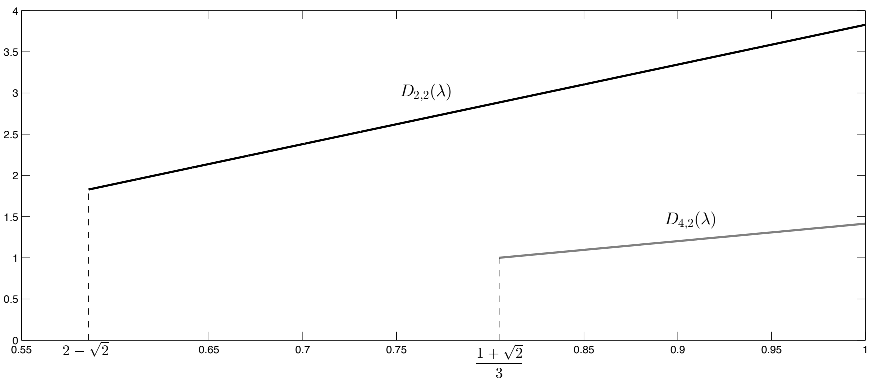

Figure 6.7: Graphs of the mappings $D_{2,2}(\lambda)$ and $D_{4,2}(\lambda)$.

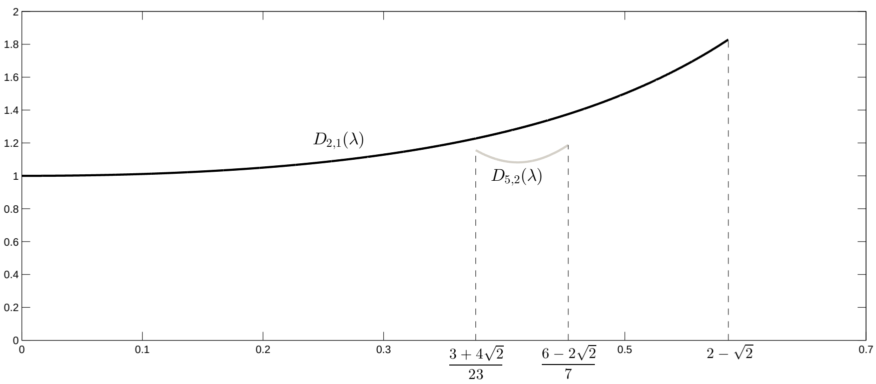

Figure 6.8: Graphs of the mappings $D_{2,1}(\lambda)$ and $D_{5,2}(\lambda)$.

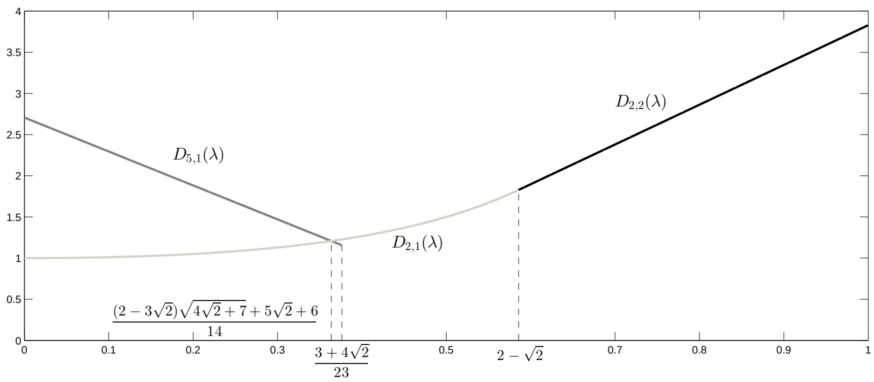

Figure 6.9: Graphs of the mappings $D_{2,1}(\lambda)$, $D_{2,2}(\lambda)$ and $D_{5,1}(\lambda)$.

Let us deal now with the polynomials

$$Q_s(x,y) = x^2 + sy^2 - 2\sqrt{2(1+s)}xy, \quad 1 \le s \le 5+4\sqrt{2}.$$

Then,

$$\begin{aligned}\nabla Q_s(x,y) &= \left(2x - 2\sqrt{2(1+s)}y,\, 2sy - 2\sqrt{2(1+s)}x\right),\\ \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} &= \sup_{0\le\theta\le\frac{\pi}{4}} \left|2x\left[\left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta + \left(s\lambda - \sqrt{2(1+s)}\right)\sin\theta\right]\right|,\end{aligned}$$

and thus

$$\sup_{1\le s\le 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(s,\theta)\in C_2} |g_\lambda(s,\theta)|,$$

with

$$g_\lambda(s,\theta) = \left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta + \left(s\lambda - \sqrt{2(1+s)}\right)\sin\theta$$

and $C_2 = [1, 5+4\sqrt{2}] \times [0, \frac{\pi}{4}]$. Again, we have several cases:

(6) $(s,\theta) \in (1, 5+4\sqrt{2}) \times (0, \frac{\pi}{4})$.

Let us first calculate the critical points of g_λ over C_2.

$$\begin{aligned}\frac{\partial g_\lambda}{\partial s}(s_0,\theta_0) &= \frac{-\lambda}{\sqrt{2(1+s_0)}}\cos\theta_0 + \left(\lambda - \frac{1}{\sqrt{2(1+s_0)}}\right)\sin\theta_0,\\ \frac{\partial g_\lambda}{\partial \theta}(s_0,\theta_0) &= \left(s_0\lambda - \sqrt{2(1+s_0)}\right)\cos\theta_0 - \left(1-\sqrt{2(1+s_0)}\lambda\right)\sin\theta_0,\end{aligned}$$

so, if $Dg_\lambda(s_0,\theta_0) = 0$, using the first expression, we obtain $\tan\theta_0 = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1}$, and, using the second one, we obtain $\tan\theta_0 = \frac{s_0\lambda-\sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda}$.
Hence, we may say

$$\frac{s_0\lambda - \sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda - 1}$$

and thus

$$s_0 = \frac{2-\lambda^2}{\lambda^2}.$$

Then, $\tan\theta_0 = \lambda$ and also, if we want to guarantee that $1 < s_0 < 5+4\sqrt{2}$, we need $\sqrt{2}-1 < \lambda < 1$.
In that case, $\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}$, and then

$$g_\lambda(s_0,\theta_0) = \frac{-1}{\sqrt{1+\lambda^2}} + \frac{-\lambda^2}{\sqrt{1+\lambda^2}} = -\sqrt{1+\lambda^2},$$

so

$$D_6(\lambda) := |g_\lambda(s_0,\theta_0)| = \sqrt{1+\lambda^2}.$$

(7) $s=1$, $0 \le \theta \le \frac{\pi}{4}$.

Apply lemma 6.5 with $a = 1-2\lambda$ and $b = \lambda - 2$. Using $0 \le \lambda \le 1$, observe that we always have $b<0$ and $b \le a$. Also, $a < \left(1-\sqrt{2}\right) b$ if and only if $\lambda > \frac{5-3\sqrt{2}}{7}$.
Putting everything together, we can say

$$\begin{aligned}
\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(1,\theta)| &= \begin{cases} 1-2\lambda & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ \frac{\sqrt{2}}{2}(1+\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1, \end{cases} \\
&=: \begin{cases} D_{7,1}(\lambda) & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ D_{7,2}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1. \end{cases}
\end{aligned}$$

(8) $s = 5+4\sqrt{2}$, $0 \le \theta \le \frac{\pi}{4}$.

Apply again lemma 6.5, this time to $a = 1-2\left(1+\sqrt{2}\right)\lambda$ and $b = \left(5+4\sqrt{2}\right)\lambda - 2\left(1+\sqrt{2}\right)$. As usual, we notice that $a<0$ if and only if $\lambda > \frac{\sqrt{2}-1}{2}$, $b<0$ if and only if $\lambda < \frac{6-2\sqrt{2}}{7}$ and $a<b$ if and only if $\lambda > \frac{3+4\sqrt{2}}{23}$. All together, we can say that, for $\frac{3+4\sqrt{2}}{23} < \lambda < \frac{6-2\sqrt{2}}{7}$, we have

$$\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| = \sqrt{a^2+b^2} = \sqrt{13+8\sqrt{2} - \left(56+40\sqrt{2}\right)\lambda + \left(69+48\sqrt{2}\right)\lambda^2}.$$

Also, notice that, for any $\lambda \in [0,1]$, we are going to have $b < -\left(1+\sqrt{2}\right)a$ and $a < \left(1-\sqrt{2}\right)b$. Hence,

$$\begin{aligned}
&\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{13+8\sqrt{2} - \left(56+40\sqrt{2}\right)\lambda + \left(69+48\sqrt{2}\right)\lambda^2} & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1, \end{cases} \\
&=: \begin{cases} D_{8,1}(\lambda) & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ D_{8,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1. \end{cases}
\end{aligned}$$

(9) $\theta = 0$, $1 \leq s \leq 5 + 4\sqrt{2}$.

We have

$$\begin{aligned} g_\lambda(s,0) &= 1 - \sqrt{2(1+s)}\lambda, \\ g_\lambda(1,0) &= 1 - 2\lambda, \\ g_\lambda(5+4\sqrt{2},0) &= 1 - 2\left(1+\sqrt{2}\right)\lambda, \\ g'_\lambda(s,0) &= -\frac{\lambda}{\sqrt{2(1+s)}} \neq 0 \text{ for } \lambda \neq 0. \end{aligned}$$

Then,

$$\begin{aligned} \sup_{1\leq s\leq 5+4\sqrt{2}} |g_\lambda(s,0)| &= \max\left\{|1-2\lambda|,\, |1-2(1+\sqrt{2})\lambda|\right\} \\ &= \begin{cases} 1-2\lambda & \text{if } 0 \leq \lambda < \frac{2-\sqrt{2}}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{2-\sqrt{2}}{2} \leq \lambda \leq 1, \end{cases} \\ &=: \begin{cases} D_{9,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2-\sqrt{2}}{2}, \\ D_{9,2}(\lambda) & \text{if } \frac{2-\sqrt{2}}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

(10) $\theta = \frac{\pi}{4}$, $1 \leq s \leq 5 + 4\sqrt{2}$.

We have

$$g_\lambda\left(s,\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left[1 + s\lambda - \sqrt{2(1+s)}(1+\lambda)\right].$$

Then

$$\begin{aligned} g_\lambda\left(1,\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}(1+\lambda), \\ g_\lambda\left(5+4\sqrt{2},\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda - \left(1+2\sqrt{2}\right)\right], \\ g'_\lambda\left(s_0,\frac{\pi}{4}\right) &= 0 \text{ if and only if } s_0 = \frac{(1+\lambda)^2}{2\lambda^2} - 1 \end{aligned}$$

and since we need to ensure that $1 < s_0 < 5+4\sqrt{2}$, we need $\frac{2\sqrt{2}-1}{7} < \lambda < 1$. In that case,

$$g_\lambda\left(s_0,\frac{\pi}{4}\right) = -\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda}.$$

Hence,

$$\begin{aligned} \sup_{1\leq s\leq 5+4\sqrt{2}} \left|g_\lambda\left(s.\frac{\pi}{4}\right)\right| &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1, \end{cases} \\ &=: \begin{cases} D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

Since (the reader can take a look at Figure 6.10)

$$D_6(\lambda) \leq \begin{cases} D_{8,2}(\lambda) & \text{if } \sqrt{2}-1 < \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \leq \lambda < 1, \end{cases}$$

we can rule out case (6). Since (see Figures 6.11 and 6.12)

$$D_{7,1}(\lambda) \le D_{10,1}(\lambda) \text{ for } 0 \le \lambda < \tfrac{5-3\sqrt{2}}{7}$$
$$D_{7,2}(\lambda) \le \begin{cases} D_{10,1}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda \le 1, \end{cases}$$

we can rule out case (7). Since

$$D_{8,1}(\lambda) = D_{10,1}(\lambda) \text{ for } 0 \le \lambda < \frac{2\sqrt{2}-1}{7}$$

we can rule out the expression $D_{8,1}(\lambda)$ of case (8). Since

$$D_{9,1}(\lambda) = D_{7,1}(\lambda) \quad \text{for } 0 \le \lambda < \tfrac{5-3\sqrt{2}}{7},$$
$$D_{9,2}(\lambda) = D_{8,3}(\lambda) \quad \text{for } \tfrac{6-2\sqrt{2}}{7} \le \lambda \le 1,$$

we can directly rule out case (9). Furthermore, since (see Figure 6.13)

$$D_{8,2}(\lambda) \le D_{10,2}(\lambda) \text{ for } \tfrac{3+4\sqrt{2}}{23} \le \lambda < \tfrac{6-2\sqrt{2}}{7},$$
$$D_{8,3}(\lambda) \le D_{10,2}(\lambda) \text{ for } \tfrac{6-2\sqrt{2}}{7} \le \lambda \le \tfrac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7},$$

we can conclude that

$$\sup_{(s,\theta)\in C_2} |g_\lambda(s,\theta)|$$
$$= \begin{cases} D_{10,1}(\lambda) & \text{if } 0 \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \le \lambda \le 1. \end{cases}$$
$$= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \le \lambda < \frac{2\sqrt{2}-1}{7}, \\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7} \le \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \le \lambda \le 1, \end{cases}$$

and hence

$$\sup_{1\le s\le 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})}$$
$$= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \le y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \le y < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x \le y \le x. \end{cases}$$

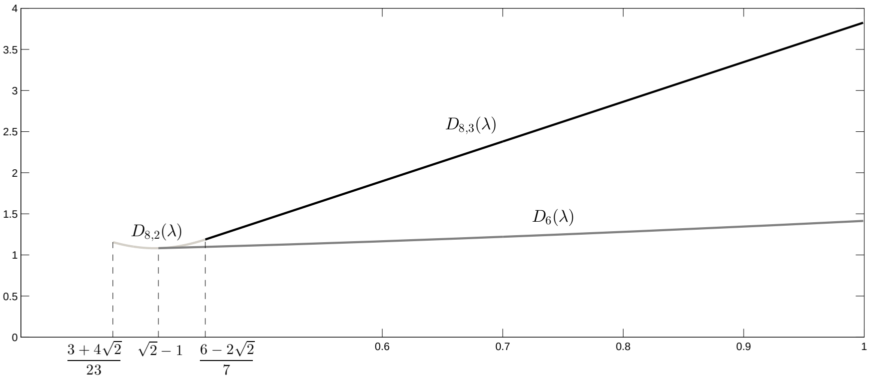

Figure 6.10: Graphs of the mappings $D_6(\lambda)$, $D_{8,2}(\lambda)$ and $D_{8,3}(\lambda)$.

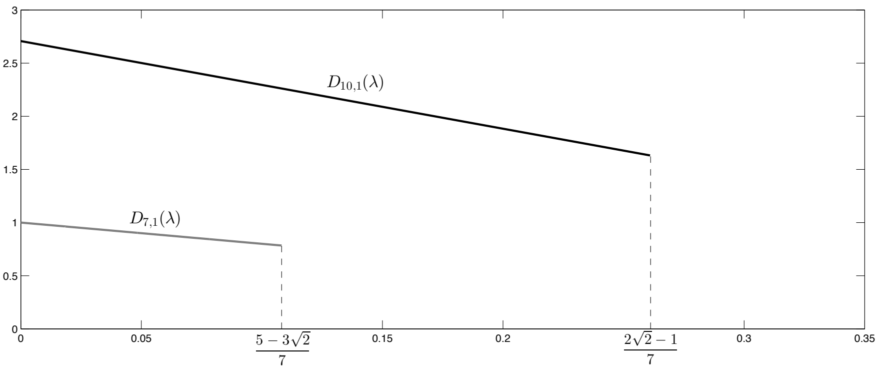

Figure 6.11: Graphs of the mappings $D_{7,1}(\lambda)$ and $D_{10,1}(\lambda)$.

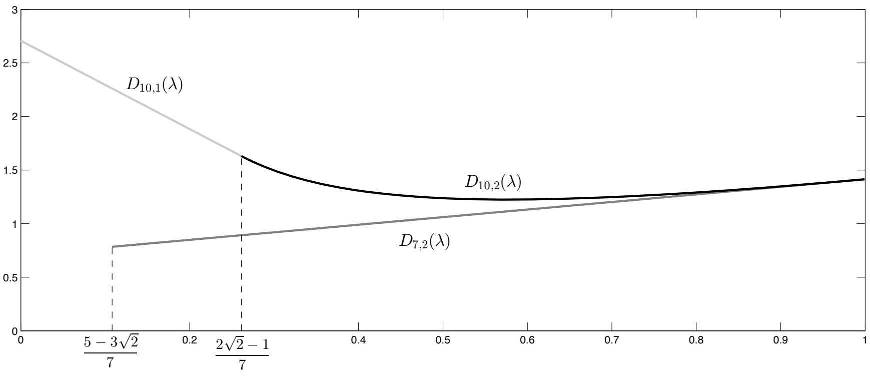

Figure 6.12: Graphs of the mappings $D_{7,2}(\lambda)$, $D_{10,1}(\lambda)$ and $D_{10,2}(\lambda)$.

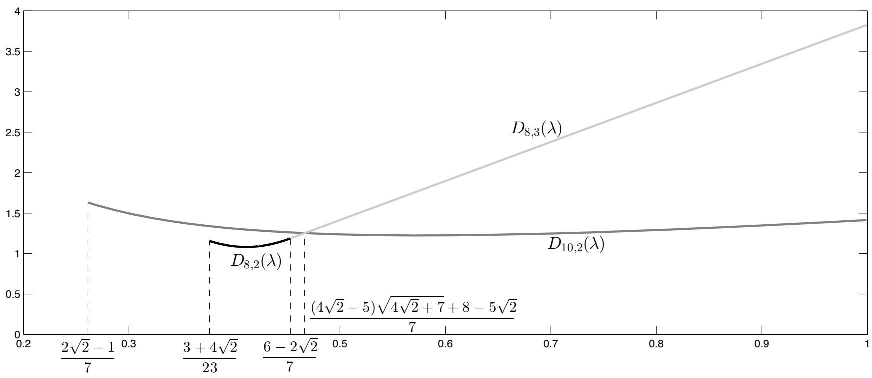

Figure 6.13: Graphs of the mappings $D_{8,2}(\lambda)$, $D_{8,3}(\lambda)$ and $D_{10,2}(\lambda)$.

Finally, if we compare the results obtained with P_t and Q_s, since $\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} \geq 1 + \frac{\lambda^2}{1-\lambda}$ whenever $\lambda \leq \sqrt{2} - 1$, we obtain

$$\Phi(x, y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \left(\sqrt{2}-1\right)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } \left(\sqrt{2}-1\right)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x. \end{cases}$$

□

We can see that $\Phi(x, y) \leq 4 + \sqrt{2}$, for all $(x, y) \in D\left(\frac{\pi}{4}\right)$. Furthermore, the maximum is attained by the polynomials

$$P_1(x, y) = x^2 + \left(5 + 4\sqrt{2}\right)y^2 - \left(4 + 4\sqrt{2}\right)xy = Q_{5+4\sqrt{2}}(x, y).$$

Corollary 6.7. *Let $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and assume $L \in \mathcal{L}^s\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is the polar of P. Then*

$$\|L\|_{D\left(\frac{\pi}{4}\right)} \leq \left(2 + \frac{\sqrt{2}}{2}\right)\|P\|_{D\left(\frac{\pi}{4}\right)}.$$

Moreover, equality is achieved for

$$P_1(x, y) = Q_{5+4\sqrt{2}}(x, y) = x^2 + \left(5 + 4\sqrt{2}\right)y^2 - \left(4 + 4\sqrt{2}\right)xy.$$

Hence, the polarization constant of the polynomial space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $2 + \frac{\sqrt{2}}{2}$.

6.3 Unconditional constants for polynomials on sectors

Here, we obtain a sharp estimate on the norm of the modulus of a polynomial in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ in terms of it norm. That sharp estimate turns out to be the unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Theorem 6.8. *The unconditional constant of the canonical basis of* $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ *is* $5+4\sqrt{2}$. *In other words, the inequality*

$$|||P|||_{D\left(\frac{\pi}{4}\right)} \leq (5+4\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. *Furthermore, the previous inequality is sharp and equality is attained for the polynomials*

$$\pm P_1(x,y) = \pm Q_{5+4\sqrt{2}}(x,y) = \pm\left[x^2 + (5+4\sqrt{2})y^2 - (4+4\sqrt{2})xy\right].$$

Proof. We just need to calculate

$$\sup\left\{|||P|||_{D\left(\frac{\pi}{4}\right)} : \ P \in \operatorname{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\}.$$

In order to calculate the above supremum we use the extreme polynomials described in Lemma 6.2. If we consider first the polynomials P_t, then

$$|P_t| = \left(|t|, 4+t+4\sqrt{1+t}, 2+2t+4\sqrt{1+t}\right).$$

Now, using Lemma 6.1 we have

$$\begin{aligned}\sup_{-1\leq t\leq 1} |||P_t|||_{D\left(\frac{\pi}{4}\right)} &= \sup_{-1\leq t\leq 1} \max\left\{|t|, \frac{1}{2}\left(|t|+4+t+4\sqrt{1+t}+2+2t+4\sqrt{1+t}\right)\right\}\\ &= \sup_{-1\leq t\leq 1} \frac{1}{2}\left(|t|+6+3t+8\sqrt{1+t}\right) = 5+4\sqrt{2}.\end{aligned}$$

Notice that the above supremum is attained at $t=1$. On the other hand, if we consider the polynomials Q_s, we have $|Q_s| = \left(1, s, 2\sqrt{2(1+s)}\right)$. Now, using Lemma 6.1 we have

$$\begin{aligned}\sup_{1\leq s\leq 5+4\sqrt{2}} |||Q_s|||_{D\left(\frac{\pi}{4}\right)} &= \sup_{1\leq s\leq 5+4\sqrt{2}} \max\left\{1, \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right)\right\}\\ &= \sup_{1\leq s\leq 5+4\sqrt{2}} \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right) = 5+4\sqrt{2}.\end{aligned}$$

Observe that the last supremum is now attained at $s = 5+4\sqrt{2}$. □

6.4 Conclusions

Comparing the results obtained in [70] and [94] for polynomials on the simplex Δ, in [69] for polynomials on the unit square □, in [77] for polynomials on the sector $D\left(\frac{\pi}{2}\right)$ and the results obtained in the previous sections, we have the following:

	$\mathcal{P}(^2\Delta)$	$\mathcal{P}\left(^2D\left(\frac{\pi}{2}\right)\right)$	$\mathcal{P}\left(^2D\left(\frac{\pi}{4}\right)\right)$	$\mathcal{P}(^2\square)$
Markov constants	$2\sqrt{10}$	$2\sqrt{5}$	$4(13+8\sqrt{2})$	$\sqrt{13}$
Polarization constants	3	2	$2+\frac{\sqrt{2}}{2}$	$\frac{3}{2}$
Unconditional Constants	2	3	$5+4\sqrt{2}$	5

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

1. $\pm(x^2+y^2-6xy)$ for the simplex.
2. $\pm(x^2+y^2-4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
3. $\pm\left(x^2+(5+4\sqrt{2})y^2-(4+4\sqrt{2})xy\right)$ for the sector $D\left(\frac{\pi}{4}\right)$.
4. $\pm(x^2+y^2-3xy)$ for the unit square.

Compare the previous table with similar results that hold for 2-homogeneous polynomials on the Banach spaces ℓ_1^2, ℓ_2^2 and ℓ_∞^2:

	$\mathcal{P}(^2\ell_1^2)$	$\mathcal{P}(^2\ell_2^2)$	$\mathcal{P}(^2\ell_\infty^2)$
Markov constants	4	2	$2\sqrt{2}$
Polarization constants	2	1	2
Unconditional Constants	$\frac{1+\sqrt{2}}{2}$	$\sqrt{2}$	$1+\sqrt{2}$

Observe that the Markov constants of the spaces $\mathcal{P}(^2\ell_1^2)$ and $\mathcal{P}(^2\ell_\infty^2)$ can be calculated taking into consideration the description of the geometry of those spaces given in [50]. Also, the Markov constant of $\mathcal{P}(^2\ell_2^2)$ is twice its polarization constant, or in other words, 2.
On the other hand, the constants appearing in the second line of the previous table are well-known results (see for instance [121]).
Finally, the unconditional constants corresponding to the third line of the previous table were calculated in Theorem 3.5, Theorem 3.19 and Theorem 3.6 of [70].

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