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Técnicas en análisis lineal (y no lineal) y aplicaciones

Linear (and non-linear) techniques and its applications

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Me gustaría agradecer el presente trabajo a mi director y codirector, que tanta paciencia han mostrado a lo largo de la elaboración del mismo y tantos consejos me han dado para mejorar su contenido, en el fondo y en la forma. Además, su apoyo y sugerencias, aún a pesar la distancia teniendo todo un océano Atlántico entre medias, en los distintos problemas fueron indispensables para poder llegar a buen puerto. También me gustaría agradecerle a mi familia el apoyo mostrado, a mi novia el que se encargara de mi nutrición en los momentos intensos previos al parto de un nuevo resultado y a mis amigos la disponibilidad para escuchar mis elucubraciones. Sin todo ello, este trabajo no habría resultado el mismo.

Contents

Introducción

La presente tesis está centrada en dos temas principales: el primero abarca el primer capítulo y el segundo se divide entre los capítulos dos y tres. En el primer capítulo estudio un problema que apareció como tal hace relativamente poco tiempo (aunque ya en la segunda mitad del pasado siglo se publicaron una serie de resultados que, con la terminología adecuada, estarían englobados dentro de esta teoría). Nos interesaremos en la búsqueda de estructuras algebraicas (como espacios vectoriales, álgebras, espacios de Banach) contenidas en subconjuntos de funciones cuyos elementos (con la posible excepción del elemento nulo) verifican ciertas propiedades anti-intuitivas (propiedades de difícil visualización). Ello nos puede conducir a la idea de cómo la intución puede engañarnos, y sugerir que, aunque se haya dedicado una ingente cantidad de esfuerzo y tiempo para encontrar un único ejemplo que verifique tales propiedades, y dicho trabajo pueda dar la idea de que no existen muchos más especímenes de similares características, de hecho existen ejemplares suficientes como para construir espacios "grandes" cuyos elementos (salvo el cero) satisfacen las mismas propiedades. Más específicamente, decimos que un subconjunto de un espacio vectorial topológico es α-lineable (dado un numero cardinal α) si podemos garantizar la existencia de un espacio vectorial de dimensión α contenido en el conjunto (unión el elemento cero, en caso de que cero no forme parte del conjunto de partida). Si el espacio vectorial es cerrado, nos referiremos a este conjunto como α-espaciable (y la propiedad que trataremos será la de α-espaciabilidad) y si la estructura en cuestión es un álgebra de Banach, entonces diremos que el conjunto es (α, β)-algebrable (donde aquí β es la cardinalidad de un conjunto minimal de generadores del álgebra). Si no se especifica ningún número cardinal, entendemos que la estructura a considerar es simplemente de dimensión infinita. Este nuevo acercamiento al estudio de las funciones y sus propiedades apareció como una teoría independiente a comienzos del presente siglo, en el artículo *Lineability and spaceability of sets of functions on* $\mathbb{R}$ (R.M. Aron, V.I. Gurariy and J.B. Seoane-Sepúlveda, Proc. Amer. Math. Soc., **133** (2005), no. 3, 795–803), y desde su aparición ha demostrado ser un campo de estudio muy fructífero tanto en la cantidad y variedad de resultados como en el interes de los mismos (véanse por ejemplo los resultados recogidos en la bibliografía que acompaña a la tesis, un compendio detallado y exhaustivo que recoge la mayor parte de los resultados publicados hasta 2014 se puede consultar en *Linear subsets of non-linear sets in topological vector spaces* (L. Bernal-González, D. Pellegrino, J.B. Seoane-Sepúlveda, Bull. Amer. Math. Soc. (N-S), **51** (2014), no. 1, 71–130.)).
Los conjuntos que se estudiarán en esta tesis, en cuanto a las propiedades anti-intuitivas en las que nos centraremos, se compondrán de funciones definidas sobre la recta real. Más concretamente, los conjuntos que estudiaremos son los siguientes:

1. Funciones continuas no diferenciables en ningún punto (donde se realiza un uso importante del ejemplo presentado por Weierstrass).

2. Funciones continuas cuya convolución consigo mismas no es diferenciable en ningún punto (destaquemos en este punto que el operador convolución es un operador que suaviza las propiedades de las funciones que intervienen, y, por ejemplo, la convolución de dos funciones de cuadrado integrable aporta una función continua).

3. Funciones continuas sobre $[0, 1]$ diferenciables salvo en un conjunto de medida nula, con derivada nula en todo punto donde la derivada está definida, pero no Lipschitz (recorde-

mos que en caso de tener diferenciabilidad en todo punto, y que la derivada sea acotada, el teorema del Valor Medio nos garantiza Lipschitzianidad).

4. Funciones diferenciables definidas sobre un conexo no convexo y que no verifican el equivalente en $\mathbb{R}^2$ del teorema del Valor Medio.

5. Funciones (de variable real) infinitamente diferenciables pero no analíticas en ningún punto (recordemos que en el caso de variable compleja, las propiedades de diferenciabilidad infinita y analiticidad son equivalentes).

En todos los casos tratados, se consigue demostrar máxima linealidad (es decir, el espacio vectorial que se consigue encontrar es de la máxima dimensión posible). En los conjuntos descritos en 3 y 5 se consiguen encontrar estructuras algebraicas más complejas: en el primer caso se consigue demostrar la existencia de un subespacio vectorial de funciones continuas sobre $[0,1]$ diferenciables salvo en un conjunto de medida nula, con derivada nula en todo punto donde la derivada está definida, pero no Lipschitz, linealmente isométrico a c_0. En el segundo, se demuestra la $(\mathfrak{c},\mathfrak{c})$-algebrabilidad del conjunto al que hacemos referencia en 5.

El segundo tema que se trata en esta tesis y que cubriremos en el resto de la misma estudia distintas normas de polinomios. En el segundo capítulo nos centraremos en la comparación de las normas de un polinomio y su derivada. Siguiendo las directrices del problema clásico que trata las desigualdades de Markov y Bernstein, para los resultados que presentamos en este capítulo segundo estudiaremos polinomios definidos sobre espacios de dimensión finita.

Aunque estos problemas se han estudiado con diligencia en el caso de polinomios sobre espacios de Banach (véanse, por ejemplo, las conocidas generalizaciones de las estimaciones de Markov y Bernstein, o la constante incondicional para un espacio de Banach), usaremos un punto de vista diferente que se empezó a desarrollar en la segunda mitad del siglo XX, y que propone estudiar espacios de polinomios, dotados de una seminorma (el supremo de los valores que toma el polinomio sobre un compacto convexo pero sin simetría central). Con este nuevo acercamiento, la pregunta que surge es si los resultados conocidos para espacios de Banach siguen siendo válidos.

En concreto (y por mor de la completitud) recogemos los resultados ya conocidos que se refieren al triángulo sólido de vértices (0,0), (0,1) y (1,0) (cuyas estimaciones aparecieron publicadas en [78]) y el cuadrado sólido de vértices $(0,0)$, $(0,1)$, $(1,1)$ y $(1,0)$ (publicado en [55]), e incluiremos los resultados (ya originales) cuando se consideran los sectores circulares de amplitud 1 y $\frac{\pi}{2}$. Para estos últimos problemas, conseguimos dar de forma explícita las constantes de polarización (comparativa entre las normas de un polinomio y su polar, esto es, la única forma multilineal simétrica que, restringida a la diagonal, nos permite recuperar el polinomio original), Markov e incondicional y la función de Bernstein. Todos los polinomios tratados están definidos sobre $\mathbb{R}^2$ y son homogéneos de grado 2.

El tríptico formado por el triángulo, el cuadrado y el sector circular de amplitud $\frac{\pi}{2}$ forma un grupo interesante, ya que cada uno de estos cuerpos es la restricción de la bola unidad de un espacio de Banach tremendamente familiar (ℓ_1^2, ℓ_∞^2 y ℓ_2^2, respectivamente) al primer cuadrante. Pondremos el acento en hasta qué punto los resultados ya conocidos para el caso general de los espacios de Banach (especialmente respeco a cotas sobre las constantes estudiadas) siguen siendo válidos y propondremos otras preguntas (no respondidas), en concreto el comportamiento que las constantes correspondientes parecen seguir cuando se considera la bola unidad de un espacio ℓ_p^2 $(1 < p < \infty)$ intersecada con el primer cuadrante.

La tabla donde se recogen los datos a los que se ha hecho referencia unas líneas más arriba se detalla a continuación:

	Δ	$D\left(\frac{\pi}{2}\right)$	$\square$
Constante de Markov	$2\sqrt{10}$	$2\sqrt{5}$	$\sqrt{13}$
Constante de Polarización	3	2	$\frac{3}{2}$
Constante Incondicional	2	3	5

Destacar además que los polinomios óptimos para estas constantes (en el sentido de tornar la desigualdad en igualdad) son los mismos, en cada espacio distinto. Así, tenemos desigualdad (en los tres problemas considerados en la tabla anterior) para los siguientes polinomios:

1. $\pm(x^2 + y^2 - 6xy)$ para el triángulo.
2. $\pm(x^2 + y^2 - 4xy)$ para el sector $D\left(\frac{\pi}{2}\right)$.
3. $\pm(x^2 + y^2 - 3xy)$ para el cuadrado unidad.

El tercer y último capítulo continúa con la comparación entre normas de polinomios. La constante de Bohnenblust-Hille relaciona la norma de un polinomio, vista como función definida sombre un espacio de Banach (de la misma forma que fue considerada en el capítulo anterior, el supremo sobre la bola unidad del espacio), y la norma $\|\cdot\|_p$ de los coeficientes del polinomio (que suele notarse como $|\cdot|_p$).
Más concretamente, el teorema clásico de Bohnenblust y Hille garantiza la existencia de una constante D_m, dependiendo sólamente en el parámetro m, de tal forma que la desigualdad

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|,$$

para todo polinomio P de grado m definido sobre n variables (nótese que la constante de Bohnenblust-Hille no depende de la dimensión n). Recordemos también que estamos usando la notación

$$\|P\| = \sup\{|P(x)| \,:\, \|x\|_{\ell_\infty^n} = 1\}.$$

También hay que destacar que el valor $\frac{2m}{m+1}$ es óptimo, en el sentido que Bohnenblust y Hille probaron que, para $p < \frac{2m}{m+1}$, cualquier constante que consideremos en una desigualdad de tipo $|P|_p \leq C\|P\|$, para todo polinomio P, la constante C tiene que depender por fuerza en la dimensión n.
El problema que estudiaremos en esta tesis es el comportamiento (puntual o asintótico) que las constantes de Bohnenblust-Hille presentan, y la primera propiedad que nos gustaría destacar es que dicho comportamiento difiere bastante dependiendo del cuerpo sobre el que el espacio de Banach está definido. Así, si el cuerpo considerado es el de los números complejos, la desigualdad es a lo sumo subexponencial, esto es, para todo $\epsilon > 0$, existe una constante $C_\epsilon > 0$ de tal forma que $D_m(1+\epsilon)^m$ para todo número natural m.
En este tercer capítulo estudiaremos primero el caso de polinomios definidos sobre los números reales, con coeficientes reales, y estudiaremos si la desigualdad de Bohnenblust-Hille para polinomios reales es subexponencial, y cuál es el crecimiento óptimo. Responderemos a estas preguntas mostrando que, para polinomios definidos sobre los números reales, se obtiene

$$\limsup_m D_m^{1/m} = 2.$$

Aunque el hecho de que los polinomios estén definidos sobre los números reales o los números complejos (con coeficientes pertenecientes a los cuerpos correspondientes) da forma a problemas completamente diferentes, todavía pueden usarse resultados probados para el caso real para obtener ciertas conclusiones en el caso complejo. Así, seremos capaces de demostrar el valor exacto para el caso de polinomios homogéneos de grado 2.

En las últimas secciones de este tercer y último capítulo, seguiremos con la misma estrategia para obtener resultados para polinomios de grados superiores. En este caso, no podemos dar el valor exacto, pero podemos dar cotas superiores, junto con evidencia simbólica de que estas estimaciones deben aproximarse bastante a la constante correspondiente de facto. Así, en estos resultados echaremos mano de cálculo simbólico computacional. Por mor de la completitud de cara a estos resultados, incluiremos los códigos empleados (más una breve explicación de qué ideas subyacen tras los algoritmos empleados) para la representación simbólica del comportamiento simbólico de las constantes de Bohnenblust-Hille (en el caso real). El programa empleado es MatLab.

Introduction

This thesis will be divided into two topics: the first one will cover the first chapter and will deal with a problem that took form little time ago (even though already in the second half of the past century there would be some results). We will be interested on finding algebraic structures (vector spaces, algebras, Banach spaces) contained in subsets of functions whose elements fulfill some anti-intuitive property, union the zero function. Thereby, we can have an idea of how the intuition may mislead us, and hint that, even though we may think that because of having to spend a huge effort in finding one example of such elements we may not find many more, in fact there are enough to consider huge spaces all whose elements except from the zero element satisfy the same property.

More specifically, we define a subset of a topological vector space to be $\alpha-$lineable (for a cardinal number α) if we can find a vector space of dimension α contained in the set (union the zero element, in case it is not included). If the vector space is closed, then we will be talking about $\alpha-$spaceability (and we will say that the set is $\alpha-$spaceable), and if the structure included is a Banach algebra then we will define the set to be $(\alpha, \beta)-$algebrable (where here β would be the cardinality of a minimal set of generators of the algebra). If no cardinal number is defined, then we will assume the structure to be infinite dimensional.

This trend was developed as an independent theory in the end of the last Century, in [5], and since its appearance it has resulted in a fruitful field of study, as the amount of results show (see for example [4], [7], [12], [24], [26] or [54], a very recent and detailed paper giving an exhausting overview of the results published until 2014 can be found in [16]).

The sets that will be considered here when studying those anti-intuitive properties will deal with functions defined over the real line, more concretely results that lie beneath the definition of differentiability (for example the relationship between bounds of the differential and the Lipschitzianity of the function). In particular, we will revisit the famous example given by Weierstrass. There will also be some sections dedicated to the analyticity of real functions and its relation with the infinite differentiability.

The second topic that we gather here and which we will deal with throughout the rest of the Ph. D. studies different norms of polynomials. On the second chapter we will focus on the comparison of the norm of a polynomial, against the norm of its differential. Following a classic problem, the Markov and Bernstein polynomial inequalities, we will be dealing with polynomials defined over a finite dimensional space.

Even though the problem has been exhaustively studied when considering the polynomials defined over Banach spaces (see the well-known Markov and Bernstein estimates or the Unconditional constants for general Banach spaces), we will use a notion that started to develop in the second half of the XXth century and that considers spaces of polynomials defined over a semi-normed space (with still the norm of the polynomial defined as the supremum over some bounded convex set), when this set is not symmetric nor balanced. The question then is to determine until what extend the results that were known for Banach spaces remain true in this new approach.

In concrete, we will gather the already known results when considering the solid triangle of vertices $(0,0)$, $(0,1)$ and $(1,0)$ (published in [78]) and the solid square of vertices $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$ (published in [55]), and we will include the original results when considering the circular sectors with radius 1 and width $\frac{\pi}{4}$ and $\frac{\pi}{2}$. All the vector spaces will be $\mathbb{R}^2$ and all the polynomials will be homogeneous ones of degree 2.

The triplet of unit balls triangle, square and sector of width $\frac{\pi}{2}$ (and explaining why the first two are specifically included) constitutes an interesting group, since each one of those subsets are the unit ball of a very well-known space (ℓ_1^2, ℓ_∞^2 and ℓ_2^2, respectively) restricted to the first quadrant. We will stress the extension to which the already known results for boundedness of the constants (theorems appeared in [67]) still hold and we will remark the unanswered question of how the constants will work when the unit ball is the corresponding unit ball for some ℓ_p^2 space ($1 < p < \infty$, p not equal to 2) intersected with the first quadrant.
The third and last chapter continues with the comparison of norms of polynomials. The Bohnenblust-Hille constant relates the norm of a polynomial considered as a functional over a Banach space and the ℓ_p-norm of the coefficients of the polynomial.
More concretely, the problem follows the search for a constant D_m, depending only on m, such that the inequality

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|,$$

for every P polynomial of degree m defined over n variables, where $|P|_p$ is used to denote the ℓ_p-norm of the coefficients of the polynomial and

$$\|P\| = \sup \left\{ |P(x)| \,:\, \|x\|_{\ell_\infty^n} = 1 \right\}.$$

The estimate must hold uniformly for every n, and the quantity $\frac{2m}{m+1}$ is optimal, in the sense that Bohnenblust and Hille showed that for $p < \frac{2m}{m+1}$, any constant fitting in that inequality depends necessarily on the number of variables.
If the field over which we are considering the problem is the field of complex numbers, the inequality is at most subexponential, that is, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $D_m \leq C_\varepsilon(1+\varepsilon)^m$ for all positive integers m.
In this third chapter, we will study first the case for the real field, and we will study whether the real polynomial Bohnenblust-Hille inequality is subexponential, and what the optimal growth of this inequality is. We will also wonder whether this growth is the optimal we can consider.
We will answer those questions by showing that, for polynomials defined over real numbers,

$$\limsup_m D_m^{1/m} = 2.$$

We shall also use the results obtained for the real case to prove some results for the complex Bohnenblust-Hille constant. In fact we will be able to provide the exact value for the case of homogeneous polynomials of degree 2.
For those last results, some extra help from computer symbolic calculation was required, and in the last pages of this dissertation, we will include the codes used (together with a short explanation of how it works) for the symbolic representation of the asymptotic behaviour of the Bohnenblust-Hille constants (the real case). The program used is MatLab.

Chapter 1

In search for linear structures

1.1 Preliminaries

As certain concepts in Mathematical Analysis were developing and new definitions were being coined, the examples that naturally arose followed certain patterns that fulfilled more requisites than those strictly required. Hence, the continuous functions that appeared in the popular domain were differentiable, everywhere except for at most a finite number of points, or the differentiable functions had to have at least an interval in which they were monotone. It was stated then the question as to whether necessary conditions for certain properties, like continuity for differentiability almost everywhere, were also sufficient. In fact, there were certain works whose aim was to achieve that.

At the end of the XIX century this was answered in the negative: Weierstrass gave an example, in a lecture at the Academy of Sciences in Berlin, of a continuous nowhere differentiable function, even though there were some other authors that suggested different examples before him but did not come with a formal proof of those functions enjoying the desired properties. The case of Weierstrass' function (that came to be popularly known as *Weierstrass' Monster*) shall be more carefully treated in section 1.2.

The idea of *pathological phenomena* for a function began to develop. A property for which finding an element that fulfills it required working with indirect tools, normally without explicit expressions and sometimes with the need of hypotheses that could be rejected by part of the Mathematical Community (e.g., the Axiom of Choice for the Vitali example of a non-measurable set). The feeling was then that there could not be many examples of that kind.

In the second half of the XX century, some authors (when working with infinite dimension vector spaces) started to find large linear structures whose elements (except for, perhaps, the null vector) fulfilled some of these *pathological properties*.

It was in the XXI century when V. I. Gurariy coined the terms that are nowadays employed and which first appeared in [5] (see, also, [58, 93]). For the following list of definitions and notations we refer the reader to [3, 5, 6, 7, 15, 47].

Definition 1.1.1 (Lineability and spaceability)**.** *Let X be a topological vector space and M a subset of X. Let μ be a cardinal number.*

(1) *M is said to be* μ-lineable (μ-spaceable) *if $M \cup \{0\}$ contains a vector space (resp. a closed*

vector space) of dimension μ. At times, we shall be referring to the set M as simply lineable *or* spaceable *if the existing subspace is infinite dimensional.*

(2) *We also let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space.*

(3) *When the above linear space can be chosen to be dense in X we shall say that M is μ-dense-lineable.*

Moreover, Bernal introduced in [15] the notion of *maximal lineable* (and those of *maximal dense-lineable* and *maximal-spaceable*), meaning that, when keeping the above notation, the dimension of the existing linear space equals $\dim(X)$. Besides asking for linear spaces one could also study other structures, such as algebrability and some related ones, which were presented in [3].

Definition 1.1.2. *Given a Banach algebra $\mathcal{A}$, a subset $\mathcal{B} \subset \mathcal{A}$ and two cardinal numbers α and β, we say that:*

(1) *$\mathcal{B}$ is* algebrable *if there is a subalgebra $\mathcal{C}$ of $\mathcal{A}$ so that $\mathcal{C} \subset \mathcal{B} \cup \{0\}$ and the cardinality of any system of generators of $\mathcal{C}$ is infinite.*

(2) *$\mathcal{B}$ is* dense-algebrable *if, in addition, $\mathcal{C}$ can be taken dense in $\mathcal{A}$.*

(3) *$\mathcal{B}$ is* (α, β)-algebrable *if there is an algebra $\mathcal{B}_1$ so that $\mathcal{B}_1 \subset \mathcal{B} \cup \{0\}$, $dim(\mathcal{B}_1) = \alpha$, $card(S) = \beta$, and S is a minimal system of generators of $\mathcal{B}_1$.*[1]

(4) *At times we shall say that $\mathcal{B}$ is, simply, κ-algebrable if there exists a κ-generated subalgebra $\mathcal{C}$ of $\mathcal{A}$ with $\mathcal{C} \subset \mathcal{B} \cup \{0\}$.*

We also say that a subset M of a linear algebra $\mathcal{L}$ is *strongly κ-algebrable* if there exists a κ-generated free algebra $\mathcal{A}$ contained in $M \cup \{0\}$ (see [11]). Other types of structures have also been considered, such as cones or modules[2]. The links between the previous concepts are as follows (in which all the below implications are strict):

(STRONG) ALGEBRABILITY
↓
SPACEABILITY ↘ MODULEABILITY ↓ CONEABILITY ↗
LINEABILITY

For instance, in 1940 B. Levine and D. Milman proved the following illustrating result, which translated into the modern terminology states the following:

Theorem 1.1.3 (B. Levine, D. Milman, [71])**.** *The subset of $C[0,1]$ of all functions of bounded variation is not spaceable.*

Or this other early result, due to V. Gurariy, in 1966 (also stated with this modern terminology):

[1] Here, by *S is a minimal set of generators of an algebra $\mathcal{D}$* we mean that $\mathcal{D} = \mathcal{A}(S)$ is the algebra generated by S, and for every $x_0 \in S$ $x_0 \notin \mathcal{A}(S \setminus \{x_0\})$.

[2] Let L be a subset of a Banach algebra (or a topological algebra) X. We say that L is *moduleable* if there exists an infinitely generated subalgebra M of X and an infinitely generated additive subgroup G of X such that G is a $(M, \mathbb{K})$-bimodule, G is $\mathbb{K}$-infinite dimensional and $L \cup \{0\} \supset G$.

Theorem 1.1.4 (V.I. Gurariy, [53]). *The set of differentiable functions on* $[0,1]$ *is not spaceable in* $(C[0,1]; \|\cdot\|_\infty)$.

This last theorem, for example, was originally published in the following terms:

Theorem 1.1.5 (V. I. Gurariy, [53]). *If all elements of a closed subspace* E *of* $C[0,1]$ *are differentiable on* $[0,1]$*, then* E *is finite-dimensional.*

The following positive result is also due to Gurariy:

Theorem 1.1.6 (V. I. Gurariy, [53]). $C[0,1]$ *has infinite-dimensional closed subspaces consisting of differentiable functions on* $(0,1)$ *(and even analytic on* $(0,1)$*).*

The latter was proved using a result due to A.F. Leont'ev, where he suggested to take the closure (in $C[0,1]$) of the linear hull of a sequence of powers $\{t^{n_k}\}_{k=1}^{\infty}$, where $n_k > 0$ and $\sum_{k=1}^{\infty} \frac{1}{n_k} < \infty$.

Gurariy also suggested the idea that the class of such infinite-dimensional vector subspaces was very narrow:

Theorem 1.1.7 (V. I. Gurariy, [53]). *If every element of a finite-dimensional subspace* E *of* $C[0,1]$ *is differentiable on* $(0,1)$*, then for every* $\epsilon > 0$ *there exists a closed subspace* E_ϵ *of* E *such that*

$$d(E_{\epsilon,c_0}) := \inf\{\|T\| \cdot \|T^{-1}\| \text{ with } T : E_\epsilon \longrightarrow c_0 \text{ is an isomorphism}\} < 1 + \epsilon,$$

which would lead to the following:

Corollary 1.1.8. *If the elements of a reflexive subspace* E *of* $C[0,1]$ *are differentiable on* $(0,1)$*, then* E *is finite-dimensional.*

This previous battery of results became even larger at the beginning of the XX century when, as mentioned above, this terminology first appeared in [5, 93]. Lineability and spaceability have a remarkable influence on many fields of mathematics, from Linear Chaos to Real and Complex Analysis, passing through Set Theory and Linear and Multilinear Algebra, or even Operator Theory, Topology, Measure Theory, Functional Analysis and even in Probability Theory. Recently a survey paper on the topic appeared (see [16]) and even a complete detailed monograph as well, [3].

1.2 Results concerning differentiability of functions

As we stated before, the seed of the conception of the pathological phenomena was introduced with the search for a continuous nowhere differentiable function. The existence of such a mapping was not clear; actually there had been some attemps to prove its non-existence.

Even though Weierstrass was the first author who gave a rigorous example of a function like that, during a lecture delivered at the Berlin Academy on July 18, 1872, there had been some other examples before him:

1. Around the year 1830, Bernard Bolzano had constructed a function that fulfilled those features, but the result remained unpublished until 1922, some years after his manuscripts were discovered.
   Bolzano's idea was to construct a sequence of continuous functions, defined between closed intervals, that converged uniformly.

2. Charles Cellélier proposed the function

$$C(x) = \sum_{k=1}^{\infty} \frac{1}{a^k} \sin(a^k x), \quad \text{with } a > 1000,$$

which was quite similar to that given by Weierstrass and conceived earlier than the year 1860, but was not published until 1890.

3. In his dissertation from 1854, Riemann used the function

$$R(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \sin(k^2 x)$$

when searching for necessary and sufficient conditions for a function in order to have a Fourier series.
Riemann is believed to have used this function as an example, later in his lectures around the year 1861, of a continuous nowhere differentiable function (even though it was proved that R had a finite derivative at the points of the form $\pi\frac{2p+1}{2q+1}$, $p, q \in \mathbb{Z}$). However, he never presented a formal proof.

In this section we study the function presented by Weierstrass, since we shall use it as a pattern to construct a linear vector space of dimension $\mathfrak{c}$ (from now on denoting the continuum) of coninuous nowhere differentiable functions.

The original Weierstrass' Monster was defined as:

$$W_{a,b}(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$ and b is an odd natural number greater than 1. The paper with the proof was published in 1875 by Paul du Bois-Reymond, after some exchanging with Weierstrass (see, e.g., [98]), even though Hardy published in 1916 a paper where more general assumptions in the parameters that appear in the Weierstrass function were considered, namely $0 < a < 1$, $ab \geq 1$ and $b > 1$ ([59]).

For our result, we shall follow the proof as it appears in the following theorem in [96]:

Theorem 1.2.1 (Weierstrass). *The function $W : \mathbb{R} \longrightarrow \mathbb{R}$ given by*

$$W(x) = \sum_{k=0}^{\infty} a^k \cos(b^k \pi x)$$

where $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$ and b is an odd natural number greater than 1, is a continuous nowhere differentiable function on $\mathbb{R}$

Related to this theorem, we shall have the following result:

Theorem 1.2.2 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [66]). *The set of all continuous nowhere differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is $\mathfrak{c}$-lineable.*

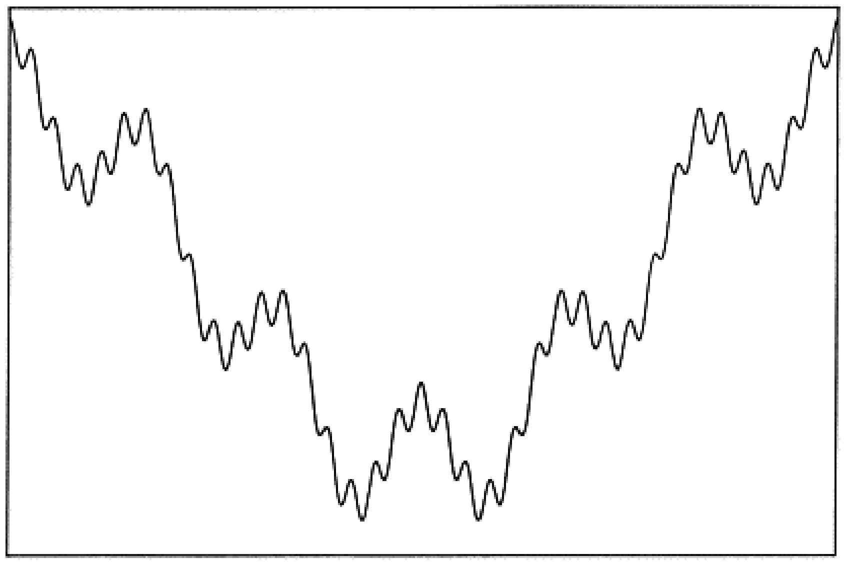


Figure 1.1: A sketch of Weierstrass' monster

Proof. Consider, for $\frac{7}{9} < a < 1$,

$$W_a(x) = \sum_{k=0}^{\infty} a^k \cos(9^k \pi x).$$

We see that each W_a is a Weierstrass function, since $9a > 7 > 1 + \frac{3}{2}\pi$.
We need to show that $\{W_a : \frac{7}{9} < a < 1\}$ is a linearly independent system that spans a space of continuous nowhere differentiable functions (with the exception of the zero element). Indeed, let $\frac{7}{9} < a_1 < a_2 < \ldots < a_l < 1$, $\alpha_1, \ldots, \alpha_l \in \mathbb{R}$, define $g(x) = \sum_{i=1}^{l} \alpha_i W_{a_i}(x)$ and assume $g = 0$. We shall prove by induction that

$$\sum_{i=1}^{l} \alpha_i a_i^n = 0 \quad \text{and} \quad \sum_{i=1}^{l} \alpha_i \frac{a_i^{n+1}}{1 - a_i} = 0,$$

for all $n \in \mathbb{N}$, which would prove $\alpha_i = 0$ for all $1 \leq i \leq l$ since we would have a Vandermonde-like determinant.
First, start with $n = 0$. We have

$$W_a(\frac{1}{3}) = \sum_{k=0}^{\infty} a^k \cos(\frac{9^k}{3}\pi) = \cos(\frac{\pi}{3}) + \sum_{k=1}^{\infty} a^k \cos(3^{k-1}3^k\pi) = \cos(\frac{\pi}{3}) - \frac{a}{1-a}$$

for every a with $\frac{7}{9} < a < 1$.
Hence,

$$g(\frac{1}{3}) = \sum_{i=1}^{l} \alpha_i \left(\cos\frac{\pi}{3} - \frac{a_i}{1 - a_i} \right) = \cos\frac{\pi}{3} \left(\sum_{i=1}^{l} \alpha_i \right) - \sum_{i=1}^{l} \frac{\alpha_i a_i}{1 - a_i} = 0.$$

Similarly,

$$g(\frac{1}{9}) = \sum_{i=1}^{l} \alpha_i \left(\cos\frac{\pi}{9} - \frac{a_i}{1-a_i} \right) = \cos\frac{\pi}{9} \left(\sum_{i=1}^{l} \alpha_i \right) - \sum_{i=1}^{l} \frac{\alpha_i a_i}{1-a_i} = 0,$$

which implies

$$\left(\sum_{i=1}^{l} \alpha_i \right) \left(\cos\frac{\pi}{9} - \cos\frac{\pi}{3} \right) = 0$$

from which $\sum_{i=1}^{l} \alpha_i = 0$ and $\sum_{i=1}^{l} \frac{a_i \alpha_i}{1-a_i} = 0$.

Assume now $\sum_{i=1}^{l} \alpha_i a_i^n = 0$ and $\sum_{i=1}^{l} \alpha_i \frac{a_i^{n+1}}{1-a_i} = 0$, for all $0 \leq n \leq m$.

Then,

$$\begin{aligned} W_{a_i}(\frac{1}{9^{m+2}}) &= \sum_{k=0}^{\infty} a_i^k \cos\frac{9^k \pi}{9^{m+2}} \\ &= \cos\frac{\pi}{9^{m+2}} + a_i \cos\frac{\pi}{9^{m+1}} + \ldots + a_i^{m+1} \cos\frac{\pi}{9} - \frac{a_i^{m+2}}{1-a_i}. \end{aligned}$$

Hence, using the induction hypothesis,

$$\begin{aligned} g(\frac{1}{9^{m+2}}) &= \sum_{i=1}^{l} \alpha_i W_{a_i}(\frac{1}{9^{m+2}}) \\ &= \sum_{i=1}^{l} \alpha_i \left[\cos\frac{\pi}{9^{m+2}} + a_i \cos\frac{\pi}{9^{m+1}} + \ldots + a_i^{m+1} \cos\frac{\pi}{9} - \frac{a_i^{m+2}}{1-a_i} \right] \\ &= \left(\sum_{i=1}^{l} \alpha_i a_i^{m+1} \right) \cos\frac{\pi}{9} - \sum_{i=1}^{l} \alpha_i \frac{a_i^{m+2}}{1-a_i} = 0, \end{aligned} \tag{1.2.1}$$

which allows us to conclude, using the induction hypothesis, that

$$\left(\sum_{i=1}^{l} \alpha_i a_i^{m+1} \right) \cos\frac{\pi}{9} - \sum_{i=1}^{l} \alpha_i \frac{a_i^{m+2}}{1-a_i} + \sum_{i=1}^{l} \alpha_i \frac{a_i^{m+1}}{1-a_i} = 0$$

and then

$$\left(\sum_{i=1}^{l} \alpha_i a_i^{m+1} \right) \cos\frac{\pi}{9} - \sum_{i=1}^{l} \alpha_i \frac{a_i^{m+1}(a_i - 1)}{1-a_i} = \left(\sum_{i=1}^{l} \alpha_i a_i^{m+1} \right) (\cos\frac{\pi}{9} + 1) = 0.$$

Using this last result we have $\sum_{i=1}^{l} \alpha_i a_i^{m+1} = 0$, which, together with the conclusion in (1.2.1), yields $\sum_{i=1}^{l} \alpha_i \frac{a_i^{m+2}}{1-a_i} = 0$. This proves the linear independency of the W_a's.

Assume now that $\alpha_1 \cdot \ldots \cdot \alpha_l \neq 0$, $\frac{7}{9} < a_l < a_{l-1} < \ldots < a_1 < 1$ and $g(x) := \sum_{i=1}^{l} \alpha_i W_{a_i}(x)$ is differentiable at $x_0 \in \mathbb{R}$.

Then, following the proof of Theorem 1.2.1, we have that for each $m \in \mathbb{N}$ and each $1 \leq i \leq l$ there exist $\epsilon_{1,m}^i$, $\epsilon_{1,m}^i \in [-1,1]$ and there exists $\eta_m^i \geq 1$ such that

$$g'(x_0) = \lim_{m\to\infty}\left[(-1)^{a_m}\sum_{i=1}^{l}\alpha_i(9a_i)^m\left(\frac{\epsilon_{1,m}^i\pi}{9a_i-1}+\eta_m^i\frac{2}{3}\right)\right]$$
$$= \lim_{m\to\infty}\left[-(-1)^{a_m}\sum_{i=1}^{l}\alpha_i(9a_i)^m\left(\frac{\epsilon_{2,m}^i\pi}{9a_i-1}+\eta_m^i\frac{2}{3}\right)\right]$$

and putting both limits together we obtain

$$\lim_{m\to\infty}\left[\sum_{i=1}^{l}\alpha_i(9a_i)^m\left(\frac{\pi}{9a_i-1}(\epsilon_{1,m}^i+\epsilon_{2,m}^i)+\frac{4}{3}\eta_m^i\right)\right]=0.$$

In other words,

$$\lim_{m\to\infty}\left[(9a_1)^m\sum_{i=1}^{l}\alpha_i\left(\frac{a_i}{a_1}\right)^m\left(\frac{\pi}{9a_i-1}(\epsilon_{1,m}^i+\epsilon_{2,m}^i)+\frac{4}{3}\eta_m^i\right)\right]=0. \tag{1.2.2}$$

Now, following again the steps of the proof of Theorem 1.2.1 where η_m^i is to appear, and keeping in mind that $1+x_{m+1}\geq\frac{1}{2}$, then we would conclude

$$\eta_m^i\frac{4}{3} = 2\sum_{k=0}^{\infty}a_i^k\frac{1+\cos(9^k\pi x_{m+1})}{1+x_{m+1}}$$
$$\leq 4\sum_{k=0}^{\infty}a_i^k[1+\cos(9^k\pi x_{m+1})] \leq 8\frac{1}{1-a_i}\leq\frac{8}{1-a_1}<\infty.$$

Hence, by the conclusion in (1.2.2) and the latter, $\frac{\pi}{9a_i-1}(\epsilon_{1,m}^i+\epsilon_{2,m}^i)+\frac{4}{3}\eta_m^i$ is bounded above and below (by a strictly positive quantity), for all $1\leq i\leq l$ and, since $0<a_i<a_1$ for all $2\leq i\leq l$ and $9a_1>1$, we get that

$$\lim_{m\to\infty}\left[(9a_1)^m\sum_{i=1}^{l}\alpha_i\left(\frac{a_i}{a_1}\right)^m\left(\frac{\pi}{9a_i-1}(\epsilon_{1,m}^i+\epsilon_{2,m}^i)+\frac{4}{3}\eta_m^i\right)\right]=\operatorname{sign}(\alpha_1)\cdot\infty,$$

contradicting the conclusion that the upper limit is null.

□

Remark 1.2.3. *The latter is an already known result: V. I. Gurariy gave first (see, e.g., [53]) a non-constructive proof of the* $\aleph_0$*-lineability of the set of continuous nowhere differentiable functions, and V.P. Font, V.I. Gurariy and M.I. Kadets gave in [50] an example of an infinite dimensional closed subspace of this set (in particular it showed* $\mathfrak{c}$*-lineability), that is, that the subset of* $C[0,1]$ *of nowhere differentiable functions is spaceable. At the end of the XX century much more was discovered about this set, and L. Rodríguez-Piazza showed in [88] that the previous subspace could be chosen so that it was isometrically isomorphic to any separable Banach space.*

It is interesting to observe that Theorem 1.2.2 is the first constructive proof of the $\mathfrak{c}$*-lineability of the space of all continuous and nowhere differentiable functions.*

The nowhere differentiability of the function proposed by Weierstrass (while being continuous) was the most shocking property that it would hold, but it also fulfills some more pathological properties.

Definition 1.2.4. *Let f and g be two functions belonging to $L^1(\mathbb{T})$ (that is, periodic functions of period 1 and such that they are integrable over any interval of length 1). We define the convolution of f and g as the function*

$$f * g(x) = \int_{[0,1]} f(y)g(x-y)dy = \int_{[0,1]} f(x-y)g(y)dy.$$

Basically, the operator convolution is an operator that takes the "smoothest" properties from both convoluted functions. For example, convoluting an absolutely integrable function with a continuously differentiable function arises a continuously differentiable function. Also, applying Hölder's inequality, we may prove the following:

Proposition 1.2.5. *If $f \in L^p(\mathbb{T})$ and $g \in L^{p'}(\mathbb{T})$ (where p' stand for the conjugate exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$), then $f * g$ is continuous. Furthermore, its Fourier series converges uniformly to it.*

In particular, the convolution of two nowhere continuous functions may be continuous. One might ask him/herself if we could have a similar result concerning differentiability. However, that is not the case with the Weierstrass Monster. To show it, we will deal with a slightly different definition of convolution (but that shall have the smoothness properties as well).

Definition 1.2.6. *For $f, g \in L^1(\mathbb{T})$, we shall define the Voltera convolution of f and g as follows:*

$$f *_V g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

We have in particular the following theorems:

Theorem 1.2.7 (P. Jiménez-Rodríguez, S. Maghsoudi, G.A. Muñoz-Fernández, [62])**.** *For $\frac{7}{9} < a_1, a_2 < 1$, we have that $W_{a_1} *_V W_{a_2}$ is nowhere differentiable.*

Proof. First of all, let us denote

$$y_{a,k}(t) = a^k \cos(9^k \pi t).$$

Now we can see that

$$|y_{a,k}(s)| \leq a^k \quad \text{and} \quad |y_{a,k}(s) - y_{a,k}(t)| \leq (9a)^k \pi.$$

Let now $t_0 > 0$ and, for every $n \in \mathbb{N}$, find an even number p_n such that

$$r_n := \frac{p_n}{9^n} \leq t_0 < \frac{p_n + 2}{9^n}$$

and define also $s_n := \frac{p_n+3}{9^n}$.

Then, for $k \geq n$

$$
\begin{aligned}
y_{a_1,k} *_V y_{a_2,k}(r_n) &= \int_0^{r_n} a_1^k \cos(9^k\pi\tau) a_2^k \cos(9^k\pi(r_n-\tau))d\tau \\
&= (a_1a_2)^k \int_0^{r_n} \cos(9^k\pi\tau)\left[\cos(9^k\pi r_n)\cos(9^k\pi\tau) + \sin(9^k\pi r_n)\sin(9^k\pi\tau)\right]d\tau \\
&= (a_1a_2)^k \int_0^{r_n} \cos^2(9^k\pi\tau)d\tau = \frac{(a_1a_2)^k}{2}\left[\tau + \frac{\sin(9^k\pi\tau)}{9^k\pi}\right]_{\tau=0}^{\tau=r_n} \\
&= \frac{(a_1a_2)^k}{2} r_n \geq \frac{(a_1a_2)^k}{2}\left(t_0 - \frac{2}{9^n}\right).
\end{aligned}
$$

Similarly, we get

$$
y_{a_1,k} *_V y_{a_2,k}(s_n) = \frac{-(a_1a_2)^k}{2} s_n \leq \frac{-(a_1a_2)^k}{2} t_0,
$$

from which

$$
\frac{y_{a_1,k} *_V y_{a_2,k}(s_n) - y_{a_1,n} *_V y_{a_2,n}(r_n)}{s_n - r_n} \leq -\frac{(a_1a_2)^k}{3}(9^n t_0 - 1),
$$

and hence

$$
\sum_{k\geq n} \frac{y_{a_1,k} *_V y_{a_2,k}(s_n) - y_{a_1,n} *_V y_{a_2,n}(r_n)}{s_n - r_n} \leq -\frac{9^n t_0 - 1}{3} \cdot \frac{(a_1a_2)^n}{1 - a_1a_2}.
$$

Then, there exists $\eta_n(a_1, a_2) \leq -1$ such that

$$
\sum_{k\geq n} \frac{y_{a_1,k} *_V y_{a_2,k}(s_n) - y_{a_1,n} *_V y_{a_2,n}(r_n)}{s_n - r_n} = \eta_n(a_1, a_2)\frac{9^n t_0 - 1}{3} \cdot \frac{(a_1a_2)^n}{1 - a_1a_2}. \tag{1.2.3}
$$

Let now $k \neq m \geq 0$. Then,

$$
\begin{aligned}
y_{a_1,k} *_V y_{a_2,m}(s_n) &= a_1^k a_2^m \int_0^{s_n} \cos(9^k\pi s_n\tau)[\cos(9^m\pi s_n)\cos(9^m\pi\tau) + \sin(9^m\pi s_n)\sin(9^m\pi\tau)]d\tau \\
&= \frac{a_1^k a_2^m}{2}\Bigg[\cos(9^m\pi s_n)\left(\frac{\sin[(9^k+9^m)\pi\tau]}{(9^k+9^m)\pi} + \frac{\sin[(9^k-9^m)\pi\tau]}{(9^k-9^m)\pi}\right)_0^{s_n} \\
&\quad - \sin(9^m\pi s_n)\left(\frac{\cos[(9^k+9^m)\pi\tau]}{(9^k+9^m)\pi} - \frac{\cos[(9^k-9^m)\pi\tau]}{(9^k-9^m)\pi}\right)_0^{s_n}\Bigg] \\
&= \frac{a_1^k a_2^m}{2\pi}\Bigg[\frac{1}{9^k+9^m}\left(\cos(9^m\pi s_n)\sin[(9^k+9^m)\pi s_n] - \sin(9^m\pi s_n)\cos[(9^k+9^m)\pi s_n]\right) \\
&\quad + \frac{1}{9^k-9^m}\left(\cos(9^m\pi s_n)\sin[(9^k-9^m)\pi s_n] + \sin(9^m\pi s_n)\cos[(9^k-9^m)\pi s_n]\right) \\
&\quad - \frac{2\cdot 9^m \sin(9^m\pi s_n)}{9^{2k}-9^{2m}}\Bigg] \\
&= \frac{a_1^k a_2^m}{2\pi}\left(\frac{1}{9^k+9^m}\sin(9^k\pi s_n) + \frac{1}{9^k-9^m}\sin(9^k\pi s_n) - \frac{2\cdot 9^m\sin(9^m\pi s_n)}{9^{2k}-9^{2m}}\right).
\end{aligned}
$$

Hence, we can put, for $m \neq k$,

$$\begin{aligned}&y_{a_1,k} *_V y_{a_2,m}(s_n) - y_{a_1,k} *_V y_{a_2,m}(r_n)\\&= \frac{a_1^k a_2^m}{2\pi}\left[\frac{2\cdot 9^k}{9^{2k}-9^{2m}}\left(\sin(9^k\pi s_n)-\sin(9^k\pi r_n)\right)+\frac{2\cdot 9^m}{9^{2k}-9^{2m}}\left(\sin(9^m\pi r_n)-\sin(9^m\pi s_n)\right)\right].\end{aligned}$$

In a similar way (for $0 \le k \le n-1$),

$$\begin{aligned}y_{a_1,k} *_V y_{a_2,k}(s_n) &= (a_1a_2)^k\int_0^{s_n}\cos(9^k\pi\tau)[\cos(9^k\pi s_n)\cos(9^k\pi\tau)+\sin(9^k\pi s_n)\sin(9^k\pi\tau)]d\tau\\&= \frac{(a_1a_2)^k}{2}\left[\cos(9^k\pi s_n)\left(\tau+\frac{\sin(2\cdot 9^k\pi\tau)}{2\cdot 9^k\pi}\right)_0^{s_n}-\sin(9^k\pi s_n)\left(\frac{\cos(2\cdot 9^k\pi\tau)}{2\cdot 9^k\pi}\right)_0^{s_n}\right]\\&= \frac{(a_1a_2)^k}{2}\left[\cos(9^k\pi s_n)\left(s_n+\frac{\sin(2\cdot 9^k\pi s_n)}{2\cdot 9^k\pi}\right)-\sin(9^k\pi s_n)\frac{\cos(2\cdot 9^k\pi s_n)-1}{2\cdot 9^k\pi}\right]\\&= \frac{(a_1a_2)^k}{2}\left(\frac{\sin(9^k\pi s_n)}{9^k\pi}+s_n\cos(9^k\pi s_n)\right),\end{aligned}$$

and hence we can write (reaching the analogous expression for $y_{a_1,k} * y_{a_2,k}(r_n)$):

$$y_{a_1,k}*_V y_{a_2,k}(s_n)-y_{a_1,k}*_V y_{a_2,k}(r_n) = \frac{(a_1a_2)^k}{2}\left[\frac{\sin(9^k\pi s_n)-\sin(9^k\pi r_n)}{9^k\pi}+s_n\cos(9^k\pi s_n)-r_n\cos(9^k\pi r_n)\right].$$

Putting everything together, we can conclude the following inequalities:

1. If $0 \le m \neq k$,

$$\left|\frac{y_{a_1,k} *_V y_{a_2,m}(s_n) - y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}\right| \le a_1^k a_2^m \frac{9^{2k}+9^{2m}}{|9^{2k}-9^{2m}|} \le 2\cdot a_1^k a_2^m. \qquad (1.2.4)$$

2. If $0 \le k \le n-1$,

$$\left|\frac{y_{a_1,k} *_V y_{a_2,k}(s_n) - y_{a_1,k} *_V y_{a_2,k}(r_n)}{s_n-r_n}\right| \le \frac{(a_1a_2)^k}{2}\left[2+9^k\pi\left(t_0+\frac{3}{9^n}\right)\right]. \qquad (1.2.5)$$

With those inequalities, we may see that

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}\sum_{m\neq k}\frac{y_{a_1,k} *_V y_{a_2,m}(s_n) - y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}+\sum_{k=0}^{n-1}\frac{y_{a_1,k} *_V y_{a_2,m}(s_n) - y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}\right|\\&\quad\le \sum_{k=0}^{\infty}\sum_{m\neq k}2\cdot a_1^k a_2^m+\sum_{k=0}^{n-1}\frac{(a_1a_2)^k}{2}\left[2+9^k\pi\left(t_0+\frac{3}{9^n}\right)\right]\\&\le \frac{2}{(1-a_1)(1-a_2)}+\frac{(a_1a_2)^n}{1-a_1a_2}+\frac{\pi}{2}\frac{(9a_1a_2)^n-1}{9a_1a_2-1}\left(t_0+\frac{3}{9^n}\right)\\&< \frac{2}{(1-a_1)(1-a_2)}+\frac{(a_1a_2)^n}{1-a_1a_2}+\frac{9^n t_0-1}{4}\cdot\frac{(a_1a_2)^n}{1-a_1a_2}.\end{aligned}$$

After all these calculations, we can guarantee, for n large enough,

$$\left|\sum_{k=0}^{\infty}\sum_{m\neq k}\frac{y_{a_1,k} *_V y_{a_2,m}(s_n)-y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}+\sum_{k=0}^{n-1}\frac{y_{a_1,k} *_V y_{a_2,m}(s_n)-y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}\right|$$
$$\leq \frac{9^n t_0-1}{4}\cdot\frac{(a_1a_2)^n}{1-a_1a_2},$$

and then, for n large enough, we can find the existence of a constant $\varepsilon_n(a_1,a_2)\in[-1,1]$ such that

$$\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\frac{y_{a_1,k} *_V y_{a_2,m}(s_n)-y_{a_1,k} *_V y_{a_2,m}(r_n)}{s_n-r_n}=\varepsilon_n(a_1,a_2)\frac{9^n t_0-1}{4}\cdot\frac{(a_1a_2)^n}{1-a_1a_2}. \tag{1.2.6}$$

In conclussion, using the identities in (1.2.3) and (1.2.6), we can say, for n large enough, that

$$\frac{W_{a_1} *_V W_{a_2}(s_n)-W_{a_1} *_V W_{a_2}(r_n)}{s_n-r_n}=\eta_n(a_1,a_2)\frac{9^n t_0-1}{3}\cdot\frac{(a_1a_2)^n}{1-a_1a_2}+\varepsilon_n(a_1,a_2)\frac{9^n t_0-1}{4}\cdot\frac{(a_1a_2)^n}{1-a_1a_2},$$

for some constants $\eta_n(a_1,a_2)\leq -1$, $\varepsilon_n(a_1,a_2),\in[-1,1]$. Using this last expression, it is easy to see that

$$\frac{W_{a_1} *_V W_{a_2}(s_n)-W_{a_1} *_V W_{a_2}(r_n)}{s_n-r_n}\xrightarrow[n\to\infty]{}-\infty.$$

□

Remark 1.2.8. *If, instead of the choice of sequences $\{r_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ we had used the following definition of sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$:*

$$v_n:=\frac{q_n}{9^n}\leq t_0<\frac{q_n+2}{9^n}, w_n:=\frac{q_n+3}{9^n},$$

for an appropriate choice of odd numbers $q_n\in\mathbb{Z}$, we would have had that

$$\frac{W_{a_1} *_V W_{a_2}(w_n)-W_{a_1} *_V W_{a_2}(v_n)}{w_n-v_n}\xrightarrow[n\to\infty]{}\infty.$$

Theorem 1.2.9 (P. Jiménez-Rodríguez, S. Maghsoudi, G.A. Muñoz-Fernández, [62])**.** *The set of all continuous functions that, convoluting with themselves, give a nowhere differentiable function, is $\mathfrak{c}$-lineable.*

Proof. Consider the set

$$\left\{W_a(x)\,:\,\frac{7}{9}<a<1\right\}.$$

Again, we shall show that this set is linearly independent and that its span is in the set we are interested in. To this aim, assume $\frac{7}{9}<a_1<a_2<\ldots<a_k<1$ and $\alpha_1,\ldots,\alpha_k\in\mathbb{R}\setminus\{0\}$ and consider the function

$$g(x)=\sum_{i=1}^{k}\alpha_i W_{a_i}(x).$$

Following the steps of the Theorem 1.2.7 and with the same definitions of sequences $\{s_n\}_{n=1}^{\infty}$, $\{r_n\}_{n=1}^{\infty}$, we find that, for n large enough,

$$\begin{aligned}\frac{g *_V g(s_n) - g *_V g(r_n)}{s_n - r_n} &= \sum_{i,j=1}^{k} \alpha_i\alpha_j \left\{ \left(\frac{\eta_n(a_i,a_j)}{3} + \frac{\varepsilon_n(a_i,a_j)}{4} \right) \cdot \frac{(a_ia_j)^n(9^n t_0 - 1)}{1 - a_1a_2} \right\} \\ &= (9a_k^2)^n \sum_{i,j=1}^{k} \alpha_i\alpha_j \left\{ \left(\frac{\eta_n(a_i,a_j)}{3} + \frac{\varepsilon_n(a_i,a_j)}{4} \right) \left(\frac{a_ia_j}{a_k^2} \right)^n \frac{9^n t_0 - 1}{9^n(1 - a_1a_2)} \right\}.\end{aligned}$$

Now, if $(i,j) \neq (k,k)$, we get that

$$\alpha_i\alpha_j \left\{ \left(\frac{\eta_n(a_i,a_j)}{3} + \frac{\varepsilon_n(a_i,a_j)}{4} \right) \left(\frac{a_ia_j}{a_k^2} \right)^n \frac{9^n t_0 - 1}{9^n(1 - a_1a_2)} \right\} \xrightarrow[n\to\infty]{} 0,$$

and if $(i,j) = (k,k)$, we get

$$\alpha_k^2 \left\{ \left(\frac{\eta_n(a_k,a_k)}{3} + \frac{\varepsilon_n(a_k,a_k)}{4} \right) \frac{9^n t_0 - 1}{9^n(1 - a_1a_2)} \right\} \xrightarrow[n\to\infty]{} x < 0.$$

Hence, we conclude

$$\frac{g *_V g(s_n) - g *_V g(r_n)}{s_n - r_n} \xrightarrow[n\to\infty]{} -\infty.$$

□

Remark 1.2.10. *If, instead of the choice of sequences $\{r_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$ we had used the definition of sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$as in Remark 1.2.8, we would have had that*

$$\frac{g *_V g(w_n) - g *_V g(v_n)}{w_n - v_n} \xrightarrow[n\to\infty]{} \infty.$$

One might think that the nowhere-differentiability of the functions is a condition strong enough to guarantee the nowhere differentiability of the convolution of the function with itself. Nevertheless, that is not the case, as we shall see in the following result. First of all, let us state a consequence of Weierstrass M-test:

Proposition 1.2.11. *Let $(f_n)_{n=0}^{\infty}$ be a sequence of differentiable functions on an interval $I = [a,b]$ and let $(a_n)_{n=0}^{\infty}$ be a sequence of numbers such that $\sum_{n=0}^{\infty} |a_n| < \infty$. Assume that $\|f_n'\|_{\infty} \leq K < \infty$ for all $n \geq 0$ and that $\sum_{n=0}^{\infty} a_n f_n(x)$ converges for at least one $x \in I$. Then, $\sum_{n=0}^{\infty} a_n f_n$ converges uniformly on I to a differentiable function f such that $f' = \sum_{n=0}^{\infty} a_n f_n'$.*

Proof. We just need to apply Weierstrass M-test to show that $\sum_{n=0}^{\infty} a_n f_n'$ converges uniformly on I. Since $\sum_{n=0}^{\infty} a_n f_n(x)$ converges for for some $x \in I$, according to a basic result on functions of one real variable, $\sum_{n=0}^{\infty} a_n f_n$ converges uniformly to a differentiable function and $\left(\sum_{n=0}^{\infty} a_n f_n\right)' = \sum_{n=0}^{\infty} a_n f_n'$. □

Next we reproduce the definition of Knopp's function. Let $0 < a < 1$, $b > 1$ with $1/a > ab > 1$ and $\Phi(z) := \text{dist}(z, \mathbb{Z})$. Observe that $\text{dist}(z, \mathbb{Z})$ is the distance from z to $\mathbb{Z}$, i.e.,

$$\text{dist}(z, \mathbb{Z}) = \inf\{|z - m| : m \in \mathbb{Z}\}.$$

If $f_k(x) = \Phi(b^k x)$ is defined over a bounded interval, say $[0, M]$, then Knopp's example is defined as $f(x) = \sum_{k=0}^{\infty} a^k f_k(x)$ for $x \in [0, M]$.

Although f is a continuous nowhere differentiable function (see [69] for the original work by Knopp or [10] for a more modern exposition), it can be proved that $f * f$ is differentiable. Indeed, we have that

$$\begin{aligned} f * f(x) &= \int_0^x f(\tau)f(x-\tau)d\tau = \int_0^x \left(\sum_{k=0}^{\infty} a^k f_k(\tau)\right)\left(\sum_{j=0}^{\infty} a^j f_j(x-\tau)\right)d\tau \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\int_0^x a^k\Phi(b^k\tau)a^{k-j}\Phi(b^{k-j}(x-\tau))d\tau \\ &= \sum_{k=0}^{\infty}\left(\frac{a^2b+1}{2}\right)^k\left(\frac{2}{a^2b+1}\right)^k\sum_{j=0}^{k} a^{2k-j}\int_0^x \Phi(b^k\tau)\Phi(b^{k-j}(x-\tau))d\tau \\ &= \sum_{k=0}^{\infty}\left(\frac{a^2b+1}{2}\right)^k g_k(x), \end{aligned}$$

with

$$g_k(x) = \left(\frac{2}{a^2b+1}\right)^k\sum_{j=0}^{k} a^{2k-j}\int_0^x \Phi(b^k\tau)\Phi(b^{k-j}(x-\tau))d\tau.$$

Each of the functions g_k is differentiable, with

$$g_k'(x) = \left(\frac{2}{a^2b+1}\right)^k\sum_{j=0}^{k} a^{2k-j}\int_0^x \Phi(b^k\tau)b^{k-j}\Phi'(b^{k-j}(x-\tau))d\tau,$$

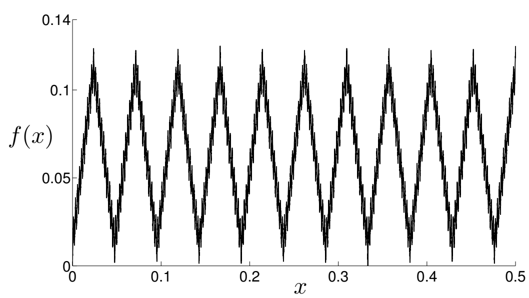
and hence,

$$|g_k'(x)| \leq \left(\frac{2a^2b}{a^2b+1}\right)^k\sum_{j=0}^{k}\left(\frac{1}{ab}\right)^j\frac{M}{2} \leq \frac{Mab}{2(1-ab)}.$$

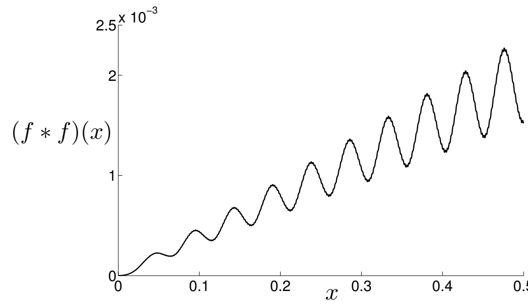
In conclusion, $|g_k'(x)| \leq \frac{Mab}{2(1-ab)}$, for all x in $[0, M]$ and for all k. Applying Proposition 1.2.11 it follows that f is differentiable. We may find interesting Figure 1.2, where we have a sketch of the graph of $f * f$ in a small interval.

Let us concentrate now in some other analytic properties for continuous functions, also related to differentiability. It is a simple exercise (by means of the Mean Value Theorem) to check that (given any interval I) a differentiable function $f : I \to \mathbb{R}$ is Lipschitz if and only if its derivative is bounded. It is natural to wonder whether this result still holds true under weaker conditions. An example of a continuous almost everywhere differentiable function on $[0, 1]$, with almost everywhere null derivative and non-Lipschitz can be found in [99] (namely, the well-known Cantor-Lebesgue function).

Example 1.2.12 (Cantor-Lebesgue function)**.** *Let C be the Cantor set and, for each $n \in \mathbb{N}$ and $1 \leq k \leq 2^n - 1$, let I_n^k be the kth open interval that is removed from $[0, 1]$ up to the nth iteration*



Graph of Knopp's function f on $[0, 1/2]$



Graph of $f * f$ on $[0, 1/2]$

Figure 1.2: We have considered f with the parameters $a = 0.2$ and $b = 21$. We have truncated the series appearing in the definition of f up to 10 terms in order to sketch the graph of f and $f * f$.

of the standard construction of C, ordered form left to right. Define, for each $n \in \mathbb{N}$, the function $f_n : [0, 1] \to [0, 1]$,

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{k}{2^n} & \text{if } x \in I_n^k, \\ 1 & \text{if } x = 1, \text{ and} \\ \text{linear} & \text{otherwise.} \end{cases}$$

Above, when we say linear*, we mean it in such a way that f_n is continuous. The sequence $\{f_n\}$ is a uniformly Cauchy sequence of continuous functions and, hence, converges uniformly to some continuous function, f, which is known as the Cantor-Lebesgue function. In the following proposition we shall recall some properties of this function that shall be useful later.*

Proposition 1.2.13. *The Cantor-Lebesgue function is a non-decreasing continuous function onto $[0, 1]$, differentiable on $(0, 1) \setminus C$, where it has null-derivative, and non-Lipschitz. It is also injective over C.*

Again, after seeing an example of such a function, one could think that there cannot be too many functions of that kind, and one more time this is indeed what has happened. More concretely, in the following result we prove the existence of a $\mathfrak{c}$-dimensional closed linear space of such functions.

Theorem 1.2.14 (P. Jiménez-Rodríguez, [61])**.** *The set of all continuous almost everywhere differentiable functions, with almost everywhere null derivative and non-Lipschitz, is spaceable in $(\mathcal{C}([0, 1]; \mathbb{R}), \| \cdot \|_\infty)$.*

Before proving the theorem, we shall state some results and definitions that might make the steps of the proof clearer and simpler.

Lemma 1.2.15. *Let $\{p_k : k \in \mathbb{N}\}$ be the natural order in the set of prime numbers. Define the function,*

$$h : c_0 \to \mathcal{C}([0, 1]; \mathbb{R})$$

as follows: for $(x_l)_{l=1}^{\infty} \in c_0$,

$$h(x_l)(y) = \begin{cases} \frac{x_{k+1}}{2^n} & \textit{if } y = \frac{1}{3^n}, \textit{ where } n \textit{ is odd and } p_k = \min\{p \textit{ prime} : p|n\}, \\ x_{\frac{n}{2}} & \textit{if } y = \frac{1}{3^n}, \textit{ where } n \textit{ is even,} \\ 0 & \textit{if } y \in \{0,1\}, \textit{ and} \\ \textit{linear} & \textit{otherwise,} \end{cases}$$

Then, the following properties hold for h:

1. *h is a linear functional.*
2. $\|h(x_l)\|_\infty = \|(x_l)\|_\infty$ *for every* $(x_l)_{l=1}^{\infty} \in c_0$ *(h is an isometry).*

Proof. 1. Let $(x_l), (z_l) \in c_0$. Then,

$$h\left((x_l) + (z_l)\right)(y) = \begin{cases} \frac{x_{k+1}+z_{k+1}}{2^n} & \text{if } y = \frac{1}{3^n}, n \text{ is odd and } p_k = \min\{p \text{ prime} : p|n\}, \\ x_{\frac{n}{2}} + z_{\frac{n}{2}} & \text{if } y = \frac{1}{3^n}, \text{ where } n \text{ is even} \\ 0 & \text{if } y \in \{0,1\}, \\ \text{linear} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{x_{k+1}}{2^n} & \text{if } y = \frac{1}{3^n}, n \text{ is odd and } p_k = \min\{p \text{ prime} : p|n\}, \\ x_{\frac{n}{2}} & \text{if } y = \frac{1}{3^n}, \text{ where } n \text{ is even} \\ 0 & \text{if } y \in \{0,1\}, \\ \text{linear} & \text{otherwise.} \end{cases}$$

$$+ \begin{cases} \frac{z_{k+1}}{2^n} & \text{if } y = \frac{1}{3^n}, n \text{ is odd and } p_k = \min\{p \text{ prime} : p|n\}, \\ z_{\frac{n}{2}} & \text{if } y = \frac{1}{3^n}, \text{ where } n \text{ is even} \\ 0 & \text{if } y \in \{0,1\}, \\ \text{linear} & \text{otherwise.} \end{cases}$$

$$= h(x_l)(y) + h(z_l)(y),$$

since the equation of the line joining $(a_1, b_1 + c_1)$ with $(a_2, b_2 + c_2)$ is the sum of the equations of the lines that join the points (a_1, b_1) with (a_2, b_2) and (a_1, c_1) with (a_2, c_2). The linearity by scalars works the same way.

2. Let $(x_l) \in c_0$ and assume $|x_{k_0}| = \|(x_l)\|_\infty$. If we set $y = \frac{1}{3^{2k_0}}$ then $|h(x_l)(y)| = |x_{\frac{2k_0}{2}}| = |x_{k_0}| = \|(x_l)\|_\infty$ and since by construction of h we also have $|h(x_l)(y)| \leq \|(x_l)\|_\infty$ for all $y \in [0,1]$, we have $\|h(x_l)\|_\infty = \|(x_l)\|_\infty$.

□

Definition 1.2.16. *Define*

$$\begin{array}{cccc} T: & c_0 & \longrightarrow & \mathcal{C}([0,1];\mathbb{R}) \\ & (x_l) & \to & h(x_l) \circ f \end{array},$$

where $f : [0,1] \to [0,1]$ *is the original Cantor-Lebesgue function from Example 1.2.12.*

Proposition 1.2.17. *Let T be the function defined in definition 1.2.16. Then, we have:*

1. *T is an isometry.*

2. *$T(x_l)$ is almost everywhere differentiable, with almost everywhere null derivative, for every $(x_l)_{l=1}^{\infty} \in c_0$.*

3. *$T(x_l)$ is non-Lipschitz, for every $(x_l) \in c_0 \setminus \{0\}$.*

Proof. 1. This comes from the fact that h (lemma 2.2) is an isometry and from the fact that f is surjective over $[0,1]$.

2. Let C be the Cantor set and $y \in (0,1) \setminus C$. Then $f(y) = \frac{k}{2^m}$, with $m \in \mathbb{N}$ and $1 \leq k \leq 2^m - 1$, in particular $f(y) \neq \frac{1}{3^n}$ for all $n \in \mathbb{N}$. Hence, $h(x_l)$ is differentiable at $f(y)$ and then $T(x_l)$ is differentiable at y. Also, if $T(x_l)$ is differentiable at $y \in (0,1) \setminus C$, we get $T(x_l)'(y) = h(x_l)'(f(y)) \cdot f'(y) = 0$.

3. Let $l_0 \geq 1$ such that $x_{l_0} \neq 0$ and consider, for every $n \in \mathbb{N}$,

$$y_n = f^{-1}\left(\frac{1}{3^{(p_{l_0-1})^n}}\right).$$

Those y_n are well-defined, being the function f injective over the Cantor set. Since f is also a non-decreasing function, we obtain

$$y_n \leq \inf\left\{f^{-1}\left(\frac{1}{2^{(p_{l_0-1})^n}}\right)\right\} \leq \frac{1}{3^{(p_{l_0-1})^n}}.$$

Then,

$$\frac{|T(x_l)(y_n) - T(x_l)(0)|}{y_n - 0} = \frac{\left|h(x_l)\left(\frac{1}{3^{(p_{l_0-1})^n}}\right)\right|}{y_n} \geq \left(\frac{3}{2}\right)^{(p_{l_0-1})^n} \cdot |x_{l_0}| \xrightarrow[n\to\infty]{} \infty.$$

□

Proof of the Theorem 1.2.14. With the conclusions obtained in the Proposition 1.2.17, we have just left to prove that if $g \in \overline{T(c_0)} \setminus \{0\}$, then g is in the set we are interested. But T is an isometry, and therefore its image is a closed set. Hence, $g \in \overline{T(c_0)} \setminus \{0\} = T(c_0) \setminus \{0\}$ means $g = T(x_n)$ and the previous lemma applies.

□

Since the operator that we have considered to prove the *spaceability* of this set is an isometry, we can actually infer a much stronger result, since what we actually have is a "special" type of spaceability by means of an isometry with the classical sequence space c_0.

Corollary 1.2.18. *c_0 is isometrically isomorphic to a subspace of continuous almost everywhere differentiable function on $[0,1]$, with almost everywhere null derivative and non-Lipschitz.*

Remark 1.2.19. *In [9] the authors are able to build a non separable closed subspace of CBV (continuous functions with bounded variation defined on $[0,1]$ and endowed with the norm $\|f\|_{CBV} = |f(0)| + Var(f)$), each non-zero element of which is a strongly singular function, that is, a continuous function with almost everywhere null-derivative and that is nowhere constant. One property of these kind of functions is the nowhere Lipschitzianity and, hence, the set the authors consider has more pathological properties that those studied in this note. However, the conclusion presented here is not a consequence of [9], since the topology in CBV is stronger than that induced by $\|\cdot\|_\infty$ (therefore, being closed in CBV does not imply being closed in $\|\cdot\|_\infty$). Also, we were able to find an isometric copy of c_0 contained in the set we were interested in (plus the zero set).*

For the previous results, we dealt with the nature of the Mean Value Theorem for functions $f : I \to \mathbb{R}$. We can extend this theorem to functions $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, where U is a convex open set. Actually, given $x, y \in U$, there is $\zeta \in [x, y] := \{\lambda x + (1-\lambda)y \,:\, \lambda \in (0,1)\}$ such that $f(y) - f(x) = Df(\zeta) \cdot (y - x)$.

On the other hand, there is not an analogous of this result for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ in general. Indeed, the mapping $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (1 - \cos t, \sin t)$ does not satisfy the Mean Value Theorem in $[0, 2\pi]$.

Here we shall construct a $\mathfrak{c}$-dimensional vector space of functions for which the Mean Value Theorem fails, even though they fulfill the hypotheses for the classical 1-dimensional version of this result. We shall also show that, for functions $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, the hypothesis of U being convex cannot be replaced with a *weaker* one such as (for instance) U being path-connected. We shall construct a $\mathfrak{c}$-dimensional vector space of differentiable functions over a path-connected open set, with bounded derivative and for which the Mean Value Theorem does not hold.

Theorem 1.2.20 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [65])**.** *The set M of differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ that do not enjoy the Mean Value Theorem is $\mathfrak{c}$-lineable.*

Proof. Given $\lambda > 0$, let $f^\lambda(x) = e^{\lambda x}(x^2 - x, x^3 - 2x^2 + x)$, which is a differentiable function. We consider $H = \{f^\lambda \,:\, \lambda > 0\}$. Let us see that H is linearly independent and that $\operatorname{span}(H) \subseteq M \cup \{0\}$. Indeed, let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_k > 0$ and define

$$g = \sum_{i=1}^{k} \alpha_i f^{\lambda_i} = \left(\sum_{i=1}^{k} \alpha_i e^{\lambda_i x}\right)(x^2 - x, x^3 - 2x^2 + x).$$

Let us assume that $g = 0$. Then there is an open interval J in which $\sum_{i=1}^{k} \alpha_i e^{\lambda_i x} = 0$, and this implies $\alpha_i = 0$ for all $1 \le i \le k$. This proves the linear independency of H.

Now, assume $\alpha_i \neq 0$ for every $1 \le i \le k$ and that g fulfills the Mean Value Theorem. It is clear that there exists an interval $(a, b) \subset [0, 1]$ with $g(x) \neq 0$ on (a, b).

If there is $x \in (b, 1)$ with $\left(\sum_{i=1}^{k} \alpha_i e^{\lambda_i x}\right) = 0$, then we define $\eta_2 = \min\{y > b \,:\, \sum_{i=1}^{k} \alpha_i e^{\lambda_i y} = 0\}$. Otherwise, we define $\eta_2 = 1$. If $x \in (0, a)$ with $\sum_{i=1}^{k} \alpha_i e^{\lambda_i x} = 0$, then we define $\eta_1 = \max\{y < a \,:\, \sum_{i=1}^{k} \alpha_i e^{\lambda_i y} = 0\}$. Otherwise, we define $\eta_1 = 0$.

Then, $(a, b) \subseteq (\eta_1, \eta_2) \subset [0, 1]$. Also, $g(\eta_1) = g(\eta_2) = (0, 0)$ and $g(x) \neq 0 \,\forall x \in (\eta_1, \eta_2)$. Now, and by assumption, there exists $x \in (\eta_1, \eta_2)$ with

$$(\eta_2 - \eta_1)Dg(x) = (\eta_2 - \eta_1)\sum_{i=1}^{k} \alpha_i(e^{\lambda_i x}(\lambda_i x^2 - (\lambda_i - 2)x - 1),$$

and

$$e^{\lambda_i x}(\lambda_i x^3 + (3 - 2\lambda_i)x^2 + (\lambda_i - 4)x + 1)) = g(\eta_2) - g(\eta_1) = (0, 0).$$

Hence,

$$\left(\sum_{i=1}^{k} \alpha_i \lambda_i e^{\lambda_i x}\right)x^2 + \left(\sum_{i=1}^{k} \alpha_i (2 - \lambda_i) e^{\lambda_i x}\right)x - \sum_{i=1}^{k} \alpha_i e^{\lambda_i x} = 0 \tag{1.2.7}$$

and

$$\left(\sum_{i=1}^{k}\alpha_i\lambda_i e^{\lambda_i x}\right)x^3+\left(\sum_{i=1}^{k}\alpha_i(3-2\lambda_i)e^{\lambda_i x}\right)x^2+\left(\sum_{i=1}^{k}\alpha_i(\lambda_i-4)e^{\lambda_i x}\right)x+$$
$$+\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}=0.$$

Adding both equations we obtain

$$\left(\sum_{i=1}^{k}\alpha_i\lambda_i e^{\lambda_i x}\right)x^3+\left(\sum_{i=1}^{k}\alpha_i(3-\lambda_i)e^{\lambda_i x}\right)x^2-2\left(\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}\right)x=0,$$

leading to

$$\left(\sum_{i=1}^{k}\alpha_i\lambda_i e^{\lambda_i x}\right)x^2+\left(\sum_{i=1}^{k}\alpha_i(3-\lambda_i)e^{\lambda_i x}\right)x-2\left(\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}\right)=0.$$

Next, combining it with equation (1.2.7), we obtain

$$\left(\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}\right)x-\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}=0.$$

Now, since $\sum_{i=1}^{k}\alpha_i e^{\lambda_i x}\neq 0$ for $x\in(\eta_1,\eta_2)$, we can conclude that $x=1$, which is a contradiction with the fact that $x<\eta_2\leq 1$. □

A function with bounded gradient on a convex set satisfies the Mean Value Theorem and, thus, it is Lipschitz. However, if the set is not convex the latter does not hold. Actually, we can even obtain lineability in this situation as we show below.

Theorem 1.2.21 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [65])**.** *Let $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}\setminus\{(x,y)\in\mathbb{R}^2 : x=0 \text{ and } y>0\}$. The set of differentiable functions $f : D\to\mathbb{R}$ with bounded gradient that are not Lipschitz (and, thus, not verifying the Mean Value Theorem) is $\mathfrak{c}$-lineable.*

Proof. We define, for every $\lambda>1$, $f_\lambda : D\longrightarrow\mathbb{R}$ as

$$f_\lambda(x,y)=\begin{cases} y^\lambda\arctan\left(\dfrac{\lambda y}{x}\right) & if\, x<0,\\ y^\lambda\left[\arctan\left(\dfrac{\lambda y}{x}\right)+\pi\right] & if\, x>0,\\ \dfrac{\pi}{2}y^\lambda & if\, x=0.\end{cases}$$

We shall prove that the set $H=\{f_\lambda : \lambda>1\}$ is linearly independent and that $\operatorname{span}(H)$ is in the set we are studying.

Assume first $g(x,y)=\sum_{i=1}^{k}\alpha_i f_{\lambda_i}(x,y)=0$ for $\lambda_1,\ldots,\lambda_k>1$ and $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$. Then, for every $-1<y<0$ we obtain $0=g(0,y)=\frac{\pi}{2}\sum_{i=1}^{k}\alpha_i y^{\lambda_i}$ which allows us to conclude $\alpha_i=0$ for every $1\leq i\leq k$, since $\{y^\lambda : \lambda>1\}$ is a linearly independent set over $(-1,0)$. To check now that

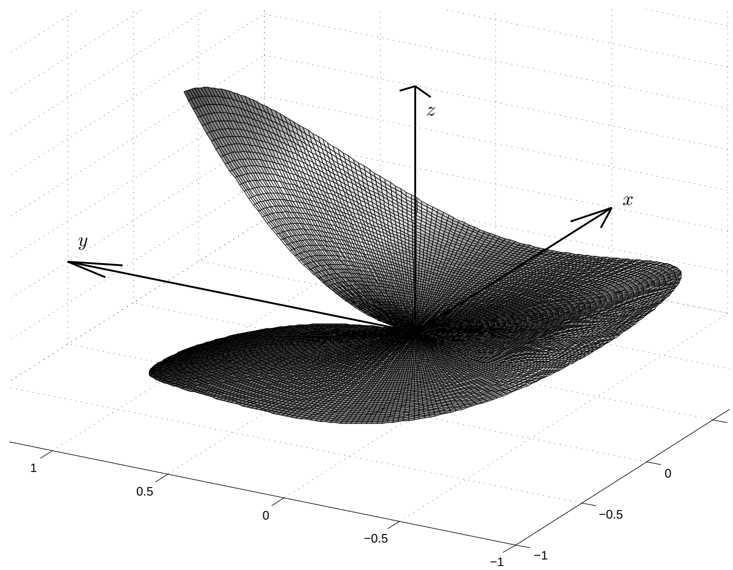


Figure 1.3: $f_2 : D \longrightarrow \mathbb{R}$.

g (assumed $\alpha_i \neq 0$, $\forall 1 \leq i \leq k$) is in our set, we shall work with the functions f_λ alone, since the arguments we are going to use can be easily extended to finite linear combinations. Let us obtain first $\frac{\partial f_\lambda}{\partial x}(x_0, y_0)$. If $x_0 \neq 0$ we can simply differentiate f_λ to get $\frac{\partial f_\lambda}{\partial x}(x_0, y_0) = -\lambda \frac{y_0^{\lambda+1}}{x_0^2+(\lambda y_0)^2}$.

For $x_0 = 0$,

$$\lim_{t\to 0^+} \frac{f_\lambda(t, y_0) - f_\lambda(0, y_0)}{t} = \lim_{t\to 0^+} \frac{y_0^\lambda \left[\frac{\pi}{2} + \arctan\left(\frac{\lambda y_0}{t}\right)\right]}{t} = \lim_{t\to 0^+} \frac{y_0^\lambda(-y_0\lambda)}{t^2 + (\lambda y_0)^2}$$
$$= -\frac{y_0^{\lambda-1}}{\lambda},$$

and identical calculations lead to the same value for $\lim_{t\to 0^-} \frac{f_\lambda(t,y_0)-f_\lambda(0,y_0)}{t}$.

Hence, we obtain

$$\frac{\partial f_\lambda}{\partial x}(x_0, y_0) = \begin{cases} -\lambda \frac{y_0^{\lambda+1}}{x_0^2+(\lambda y_0)^2} & if\, x \neq 0 \\ -\frac{y_0^{\lambda-1}}{\lambda} & if\, x_0 = 0 \end{cases}.$$

Analogously one can obtain

$$\frac{\partial f_\lambda}{\partial y}(x_0, y_0) = \begin{cases} \lambda y_0^{\lambda-1}\left(\arctan\left(\frac{\lambda y_0}{x_0}\right) + \frac{x_0 y_0}{x_0^2 + (\lambda y_0)^2}\right) & if\, x < 0, \\ \lambda y_0^{\lambda-1}\left(\arctan\left(\frac{\lambda y_0}{x_0}\right) + \frac{x_0 y_0}{x_0^2 + (\lambda y_0)^2} + \pi\right) & if\, x > 0, \\ \frac{\pi}{2}\lambda y_0^{\lambda-1} & if\, x = 0. \end{cases}$$

Thus, the partial derivatives exist. Let us now see that they are continuous (which would prove $f_\lambda \in C^1(\mathbb{R}^2;\mathbb{R})$), for which we shall only have to focus on the points of the form $(0, y_0)$. Indeed,

$$\lim_{(x,y)\to(0,y_0)} \left| \frac{\partial f_\lambda}{\partial x}(x,y) - \frac{\partial f_\lambda}{\partial x}(0,y_0) \right| = \lim_{(x,y)\to(0,y_0)} \left| \frac{y_0^{\lambda-1}}{\lambda} - \frac{\lambda y^{\lambda+1}}{x_0^2 + (\lambda y_0)^2} \right| = 0.$$

Analogously,

$$\lim_{(x,y)\to(0,y_0)} \left| \frac{\partial f_\lambda}{\partial y}(x,y) - \frac{\partial f_\lambda}{\partial y}(0,y_0) \right| = 0,$$

from which it follows that the partial derivatives are continuous.
Assume now that $x_0 \neq 0$. Then $\left|\frac{\partial f_\lambda}{\partial x}(x_0,y_0)\right| \le \lambda |y_0|^{\lambda-1} \le \lambda$. Similarly $\left|\frac{\partial f_\lambda}{\partial x}(0,y_0)\right| \le \frac{1}{\lambda} < 1 < \lambda$, which gives us $\left|\frac{\partial f_\lambda}{\partial x}(x_0,y_0)\right| \le \lambda$ for every $(x_0,y_0) \in D$. In an analogous way, $\left|\frac{\partial f_\lambda}{\partial y}(x_0,y_0)\right| \le \lambda \frac{1+3\pi}{2}$, $\forall (x_0,y_0) \in D$ and hence we deduce that Df_λ is bounded on D.
Finally, suppose that f_λ is Lipschitz with constant $K > 0$. Thus, given (x,y), $(\hat{x},\hat{y}) \in D$ we have

$$|f_\lambda(\hat{x},\hat{y}) - f_\lambda(x,y)| \le K \|(\hat{x}-x, \hat{y}-y)\|_2.$$

Now, if we fix $\hat{y} = y > 0$ and force $x > 0$ and $\hat{x} < 0$ we obtain

$$\begin{aligned} |f_\lambda(\hat{x},y) - f_\lambda(x,y)| &= \left| y^\lambda \left[\arctan\left(\frac{\lambda y}{\hat{x}}\right) - \arctan\left(\frac{\lambda y}{x}\right) - \pi \right] \right| \\ &\le K \|(\hat{x}-x, 0)\|_2, \end{aligned}$$

but $|f_\lambda(\hat{x},y) - f_\lambda(x,y)| \xrightarrow[\hat{x}\to 0, x\to 0]{} 2\pi |y|^\lambda \neq 0$ for $y \neq 0$ and

$$K\|(\hat{x}-x,0)\|_2 \xrightarrow[\hat{x}\to 0, x\to 0]{} 0,$$

which makes it impossible for f_λ to be Lipschitz.

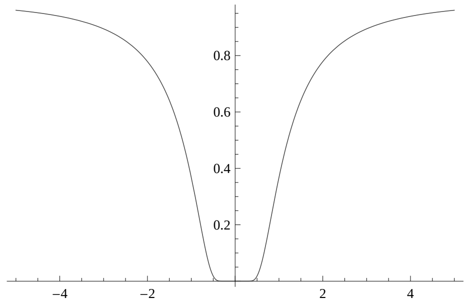
□

1.3 When the Identity Theorem *seems* to fail

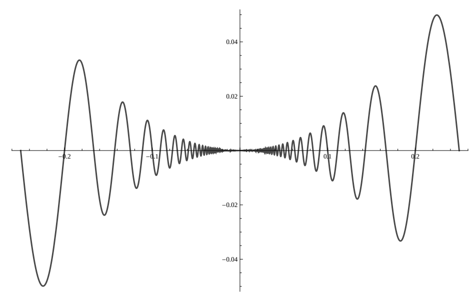
In Complex Analysis, the *Identity Theorem* states that, if two holomorphic functions f and g defined on a domain (connected open subset) $D \subset \mathbb{C}$ agree on a set A which has an accumulation point in D, then $f = g$ all over D. Of course, one amazing consequence of this fact is that any analytic function is completely determined by its values on any neighborhood V in D, no matter *how small* V is.

On a totally different framework, a real function is said to be real analytic if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point. Of course, the *Identity Theorem* also holds for real analytic functions but one needs to be careful when applying it, since (in $\mathbb{R}$) one can have $\mathcal{C}^\infty$ functions that are not analytic, as the following *well-known* function shows (see Figure 1.4):

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$



Graph of f.



Graph of g.

Figure 1.4:

As some simple calculations would entail, the above function only agrees with its Taylor series expansion at $x = 0$.

On the other hand, Weierstrass' factorization theorem states that, if $f \in \mathcal{H}(\mathbb{C})$, then f has only countably many zeros (possibly only finitely many), counting multiplicities. It also follows that, if f has only finitely many zeroes, then f is of the form $p(z)e^{h(z)}$ for some $h \in \mathcal{H}(\mathbb{C})$ and some polynomial $p \in \mathbb{C}[z]$.

Of course, if an entire function has infinitely many zeroes with an accumulation point, then (by the Identity Theorem) it must be the zero function. For real functions this does not hold. For instance, the differentiable function $g : \mathbb{R} \to \mathbb{R}$ given by (see Figure 1.4)

$$g(x) = \begin{cases} x^2 \sin(\pi x^{-1}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

has the infinite set $Z = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ as its set of zeroes, Z has an accumulation point (0) but, obviously, $g \neq 0$.

After all the above, the following question comes out naturally:

> *Are there non-zero real valued differentiable functions with infinitely many zeroes, possessing derivatives of all orders, and also non-analytic? And, how big is this set of functions? What algebraic/linear structure does this set possess?*

In the following results we shall answer these questions positively. Moreover, we shall even show more: We shall construct an algebra $\mathcal{A}$ of real valued functions enjoying, simultaneously, each of the following properties:

(i) $\mathcal{A}$ is uncountably infinitely generated. That is, the cardinality of a minimal system of generators of $\mathcal{A}$ is $\mathfrak{c}$.

(ii) Every non-zero element of $\mathcal{A}$ is nowhere analytic.

(iii) $\mathcal{A} \subset \mathcal{C}^\infty(\mathbb{R})$.

(iv) Every element of $\mathcal{A}$ has infinitely many zeroes in $\mathbb{R}$.

(v) For every $f \in \mathcal{A}$ and $n \in \mathbb{N}$, $f^{n)}$ (the n-th derivative of f) is also in $\mathcal{A}$.

Functions with infinitely many zeros in a closed finite interval are known as *annulling functions.* The question on the existence of an algebra of such functions inside of $C[0,1]$ is what shall also be solved here.

Let $\mathcal{H}$ be a Hamel basis of $\mathbb{R}$. That is, a basis of the real numbers $\mathbb{R}$, considered as a $\mathbb{Q}$-vector space. Furthermore, without loss of generality, we can assume that $\mathcal{H}$ consists only of positive real numbers.

Let us now consider the minimum algebra of $\mathcal{C}(\mathbb{R})$ that contains the family of functions $\{\rho_\alpha\}_{\alpha\in\mathcal{H}}$ with $\rho_\alpha : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\rho_\alpha(x) = \sum_{j=1}^{\infty} \lambda_j(x)\phi_\alpha\left(2^j x - [2^j x]\right)^j ,$$

where $[\cdot]$ denotes the *greatest integer function*, $\phi_\alpha : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi_\alpha(x) = \begin{cases} e^{\frac{-\alpha}{x^2}} \cdot e^{\frac{-\alpha}{(x-1)^2}} & \text{if } 0 < x < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

and, for $j \in \mathbb{N}$,

$$\lambda_j(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{2^j}, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.3.1 (J. A. Conejero, P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, J. B. Seoane-Sepúlveda, [31])**.** *All functions $\{\rho_\alpha\}_{\alpha>0}$ are $\mathcal{C}^\infty$ and nowhere analytic. Moreover, all the derivatives and the function itself vanish at the points $\{2^{-j}\}_{j\in\mathbb{N}_0}$.*

Proof. For every $\alpha > 0$, the function $\phi_\alpha(x)$ is smooth everywhere and analytic except at $x = 0$ and $x = 1$. Moreover, the function $\phi_\alpha(x)$ is flat at both of these points, that is, all the derivatives and the function ϕ_α itself evaluated at those points are also 0. Replacing x by $2^j x - [2^j x]$ the behaviour of $\phi_\alpha(x)^j$ over the interval $[0,1]$ is replicated by $\phi_\alpha(2^j x - [2^j x])^j$ on any dyadic interval of the form $[(m-1)/2^j, m/2^j]$ for all $m \in \mathbb{Z}$. Thus, $\phi_\alpha(2^j x - [2^j x])^j$ is smooth everywhere and analytic in $\mathbb{R}$ except the points $x = m/2^j$ for all $m \in \mathbb{Z}$.

For any $x \neq 0$, there is some $j_0 \in \mathbb{N}$ such that $x \geq \frac{1}{2^j}$ for all $j \geq j_0$, therefore

$$\rho_\alpha(x) = \sum_{j=j_0}^{\infty} \lambda_j(x)\phi_\alpha\left(2^j x - [2^j x]\right)^j .$$

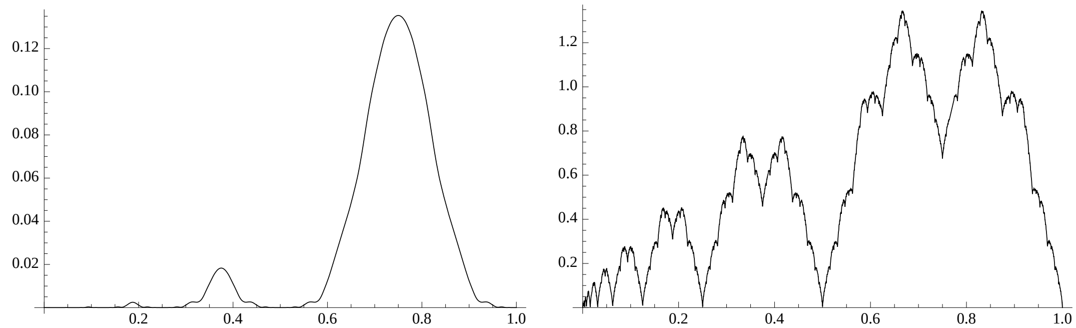
Thus, the same proof of the infinite differentiability and nowhere analyticity of p_α at any point $x \neq 0$ follows the very same strategy of the proof of the infinite differentiability and nowhere analyticity of the functions

$$\sum_{j=1}^{\infty} \frac{1}{j!}\phi_\alpha(2^j x - [2^j x])$$

from [68] (Theorem 1).

□

Theorem 1.3.2 (J. A. Conejero, P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, J. B. Seoane-Sepúlveda, [31])**.** *Let $\mathcal{A}$ be the algebra generated by $\{\rho_\alpha\}_{\alpha\in\mathcal{H}}$. Then:*



Graph of $\rho_{0.25}$. Graph of $\rho_{0.05}$.

Figure 1.5:

(i) $\mathcal{A}$ *is uncountably infinitely generated.*

(ii) Every non-zero element of $\mathcal{A}$ *is nowhere analytic.*

(iii) $\mathcal{A} \subset \mathcal{C}^{\infty}(\mathbb{R})$.

(iv) Every non-zero element of $\mathcal{A}$ *is an annulling function on* $\mathbb{R}$.

(v) For every $f \in \mathcal{A}$ *and* $n \in \mathbb{N}$, $f^{n)}$ *is also in* $\mathcal{A}$.

Proof. Any element $h \in \mathcal{A}$ can be written as $h(x) = \sum_{k=1}^{n} \beta_i \rho_{\alpha_k}^{m_k}(x)$ with $\alpha_k \in \mathcal{H}, m_k \in \mathbb{N}$, for $k = 1, \ldots, n$. Let us suppose that $h \equiv 0$, that is, for every $x \in \mathbb{R}$, we have $h(x) = 0$. Let us evaluate $h(x)$ at the points $x_j = \frac{3}{2^{j+1}}$, $j = 1, \ldots, n$. Evaluating the function $\rho_{\alpha_k}^{m_k}$ at the points x_j, the sum that gives its definition is reduced to just one single term:

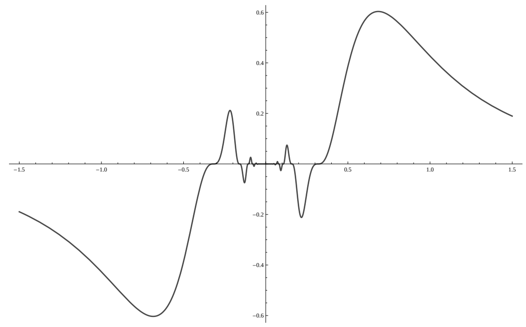
$$\rho_{\alpha_k}^{m_k}\left(\frac{3}{2^{j+1}}\right) = \phi_{j\alpha_k m_k}\left(\frac{1}{2}\right) = e^{-8j\alpha_k m_k}.$$

Therefore, if we consider the system of equations obtained from the conditions $h\left(\frac{3}{2^{j+1}}\right) = 0$ for $j = 1, \ldots, n$, we obtain the following:

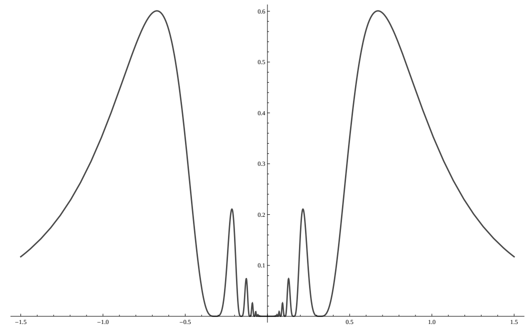
$$\begin{pmatrix} e^{-8\alpha_1 m_1} & e^{-8\alpha_2 m_2} & \cdots & e^{-8\alpha_n m_n} \\ e^{-16\alpha_1 m_1} & e^{-16\alpha_2 m_2} & \cdots & e^{-16\alpha_n m_n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-8n\alpha_1 m_1} & e^{-8n\alpha_2 m_2} & \cdots & e^{-8n\alpha_n m_n} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If (for all $j = 1, \ldots, n$) we multiply the j-column of the above matrix by $e^{8\alpha_j m_j}$, we have that the former system is equivalent to a system with the following matrix,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ e^{-8\alpha_1 m_1} & e^{-8\alpha_2 m_2} & \cdots & e^{-8\alpha_n m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \left(e^{-8\alpha_1 m_1}\right)^{n-1} & \left(e^{-8\alpha_2 m_2}\right)^{n-1} & \cdots & \left(e^{-8\alpha_n m_n}\right)^{n-1} \end{pmatrix},$$



Graph of $f_{\lambda,m}(x)$ with $\lambda = 1/3$ and $m = 3$



Graph of $f_{\lambda,m}(x)$ with $\lambda = 1/3$ and $m = 4$

Figure 1.6: Graphs of $f_{\lambda,m}$ for some choices of m and λ.

which is non-singular since it is a Vandermonde-type matrix (and also because the α_k's are different elements of the Hamel basis $\mathcal{H}$). Therefore $\beta_i = 0$ for $i = 0, \ldots, k$. The rest of the statements yield directly from Theorem 1.3.1. □

Remark 1.3.3. *I would like to finish this note by mentioning that the result in Theorem 1.3.2 is the best possible in the following sense:*

(a) The dimension of $\mathcal{A}$ (as a vector space) is the largest *possible, $\mathfrak{c}$, since the dimension of the space of continuous functions is also $\mathfrak{c}$. Also, the cardinality of the system of generators of $\mathcal{A}$ is the* biggest *possible for the same reason.*

(b) If we restrict ourselves to the interval $[0,1]$ (or to any compact interval for that matter), the corresponding algebra $\mathcal{A}$ cannot be constructed being close in $\mathcal{C}[0,1]$. This is due to the fact that Gurariy showed in [53] that the set of differentiable functions on $[0,1]$ does not contain an infinite dimensional closed subspace.

To summarize, there is no way to improve the "size" of $\mathcal{A}$ or its topological structure by making it close.

Chapter 2

Inequalities in three dimensional polynomial spaces

The field of polynomial inequalities comprises of an extremely vast range of problems. Our contribution in this chapter focuses on three specific types of inequalities. Namely, Bernstein and Markov type inequalities, inequalities involving unconditional constants and inequalities that arise from the notion of polarization constant of a polynomial space. These three problems have been previously studied with great generality (see for instance [19, 32, 52, 55, 60, 67, 70, 74, 79, 80, 81, 89, 90, 91]), which serves as motivation and inspiration to deepen into this type of questions.

2.1 Motivation and preliminaries

Bernstein and Markov inequalities

The first of the inequalities we will study is named after the following result published by one of the Markov brothers in 1889:

Theorem 2.1.1 (A.A. Markov)**.** *If P is a real polynomial of degree n and $|P(x)| \leq 1$ on $[-1,1]$, then $|P'(x)| \leq n^2$ on $[-1,1]$.*
Moreover, equality is attained by $P(x) = \pm T_n(x)$, where T_n is the nth Chebishev polynomial, defined as

$$T_n(x) = \begin{cases} 1 & \text{for } n = 0, \\ x & \text{for } n = 1, \\ 2xT_{n-1}(x) - T_{n-2}(x) & \text{otherwise.} \end{cases}$$

It is interesting to observe that the previous problem was told to Markov by D. Mendeleev, father of the periodic table of the elements. Actually, it seems that Mendeleev was able to solve the problem for quadratic polynomials when studying the results obtained after one of his experiments. For a more complete description on Mendeleev's experiment, see for instance [19].

As for an analogue of Markov's estimate for polynomials on the complex unit disk, the mathematician S. Bernstein concluded a very similar result:

Theorem 2.1.2 (S. Bernstein)**.** *If P is a complex polynomial of degree n such that $|P'(z)| \leq 1$ on $\mathbb{D}$, then $|P'(z)| \leq n$ for every $z \in \mathbb{D}$. Moreover, equality is attained for $P(z) = z^n$.*

From this result, Bernstein was able to conclude the following inequality for real polynomials:

Corollary 2.1.3 (Bernstein inequality for real polynomials)**.** *If P is a real polynomial of degree n such that $|P(x)| \leq 1$ for every $x \in (-1, 1)$, then*

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}}.$$

Along the XX century, both Markov and Bernstein's theorems were studied in the more general setting of polynomials in many variables. For instance, O.G. Kellogg found the following estimate on the length of the gradient of a polynomial:

Theorem 2.1.4 (O.G. Kellogg)**.** *Let P be a polynomial in m real variables of degree at most n. Then,*

$$\|\nabla P(x)\|_2 \leq n^2,$$

where $\| \cdot \|_2$ stands for the Euclidean length, $\|x\|_2 \leq 1$ and P is bounded by 1 over the vectors of Euclidean length not exceeding 1.

Markov type inequalities were also studied for polynomials on a non centrally symmetric convex set. As an example of a result of this kind we mention Whilhemsen estimate on the one hand, and Kroo and Revesz's results on the other.

Theorem 2.1.5 (R.D. Wilhelmsen)**.** *Let K be a convex body and P a real polynomial in m variables bounded by 1 on K. Then, for every $x \in K$ we have*

$$\|\nabla P(x)\|_2 \leq \frac{4m^2}{\operatorname{diam}(K)}, \tag{2.1.1}$$

where $\operatorname{diam}(K)$ *stands for the diameter of K, that is,* $\operatorname{diam}(K) := \sup\{\|x - y\| \ : \ x, y \in K\}$.

Theorem 2.1.6 (A. Kroó, S. Révesz, [70])**.** *Under the same hypotheses of the Theorem 2.1.5 we have*

$$\|\nabla P(x)\|_2 \leq \frac{4m^2 - 2m}{\operatorname{diam}(K)},$$

for every $x \in K$

The previous results admit a further extension to polynomials on a general Banach space. In order to discuss polynomials on a normed space we need first some definitions, notations and results.

Definition 2.1.7. *Let $(E, \| \cdot \|_E)$ and $(F, \| \cdot \|_F)$ be two normed spaces over the field $\mathbb{K}$ (real or complex numbers). We say that $P : E \to F$ is a homogeneous polynomial of degree n if there exists a symmetric multilinear mapping $L : E^n \to F$ such that $P(x) = L(x, \ldots, x)$ for all $x \in E$. We will use the following notations:*

$$\mathcal{P}(^nE; F) = \{P : E \to F \ : \ P \textit{ is a continuous } n\textit{-homogeneous polynomial}\}$$
$$\mathcal{L}(^nE; F) = \{L : E^n \to F \ : \ L \textit{ is a continuous multilinear function}\},$$
$$\mathcal{L}_s(^nE; F) = \{L : E^n \to F \ : \ L \textit{ is a continuous symmetric multilinear function}\}.$$

For simplicity, when $F = \mathbb{K}$ *we use* $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ *and* $\mathcal{L}_s(^nE)$ *instead of* $\mathcal{P}(^nE;\mathbb{K})$, $\mathcal{L}(^nE;\mathbb{K})$ *and* $\mathcal{L}^s(^nE;\mathbb{K})$ *respectively.*

As usual, B_E *will denote the open unit ball of* E.

It can be proved (see for instance [44]) that a homogeneous polynomial (or a multilinear mapping) is continuous if and only if it is bounded on B_E. *We will consider the following natural norms on the spaces of continuous homogeneous polynomials and continuous multilinear functions:*

$$\|P\| := \sup\{\|P(x)\|_F \ : \ \|x\|_E \leq 1\},$$
$$\|L\| := \sup\{\|L(x_1,\ldots,x_n)\|_F \ : \ \|x_k\|_E \leq 1, \ k = 1,\ldots,n\},$$

for all $P \in \mathcal{P}(^nE;F)$ *and all* $L \in \mathcal{L}(^nE;F)$.

Theorem 2.1.8 (Y. Sarantopoulos)**.** *Let* E *be a Banach space and* $P \in \overline{B}_{\mathcal{P}(^nE;\mathbb{R})}$. *Then,*

$$\|DP(x)\| \leq \max\left\{n^2, \frac{n}{\sqrt{1-\|x\|^2}}\right\} \quad \text{for all } x \in B_E.$$

This allows us to generalize what is popularly known as Markov inequality (uniform estimates between norms of polynomials and its differentials) and Bernstein inequalities (pointwise estimates) to polynomials defined over Banach spaces.

Polarization constants

It is widely known that for every $P \in \mathcal{P}^n(E;F)$ there exists a unique symmetric linear form $L \in \mathcal{L}(^nE,F)$ such that $P(x) = L(x,\ldots,x)$. We will denote it as $L = \check{P}$ or equivalently $P = \widehat{L}$. We call L the **polar** of P. Actually we can recover the polar if we only know the polynomial by using the so called polarization formula:

$$\check{P}(x_1,\ldots,x_n) = \frac{1}{2^n n!}\sum_{\epsilon_i=\pm 1} P(\epsilon_1 x_1 + \cdots + \epsilon_n x_n).$$

Theorem 2.1.9 (Martin, [74])**.** *If* E *is a real or complex normed space and* $P \in \mathcal{P}(^nE)$, *then*

$$\|P\| \leq \|\check{P}\| \leq \frac{n^n}{n!}\|P\|.$$

Moreover, equality is attained for the Banach space ℓ_1^n *and the polynomial* $P(x_1,\ldots,x_n) = x_1\cdots x_n$.

This last theorem holds for every polynomial defined over a normed space. However, for a specific space E the constant $\frac{n^n}{n!}$ can be improved. This motivates the following definition:

Definition 2.1.10. *If* E *is a normed space over* $\mathbb{K}$, *we define its* n*th-***polarization constant** *as*

$$\mathbb{K}(n,E) := \inf\{M > 0 \ : \ \|\check{P}\| \leq M\|P\| \text{ for all } P \in \mathcal{P}(^nE)\}.$$

The problem of calculating exactly the value of a polarization constant for a specific space is very complicated. However, some progress has been done since this question began to be studied.

It is worth mentioning the contribution done by Y. Sarantopoulos in [89, 90], where the author studies $\mathbb{K}(n, E)$ for E being an L^p space.

Notice that having a (sharp) Markov-type inequality may very easily provide estimations for the Polarization constant, since

$$DP(x)(v) = n\check{P}(x, \overbrace{\ldots}^{n-1}, x, v), \quad \text{for all } x, v \in E.$$

Observe that when $n = 2$, we have that $DP(x)(v) = n\check{P}(x, v)$, which provides an explicit relationship between the Markov and the polarization constants for quadratic polynomials.

Unconditional constants

The last of the four polynomial inequalities that will be treated in this chapter involves the so called unconditional constants. First, let us consider the following definition:

Definition 2.1.11 (Unconditional bases)**.** *If E is a real Banach space, a basis $\{x_n\} \subseteq E$ is said to be unconditional if $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally for every $\{a_n\} \subseteq \mathbb{K}$. Recall that a series $\sum_{n=1}^{\infty} y_n$ in E converges unconditionally if $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges for every permutation σ of $\mathbb{N}$. Notice that it can be proved that if $\sum_{n=1}^{\infty} \theta_n y_n$ converges for every choice of signs $\theta_n = \pm 1$, then $\sum_{n=1}^{\infty} y_n$ converges unconditionally.*

Definition 2.1.12 (Unconditional constants)**.** *Let E be a Banach space. If $\{x_n\} \subseteq E$ is an unconditional basis and we set*

$$M_\theta(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n=1}^{\infty} \theta_n a_n x_n,$$

for every choice of signs $\theta = \{\theta_n\}_{n=1}^{\infty}$, the number $\chi(E) := \sup_\theta \|M_\theta\|$ is called the **unconditional constant** *of $\{x_n\}$.*

A considerable effort has been done in order to calculate unconditional constants in polynomial spaces. The following result is a good example:

Theorem 2.1.13 (A. Defant, J.C. Díaz, D. García, M. Maestre, [32])**.** *The unconditional basis constants of all m-homogeneous polynomials on ℓ_p^n have the following asymptotic behavior:*

$$\chi(\mathcal{P}(^m\ell_p^n)) \overset{m}{\asymp} \begin{cases} n^{\frac{m-1}{2}} & \text{if } 2 < p \le \infty, \\ n^{\frac{m-1}{q}} & \text{if } 1 \le p \le 2, \end{cases}$$

where q is the conjugate exponent of p and $a_{n,m} \overset{m}{\asymp} b_{n,m}$ means that for every $m \in \mathbb{N}$ there exist constants $A_m > 0$ and $B_m > 0$ such that $a_{n,m} \le A_m b_{n,m}$ and $b_{n,m} \le B_m a_{n,m}$.

Using the standard notation for multiindices, let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, where $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_k \in \mathbb{N} \cup \{0\}$, $1 \le k \le m$. Now let $\mathcal{B}_n = \{\mathbf{x}^\alpha : |\alpha| = n\}$ be the canonical basis of $\mathcal{P}(^n\mathbb{R}^m)$, and consider $\mathcal{S} = \{\mathbf{x}^{\alpha_k} : |\alpha_k| = n, 1 \le k \le r\}$ any subset of $\mathcal{B}_n$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ a choice of signs, $P(\mathbf{x}) = a_{\alpha_1}\mathbf{x}^{\alpha_1} + \ldots + a_{\alpha_r}\mathbf{x}^{\alpha_r}$ and $P_\varepsilon(\mathbf{x}) = \varepsilon_1 a_{\alpha_1}\mathbf{x}^{\alpha_1} + \ldots + \varepsilon_r a_{\alpha_r}\mathbf{x}^{\alpha_r}$. Then, if B is a convex set in $\mathbb{R}^m$, we have

$$\|P_\varepsilon\|_{\mathsf{B}} := \sup_{\mathbf{x} \in \mathsf{B}} |P_\varepsilon(\mathbf{x})| \le \sup_{\mathbf{x} \in \mathsf{B}} |a_{\alpha_1}||\mathbf{x}|^{\alpha_1} + \ldots + |a_{\alpha_r}||\mathbf{x}|^{\alpha_r} = \||P|\|_{\mathsf{B}},$$

where $|P|(\mathbf{x}) = |a_{\alpha_1}|\mathbf{x}^{\alpha_1} + \ldots + |a_{\alpha_r}|\mathbf{x}^{\alpha_r}$. Moreover, if $\varepsilon_k = \text{sign}(a_{\alpha_k})$, then $\|P_\varepsilon\|_{\mathsf{B}} = \||P|\|_{\mathsf{B}}$. This shows that the unconditional constant of $\mathcal{S}$ coincides with the best possible constant $C_{\mathsf{B},\mathcal{S}}$ in the inequality

$$\||P|\|_{\mathsf{B}} \leq C_{\mathsf{B},\mathcal{S}}\|P\|_{\mathsf{B}}, \tag{2.1.2}$$

for every P in the space generated by $\mathcal{S}$.

It is interesting to note that already in 1914, H. Bohr [23] studied this type of inequalities for infinite complex power series. Actually, the study of Bohr radii is nowadays a fruitful field (see for instance [14, 21, 33, 36, 37, 41]).

Observe that the relationship between unconditional constants in polynomial spaces and inequalities of the type (2.1.2) was already noticed in [36].

2.2 A few final considerations before the results

For the estimates that we will present in the following theorems, we will make use of a straightforward consequence of the Krein-Milman Theorem, for which any convex function defined over a convex set attains its maximum value at the extreme points of the set (that is, the points for which it is impossible to find a convex combination out of the trivial one).

Also, in the case of Markov and Bernstein type inequalities, we will be providing estimates on the length of $\nabla P(x,y)$. In this context, the norm of $DP(x,y)$ as a linear form on $\mathbb{R}^m$, denoted by $\|DP(x,y)\|_2$, coincides with the length of the gradient $\nabla P(x,y)$, denoted by $\|\nabla P(x,y)\|_2$. This justifies the simultaneous use of the notations $\|DP(x,y)\|_2$ and $\|\nabla P(x,y)\|_2$.

Having all these considerations in mind, we know that the unit ball in a Banach space is a balanced convex body (that is, a convex, bounded, closed set with non-empty interior) with symmetry with respect to the origin. We can then have

$$\|P\|_D = \sup\{|P(x)| \,:\, x \in D\},$$

for any convex set, D, and then we can apply to $\mathcal{P}(^nE;F)$ all the theory we have summarized at the beginning of the chapter.

We wonder how important the feature of the body being symmetric is. To this aim, we are going to consider, in a vector space, a convex body, C, without symmetry with respect to the origin. We may, in the same fashion, consider $\|P\|_C = \sup\{|P(x)| \,:\, x \in C\}$ (which is no longer a norm, but a seminorm) and study the space $(\mathcal{P}(^nE;F), \|\cdot\|_C)$.

As a final consideration, given a convex non-symmetric convex set C, we will be working on $\mathcal{P}(^2C,\mathbb{R})$, identifying $ax^2 + by^2 + cxy = (a,b,c) \in \mathbb{R}^3$ and $\|(a,b,c)\|_C = \sup\{|ax^2 + by^2 + cxy| \,:\, (x,y) \in C\}$.

Since we are going to consider polynomials taking real values, we shall denote $\mathcal{P}(^2(C))$ to be the space of homogeneous polynomials defined over the vector space where C is contained, endowed with the norm

$$\|P\|_C = \sup\{|P(x)| \,:\, x \in C\}.$$

We will study in detail the polynomials of degree 2 over $\mathbb{R}^2$, where the norm will be consider as maximum taken over the simplex, the unit square (of vertices $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$) and sectors of different amplitude β (the sets $D(\beta) = \{re^{it} \,:\, 0 \leq t \leq \beta,\, 0 \leq r \leq 1\}$).

We shall denote

$$B_C = \left\{P \in \mathcal{P}(^2E) \,:\, \|P\|_C < 1\right\}$$

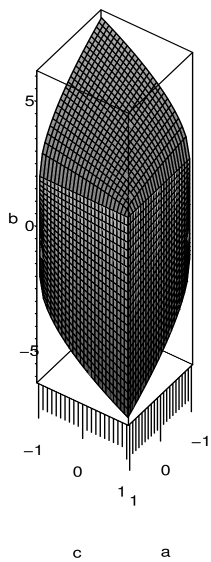


Figure 2.1: Unit ball of $(\mathbb{R}^3, \|\cdot\|_\Delta)$.

2.3 Polynomials over the simplex

Let Δ be the convex hull of the points $(0,0),(0,1)$ and $(1,0)$. Remember that we will identify $ax^2+by^2+cxy = (a,b,c) \in \mathbb{R}^3$ and we will consider $\|(a,b,c)\|_\Delta = \sup\{|ax^2+by^2+cxy| : (x,y) \in \Delta\}$.

2.3.1 The geometry of $\mathcal{P}(^2\Delta)$.

First of all we find a formula for $\|\cdot\|_\Delta$ in the following result.

Theorem 2.3.1 (G.A. Muñoz-Fernández, S.G. Révész, J.B. Seoane-Sepúlveda, [78])**.** *Let $a,b,c \in \mathbb{R}$ and $P(x,y) = ax^2+by^2+cxy$. Then*

$$\|P\|_\Delta = \begin{cases} \max\left\{|a|,|b|,\left|\frac{c^2-4ab}{4(a-c+b)}\right|\right\} & \text{if } a-c+b\neq 0 \text{ and } 0<\frac{2b-c}{2(a-c+b)}<1, \\ \max\{|a|,|b|\} & \text{otherwise.} \end{cases} \tag{2.3.1}$$

In order to parametrize S_Δ it will be useful to know what the projection of S_Δ onto any of the coordinate planes looks like.

Theorem 2.3.2 (G.A. Muñoz-Fernández, S.G. Révész, J.B. Seoane-Sepúlveda, [78])**.** *The projection of S_Δ onto the $ab-$plane is $\mathsf{B}_{\ell_\infty^2}$.*

With the help of Theorems 2.3.1 and 2.3.2, it will be possible to characterize the extreme points of B_Δ.

Theorem 2.3.3 (G.A. Muñoz-Fernández, S.G. Révész, J.B. Seoane-Sepúlveda, [78]). *If we define the mappings*

$$f_+(a,b) = 2 + 2\sqrt{(1-a)(1-b)}$$

and

$$f_-(a,b) = -f_+(-a,-b) = -2 - 2\sqrt{(1+a)(1+b)},$$

for every $(a,b) \in \mathsf{B}_{\ell_\infty^2}$ *and the set*

$$F = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \mathsf{S}_{\ell_\infty^2} \text{ and } f_-(a,b) \le c \le f_+(a,b)\},$$

then

(a) $\mathsf{S}_\Delta = \operatorname{graph}(f_+|_{\mathsf{B}_{\ell_\infty^2}}) \cup \operatorname{graph}(f_-|_{\mathsf{B}_{\ell_\infty^2}}) \cup F.$

(b) $\operatorname{ext}(\mathsf{B}_\Delta) = \{\pm(1, -2-2\sqrt{2(1+t)}, t), \pm(t, -2-2\sqrt{2(1+t)}, 1) : t \in [-1,1]\}.$

Figure 2.1 shows what the unit ball of $(\mathbb{R}^3, \|\cdot\|_\Delta)$ looks like.

2.3.2 Markov Inequality in $\mathcal{P}(^2\Delta)$

Remark that Markov Inequality in $\mathcal{P}(^2\Delta)$ is about the size, i.e. norm, of the linear functional $DP(x,y)$, i.e., the sup of the values attained over the set $B_{\ell_2^2}$, uniformly for all $(x,y) \in \Delta$, as mentioned before. It can also be expressed as the Euclidean norm of the gradient vector $\nabla P(x,y) := \left(\frac{\partial}{\partial x}P(x,y), \frac{\partial}{\partial y}P(x,y)\right)$. In [78], the authors obtained what here we state as main result of this subsection:

Theorem 2.3.4 (G.A. Muñoz-Fernández, S.G. Révész, J.B. Seoane-Sepúlveda, [78]). *Let* $P \in \mathcal{P}(^2\Delta)$ *be arbitrary. Then for any* $(x,y) \in \Delta$ *we have*

$$\|DP(x,y)\|_2 \le 2\sqrt{10} \cdot \|P\|_\Delta,$$

and equality occurs for the polynomial $\pm(x^2+y^2-6xy)$ *and at the points* $(0,1)$ *and* $(1,0)$.

2.3.3 Polarization constant of $\mathcal{P}(^2\Delta)$.

First we find a pointwise gradient estimate on the whole plane $\mathbb{R}^2$ for polynomials in $\mathcal{P}(^2\Delta)$ – so, in a sense a *Bernstein type* inequality – but considering the sup norm over Δ for the gradient. This inequality, restricted to Δ, will provide in Corollary 2.3.6 a sharp *Markov type* inequality for polynomials on Δ and the polarization constant of the space $\mathcal{P}(^2\Delta)$. The latter will be used to show that Martin's inequality (theorem 2.1.9) does not hold with the same constant when working on non symmetric convex bodies.

Theorem 2.3.5 (G.A. Muñoz-Fernández, S.G. Révész, J.B. Seoane-Sepúlveda, [78]). *If for each* $(x,y) \in \mathbb{R}^2$, $\Psi_\Delta(x,y)$ *represents the best constant in*

$$\|DP(x,y)\|_\Delta \le \Psi_\Delta(x,y)\|P\|_\Delta, \qquad \text{for every } P \in \mathcal{P}(^2\Delta), \tag{2.3.2}$$

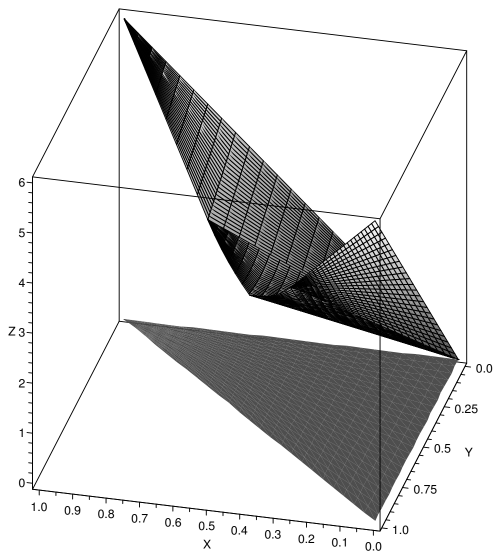


Figure 2.2: $\Psi_\Delta(x, y)$ on the simplex.

then

$$\Psi_\Delta(x,y) = \begin{cases} |2x - 6y| & \text{if } x = 0 \text{ or } x \neq 0 \text{ and } (\frac{y}{x} \leq -1 \text{ or } \frac{y}{x} \geq 2), \\ |2x + 2y + y^2/x| & \text{if } x \neq 0 \text{ and } 1 \leq \frac{y}{x} \leq 2, \\ |2x + 2y + x^2/y| & \text{if } y \neq 0 \text{ and } 1 \leq \frac{x}{y} \leq 2, \\ |6x - 2y| & \text{if } y = 0 \text{ or } y \neq 0 \text{ and } (\frac{x}{y} \leq -1 \text{ or } \frac{x}{y} \geq 2). \end{cases}$$

As a consequence of the previous result we can establish the following Markov type estimate for polynomials in $\mathcal{P}(^2\Delta)$.

Corollary 2.3.6. *If* $P \in \mathcal{P}(^2\Delta)$ *then*

$$\max_{(x,y)\in\Delta} \|DP(x,y)\|_\Delta \leq 6\|P\|_\Delta. \tag{2.3.3}$$

Furthermore, 6 *is optimal in* (2.3.3) *since equality holds for the polynomial* $P(x,y) = x^2 + y^2 - 6xy$.

Remark 2.3.7. *As stated before, if* $P \in \mathcal{P}(^2\Delta)$*, then* $DP(\mathbf{x}) = 2\check{P}(\mathbf{x}, \cdot)$ *for every* $\mathbf{x} \in \mathbb{R}^2$*. This shows that* $\|DP(\mathbf{x})\|_\Delta = 2\|\check{P}\|_\Delta$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *and, hence using* (2.3.3) *we derive*

$$\|\check{P}\|_\Delta \leq 3\|P\|_\Delta.$$

Furthermore, as the constant 3 *here is sharp in view of Corollary 2.3.6, Martin's inequality* (2.1.9) *does not hold for polynomials on a non symmetric convex body.*

2.3.4 Unconditional constant of the simplex

Theorem 2.3.8 (B.C. Grecu, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [52]). *If $P \in \mathcal{P}(^2\Delta)$ then*

$$\||P|\|_\Delta \leq 2\|P\|_\Delta,$$

and 2 is optimal in the previous inequality. Therefore the unconditional constant (referred to the canonical basis) of $\mathcal{P}(^2\Delta)$ is 2.

2.4 Inequalities on the unit square

Define $\square$ to be the convex hull of the points $(0,0),(0,1),(1,0),(1,1)$. Again, we will be working on $\mathcal{P}(^2\square;\mathbb{R})$ and we will identify $(\mathcal{P}(^2\square;\mathbb{R}),\|\cdot\|_\square)$ with $(\mathbb{R}^3,\|\cdot\|_\square)$, setting $\|(a,b,c)\|_\square = \|ax^2+by^2+cxy\|_\square = \sup\{|ax^2+by^2+cxy : (x,y)\in\square\}$

2.4.1 The geometry of $\mathcal{P}(^2\square)$

First of all we obtain a formula for $\|\cdot\|_\square$:

Theorem 2.4.1 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55]). *If $P(x,y) = ax^2+by^2+cxy$, then*

$$\|P\|_\square = \begin{cases} \max\left\{|a|,|b|,|a+b+c|,\frac{c^2-4ab}{4|b|}\right\} & \text{if } c^2-4ab>0,\ b\neq 0 \text{ and } -\frac{c}{2b}\in(0,1). \\ \max\left\{|a|,|b|,|a+b+c|,\frac{c^2-4ab}{4|a|}\right\} & \text{if } c^2-4ab>0,\ a\neq 0 \text{ and } -\frac{c}{2a}\in(0,1). \\ \max\{|a|,|b|,|a+b+c|\} & \text{otherwise.} \end{cases}$$

In order to sketch $\mathsf{S}_\square$ and obtain the extreme points of $\mathsf{B}_\square$, it is important to have a parametrization of $\mathsf{S}_\square$. This parametrization can be constructed by projecting $\mathsf{B}_\square$ onto the ab-plane. For this matter we will use the subsets A, B and C of $[-1,1]^2$ (see Figure 2.3):

$$\begin{aligned} A &:= \{(a,b)\in[-1,1]^2 : -1\leq a\leq 0 \text{ and } a+1\leq b\leq 1\}, \\ B &:= \left\{(a,b)\in[-1,1]^2 : -1\leq a\leq 1 \text{ and } \max\{-1,a-1\}\leq b\leq \min\{1,a+1\}\right\}, \\ C &:= \{(a,b)\in[-1,1]^2 : 0\leq a\leq 1 \text{ and } -1\leq b\leq a-1\}. \end{aligned}$$

Theorem 2.4.2 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55]). *The projection of $\mathsf{S}_\square$ onto the $ab-$plane is $[-1,1]^2$.*

Theorem 2.4.3 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55]). *If for every $(a,b)\in[-1,1]^2$ we define the mappings*

$$F(a,b) = \begin{cases} 2\sqrt{ab+|a|} & \text{if } (a,b)\in A, \\ 2\sqrt{ab+|b|} & \text{if } (a,b)\in C, \\ 1-a-b & \text{if } (a,b)\in B, \end{cases}$$

$$G(a,b) = -F(-a,-b),$$

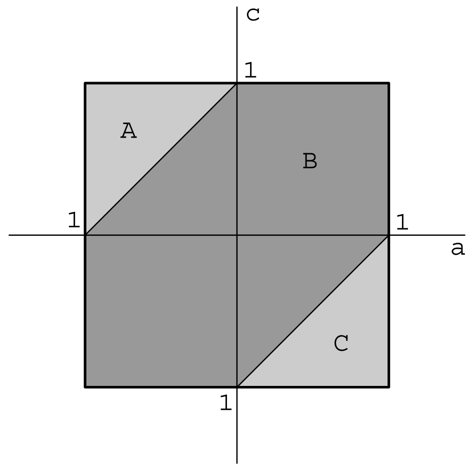


Figure 2.3: projection of $\mathsf{B}_\square$) onto the $ab-$plane.

where A, B and C are as in Figure 2.3 and the set

$$H = \{(a,b,c) \in \mathbb{R}^3 : (a,b) \in \partial[-1,1]^2 \text{ and } G(a,b) \le c \le F(a,b)\},$$

then

(a) $\mathsf{S}_\square = \mathrm{graph}(F) \cup \mathrm{graph}(G) \cup H$.

(b) *The extreme points of* $\mathsf{B}_\square$ *have the form*

$$\pm(t,-1,2\sqrt{1-t}) \quad \text{and} \quad \pm(-1,t,2\sqrt{1-t}) \quad \text{with } t \in [0,1]$$

or

$$\pm(1,1,-1),\ \pm(1,1,-3),\ \pm(1,0,0),\ \pm(0,1,0).$$

2.4.2 Markov and Bernstein inequalities in $\mathcal{P}(^2\square)$

Theorem 2.4.4 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55])**.** *If* $P \in \mathcal{P}(^2\square)$ *then we have*

$$\|DP(x,y)\|_2 \le \mathcal{M}(x,y) \cdot \|P\|_\square, \tag{2.4.1}$$

for every $(x,y) \in \square$, *where*

$$\mathcal{M}(x,y) = \begin{cases} \sqrt{\frac{24y^4+12x^2y^2+x^4+x(8y^2+x^2)^{\frac{3}{2}}}{8y^2}} & \text{if } 0 < \alpha_0 x \le y \le x, \\ \sqrt{\frac{24x^4+12x^2y^2+y^4+y(8x^2+y^2)^{\frac{3}{2}}}{8x^2}} & \text{if } 0 < x \le y \le \frac{x}{\alpha_0}, \\ \sqrt{13x^2+13y^2-24xy} & \text{otherwise,} \end{cases}$$

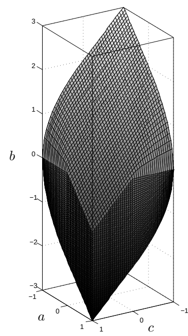


Figure 2.4: The unit sphere $\mathsf{B}_\square$.

and $\alpha_0 \approx 0.4029036618$ *is the unique root of the equation*

$$80\alpha^4 - 192\alpha^3 + 92\alpha^2 - 1 = (8\alpha^2 + 1)^{\frac{3}{2}}$$

in the interval $[\frac{3-\sqrt{5}}{2}, \frac{12-3\sqrt{3}}{13}]$*. Moreover, the inequality is sharp.*

Corollary 2.4.5. *If* $P \in \mathcal{P}(^2\square)$ *then for any* $(x, y) \in \square$ *we have*

$$\|DP(x, y)\|_2 \leq \sqrt{13} \cdot \|P\|_\square.$$

Equality occurs for the polynomials $\pm(x^2 + y^2 - 3xy)$ *at the points* $(0, 1)$ *and* $(1, 0)$*.*

2.4.3 Polarization constant of $\mathcal{P}(^2\square)$

First we obtain the following Bersntein type inequality where only the norm $\|\cdot\|_\square$ is considered.

Theorem 2.4.6 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55])**.** *If for each* $(x, y) \in \square$*,* $\Psi_\square(x, y)$ *represents the best constant in*

$$\|DP(x, y)\|_\square \leq \Psi_\square(x, y)\|P\|_\square, \qquad \text{for every } P \in \mathcal{P}(^2\square), \tag{2.4.2}$$

then

$$\Psi_\square(x, y) = \begin{cases} 3x - 2y & \text{if } y \leq (\sqrt{2} - 1)x, \\ \frac{5}{2}x - y + \frac{y^2}{2x} & \text{if } x \neq 0 \text{ and } (\sqrt{2} - 1)x \leq y \leq \frac{1}{2}x, \\ 2x + \frac{y^2}{2x} & \text{if } x \neq 0 \text{ and } \frac{1}{2}x \leq y \leq x, \\ 2y + \frac{x^2}{2y} & \text{if } y \neq 0 \text{ and } x \leq y \leq 2x, \\ \frac{5}{2}y - x + \frac{x^2}{2y} & \text{if } y \neq 0 \text{ and } 2x \leq y \leq (\sqrt{2} + 1)x, \\ 3y - 2x & \text{if } (\sqrt{2} + 1)x \leq y. \end{cases}$$

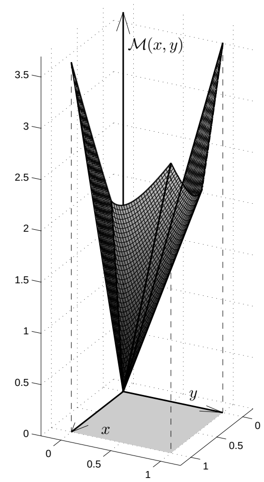


Figure 2.5: The Bernstein mapping $\mathcal{M}(x,y)$ for $\square$.

Corollary 2.4.7. *If* $P \in \mathcal{P}(^2\square)$ *then*

$$\max_{(x,y)\in\square} \|DP(x,y)\|_\square \le 3\|P\|_\square. \tag{2.4.3}$$

Furthermore, 3 *is optimal in* (2.4.3) *since equality holds for the polynomials* $P(x,y) = \pm(x^2+y^2-3xy)$.

Corollary 2.4.8. *If* $P \in \mathcal{P}(^2\square)$ *and* $L \in \mathcal{L}_s(^2\square)$ *is the polar of* P, *then*

$$\|L\|_\square \le \frac{3}{2}\|P\|_\square. \tag{2.4.4}$$

Furthermore, $\frac{3}{2}$ *is optimal in* (2.4.4) *since equality holds for the polynomials* $P(x,y) = \pm(x^2+y^2-3xy)$.

2.4.4 Unconditional constant of $\mathcal{P}(^2\square)$

Theorem 2.4.9 (J.L. Gámez-Merino, G.A. Muñoz-Fernández, V. Sánchez, J.B. Seoane-Sepúlveda [55])**.** *If* $P \in \mathcal{P}(^2\square)$ *then*

$$|||P|||_\square \le 5\|P\|_\square.$$

Equality is attained for the polynomials $P(x,y) = \pm(x^2+y^2-3xy)$. *Therefore the unconditional constant of the canonical basis of* $\mathcal{P}(^2\square)$ *is* 5.

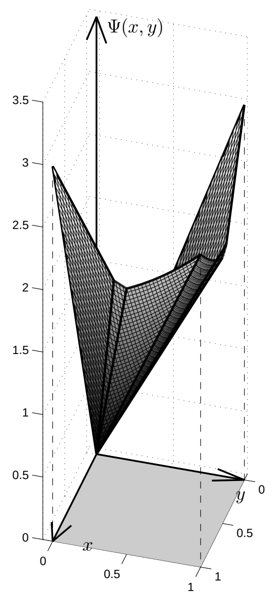


Figure 2.6: $\Psi_{\square}(x, y)$ for $(x, y) \in \square$.

2.5 Polynomials on $B_{D\left(\frac{\pi}{4}\right)}$

In the following two sections, we will be considering the sets

$$D\left(\frac{\pi}{4}\right)=\left\{re^{i\theta}\,:\,0\leq r\leq 1,\,0\leq\theta\leq\frac{\pi}{4}\right\},$$
$$D\left(\frac{\pi}{2}\right)=\left\{re^{i\theta}\,:\,0\leq r\leq 1,\,0\leq\theta\leq\frac{\pi}{2}\right\}.$$

Identify $(\mathcal{P}(^2D(\cdot)),\|\cdot\|_{D(\cdot)})=(\mathbb{R}^3,\|\cdot\|_{D(\cdot)})$, as usual, with $ax^2+by^2+cxy=(a,b,c)$ and $\|(a,b,c)\|_{D(\cdot)}=\sup\{|ax^2+by^2+cxy|\,:\,(x,y)\in D(\cdot)\}$.
Even though we will be dealing with the sectors of width angle $\frac{\pi}{4}$ and $\frac{\pi}{2}$, respectively, in [77] the authors study the geometry of sectors of arbitrary width, paying also special attention to the amplitude $\frac{3\pi}{4}$.

2.5.1 The geometry of $D\left(\frac{\pi}{4}\right)$

We shall follow an analogous procedure as in sections 2.3 and 2.4: first we shall have a complete description of the set that constitutes the extremal points of the unit ball, and then we shall consider those to simplify the calculations when searching for the maximum of the different functions that appear.
To this aim, we obtain first a formula for $\|ax^2+by^2+cxy\|_{D(\beta)}$, where $\beta=\frac{\pi}{4},\frac{\pi}{2}$ and $a,b,c\in\mathbb{R}$.

Theorem 2.5.1 (G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, A. Weber, [77])**.** *If $a,b,c\in\mathbb{R}$ and $P(x,y)=ax^2+by^2+cxy$ then $\|P\|_{D(\frac{\pi}{4})}$ and $\|P\|_{D(\frac{\pi}{2})}$ are given, respectively, by*

$$\begin{cases}\max\left\{|a|,\frac{1}{2}|a+b+c|,\frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\} & \text{if } c(a-b)\geq 0,\\ \max\{|a|,\frac{1}{2}|a+b+c|\} & \text{if } c(a-b)\leq 0,\end{cases}$$

$$\max\left\{|a|,|b|,\frac{1}{2}|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}|\right\}.$$

Proof. As usual, the Krein-Milman theorem tells us that the supremum of $|P|$ over $D(\beta)$ is obviously attained on the set $\{(\cos\theta,\sin\theta):0\leq\theta\leq\beta\}$. It is easy to see that, when restricted to that set, P is given by

$$f(\theta)=\frac{1}{2}\left[a+b+(a-b)\cos 2\theta+c\sin 2\theta\right],$$

for $\theta\in[0,\beta]$. Let

$$g(\theta)=\frac{1}{2}\left[a+b+(a-b)\cos\theta+c\sin\theta\right],$$

for $\theta\in[0,2\beta]$. Then obviously $\|P\|_{D(\beta)}=\sup_{\theta\in[0,2\beta]}|g(\theta)|$. The study of the simpler case $c(a-b)=0$ is solved by elementary calculations (observe that in this case the formula of the norm for both sectors coincide). If now $c(a-b)\neq 0$, then g' vanishes at the points θ_0 such that $\tan\theta_0=\frac{c}{a-b}$.

Assume first that $\beta=\frac{\pi}{4}$. Then $\tan\theta=\frac{c}{a-b}$ has exactly one root $\theta_0\in[0,\frac{\pi}{2}]$ if and only if $\frac{c}{a-b}>0$, in which case

$$\cos\theta_0=\frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}}\qquad\text{and}\qquad\sin\theta_0=\frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}.$$

Hence

$$\begin{aligned} g(\theta_0) &= \frac{1}{2}\left[a+b+(a-b)\frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}}+c\frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}\right] \\ &= \frac{1}{2}\left[a+b+\frac{(a-b)|a-b|}{\sqrt{c^2+(a-b)^2}}+\frac{c^2\operatorname{sign}(a-b)}{\sqrt{c^2+(a-b)^2}}\right] \\ &= \frac{1}{2}\left[a+b+\operatorname{sign}(a-b)\frac{(a-b)^2+c^2}{\sqrt{c^2+(a-b)^2}}\right] \\ &= \frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right]. \end{aligned}$$

This, together with the fact that $g(0)=a$ and $g(\frac{\pi}{2})=\frac{1}{2}(a+b+c)$, proves the first formula.

Suppose that now $\beta=\frac{\pi}{2}$. Then $\tan\theta=\frac{c}{a-b}$ has exactly one solution $\theta_0\in[0,\pi]$. If $c(a-b)>0$ then $\theta_0\in[0,\frac{\pi}{2}]$ and we have already seen that $g(\theta_0)=\frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right]$. If $c(a-b)<0$ then $\theta_0\in[\frac{\pi}{2},\pi]$ and

$$\cos\theta_0=-\frac{1}{\sqrt{1+\frac{c^2}{(a-b)^2}}} \qquad \text{and} \qquad \sin\theta_0=-\frac{\frac{c}{a-b}}{\sqrt{1+\frac{c^2}{(a-b)^2}}}.$$

It is finally easy to check, as above, that in this case we obtain again

$$g(\theta_0)=\frac{1}{2}\left[a+b+\operatorname{sign}(c)\sqrt{c^2+(a-b)^2}\right].$$

This, together with the fact that $g(0)=a$ and $g(\pi)=b$, proves the second formula.

□

Before starting to compute the different constants in this section, we shall prove the following technical result:

Lemma 2.5.2. *Let us define P_1 and P_2 by*

$$P_1:=\{(a,b)\in\mathbb{R}^2:\ a\geq -1 \text{ and } 4+a-4\sqrt{1+a}\leq b\leq 4+a+4\sqrt{1+a}\},$$
$$P_2:=\{(a,b)\in\mathbb{R}^2:\ a\leq 1 \text{ and } -4+a-4\sqrt{1-a}\leq b\leq -4+a+4\sqrt{1-a}\}.$$

Then:

(a) The inequality

$$\frac{1}{2}\left|a+b-\sqrt{2\left[(a-1)^2+(b-1)^2\right]}\right|\leq 1, \tag{2.5.1}$$

holds if and only if $(a,b)\in P_1$.

(b) The inequality

$$\frac{1}{2}\left|a+b+2\sqrt{(1-a)(1-b)}\right|\leq 1, \tag{2.5.2}$$

holds if and only if $(a,b)\in P_2$.

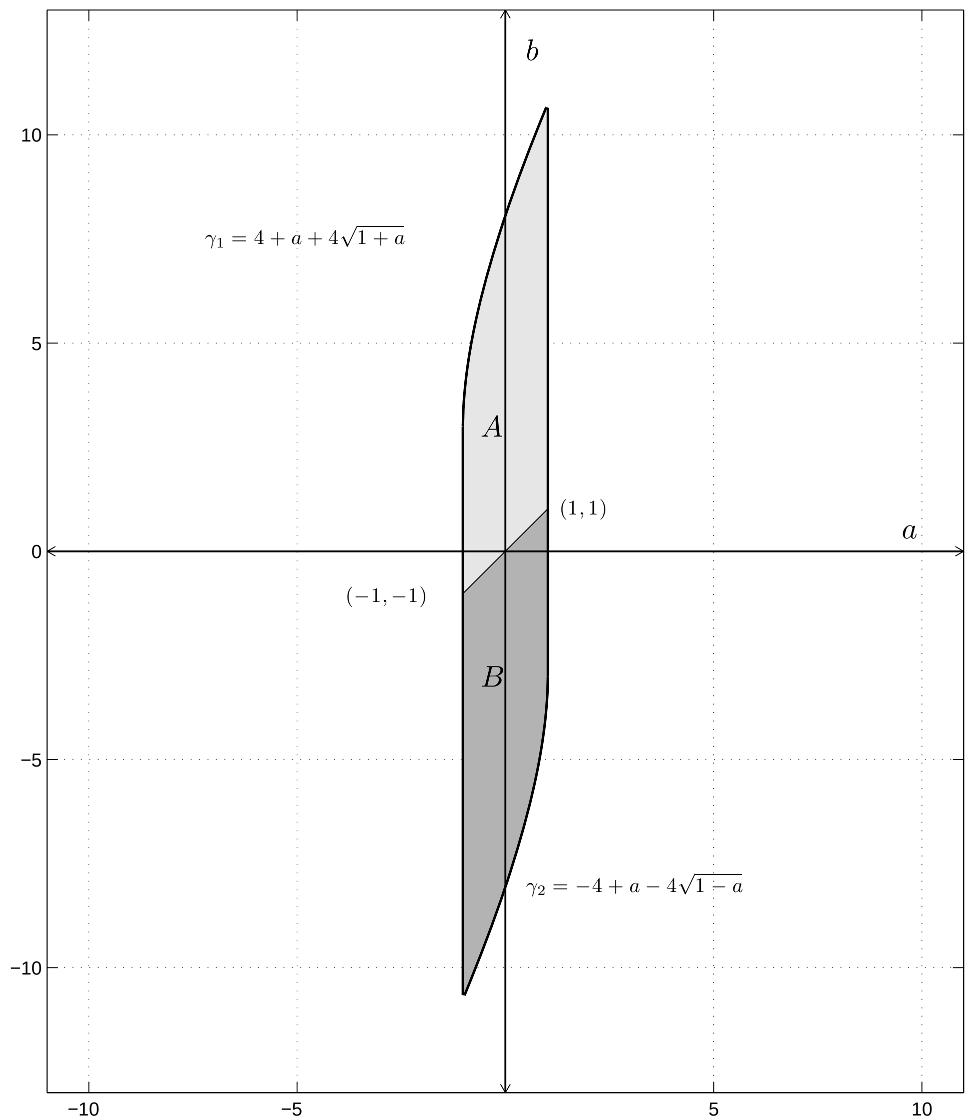


Figure 2.7: Projection of $\mathsf{S}_{D(\frac{\pi}{4})}$ onto the ab-plane

Theorem 2.5.3 (G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, A. Weber, [77]). *Let A and B be as in Figure 2.7, namely*

$$A = \{(a,b) : \ a \in [-1,1], \ a < b \le \gamma_1(a)\},$$
$$B = \{(a,b) : \ a \in [-1,1], \ \gamma_2(a) \le b \le a\},$$

where γ_1, γ_2 are defined by

$$\gamma_1(a) = 4 + a + 4\sqrt{1+a},$$
$$\gamma_2(a) = -\gamma_1(-a) = -4 + a - 4\sqrt{1-a},$$

for $a \in [-1,1]$. Then, the projection of $\mathsf{S}_{D(\frac{\pi}{4})}$ over the ab-plane is

$$\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})}) = \{(a,b) : \ a \in [-1,1], \ \gamma_2(a) \le b \le \gamma_1(a)\}.$$

Proof. Let $(a,b) \in A$ and set $c = 2 - a - b$. Suppose first that $2 - a - b > 0$. Then $c(a-b) < 0$, and therefore, according to the first formula in Theorem 2.5.1 we have

$$\|(a,b,2-a-b)\|_{D(\frac{\pi}{4})} = \max\left\{|a|, \frac{1}{2}|a+b+c|\right\} = \max\{|a|,1\} = 1.$$

On the other hand, if $2 - a - b \le 0$, then $c(a-b) \ge 0$. Applying now the first formula in Theorem 2.5.1 again we obtain

$$\begin{aligned}\|(a,b,2-a-b)\|_{D(\frac{\pi}{4})} &= \max\left\{|a|, 1, \frac{1}{2}\left|a+b-\sqrt{(a-b)^2+(2-a-b)^2}\right|\right\}\\ &= \max\left\{|a|, 1, \frac{1}{2}\left|a+b-\sqrt{2[(a-1)^2+(b-1)^2]}\right|\right\}.\end{aligned}$$

Recall that here (a,b) satisfies $|a| \le 1$ and $2-a \le b \le 4+a+4\sqrt{1+a}$. Since $4+a-4\sqrt{1+a} \le 2-a$ for $-1 \le a \le 1$, it follows from Lemma 2.5.2, part (a), that

$$\frac{1}{2}\left|a+b-\sqrt{2[(a-1)^2+(b-1)^2]}\right| \le 1,$$

proving that $\|(a,b,2-a-b)\|_{D(\frac{\pi}{4})} = 1$.

Now assume that $(a,b) \in B$(see Figure 2.7), and define $c = 2\sqrt{(1-a)(1-b)}$. Notice that if $(a,b) \in B$ then $2 - a - b \ge 0$, from which

$$\begin{aligned}\frac{1}{2}\left|a+b+\operatorname{sign}(c)\sqrt{(a-b)^2+c^2}\right| &= \frac{1}{2}\left|a+b+\sqrt{(a-b)^2+4(1-a)(1-b)}\right|\\ &= \frac{1}{2}\left|a+b+\sqrt{(2-a-b)^2}\right| = 1.\end{aligned}$$

In this case $c(a-b) \ge 0$, so using the first formula in Theorem 2.5.1 once again, we arrive at

$$\left\|(a,b,2\sqrt{(1-a)(1-b)})\right\|_{D(\frac{\pi}{4})} = \max\left\{|a|, \frac{1}{2}\left|a+b+2\sqrt{(1-a)(1-b)}\right|, 1\right\}.$$

However, under the given conditions, Lemma 2.5.2, part (b) asserts that

$$\frac{1}{2}\left|a+b+2\sqrt{(1-a)(1-b)}\right| \leq 1,$$

which implies that $\left\|(a,b,2\sqrt{(1-a)(1-b)})\right\|_{D(\frac{\pi}{4})} = 1$.

We have proved so far that

$$\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})}) \subset \{(a,b):\ a\in[-1,1],\ \gamma_2(a)\leq b\leq \gamma_1(a)\}.$$

On the other hand, if $(a,b)\notin\{(a,b):\ a\in[-1,1],\ \gamma_2(a)\leq b\leq\gamma_1(a)\}$ then either $|a|>1$ or one of the inequalities $\gamma_2(a)\leq b\leq\gamma_1(a)$ fail with $-1\leq a\leq 1$. In the first case we would have $\|(a,b,c)\|_{D(\frac{\pi}{4})}\geq |a|>1$ for all $c\in\mathbb{R}$, i.e., $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. Now if $|a|\leq 1$ and $\gamma_1(a)<b$, then $a-b<0$. If $c\leq 0$, from the first formula in Theorem 2.5.1 we would have

$$\|(a,b,c)\|_{D(\frac{\pi}{4})} \geq \max\left\{\frac{1}{2}|a+b+c|,\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|\right\}.$$

It can be easily checked that $\frac{1}{2}|a+b+c|>1$ if $0\geq c>2-a-b$ and $\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|>1$ if $c\leq 2-a-b$. This last statement follows from the fact that $\frac{1}{2}\left|a+b-\sqrt{(a-b)^2+c^2}\right|>1$ is equivalent to the expression $c^2>4(1+a)(1+b)$ and, in this case, $c^2\geq(2-a-b)^2>4(1+a)(1+b)$.

Now if $c>0$, Theorem 2.5.1 allows us to show that

$$\|(a,b,c)\|_{D(\frac{\pi}{4})} \geq \frac{1}{2}|a+b+c| = \frac{1}{2}(a+b+c) > 2+a+2\sqrt{1+a} \geq 1,$$

whenever $-1\leq a\leq 1$. In any case $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$.

Finally, using the symmetry of the unit ball and the previous case, if $\gamma_2(a)>b$ with $-1\leq a\leq 1$ then $\|(a,b,c)\|_{D(\frac{\pi}{2})}>1$ for all $c\in\mathbb{R}$, from which $(a,b)\notin\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. This concludes the proof. □

Remark 2.5.4. *In the proof of the first part of the following result, it will be useful to observe that*

$$\frac{1}{2}|b+c-1|\leq 1 \Leftrightarrow -1-b\leq c\leq 3-b.$$

Also, if $b\leq -1$, *then*

$$\frac{1}{2}\left|b-1+\sqrt{(b+1)^2+c^2}\right|\leq 1 \Leftrightarrow |c|\leq 3-b.$$

Theorem 2.5.5 (G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, A. Weber, [77])**.** *Let* A *and* B *be as in Theorem 2.5.3 (see also Figure 2.7) and define*

$$F_1(a,b)=\begin{cases}2-a-b & \text{if } (a,b)\in A,\\ 2\sqrt{(1-a)(1-b)} & \text{if } (a,b)\in B,\end{cases}$$

and $F_2(a,b)=-F_1(-a,-b)$ *for all* $(a,b)\in\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})})$. *If*

$$\Gamma=\{(\pm 1,b,c)\in\mathbb{R}^2:\ (\pm 1,b)\in\partial\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{4})}),\ F_2(\pm 1,b)\leq c\leq F_1(\pm 1,b)\},$$

then

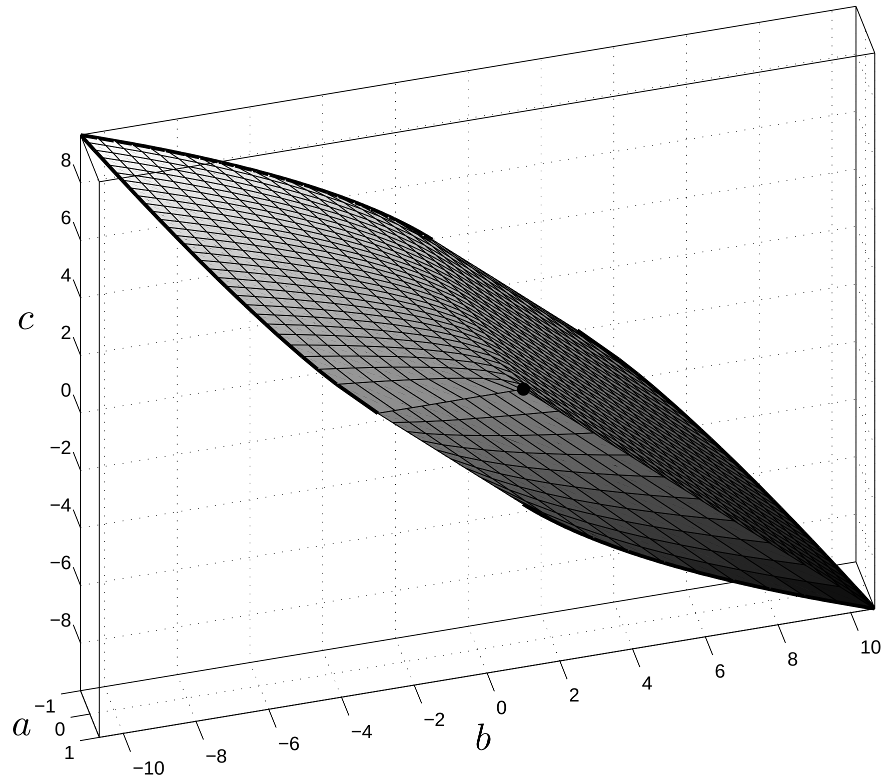


Figure 2.8: $\mathsf{S}_{D(\frac{\pi}{4})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{4})}$ are drawn with a thicker line and dots

(a) $\mathsf{S}_{D(\frac{\pi}{4})} = \mathrm{graph}(F_1) \cup \mathrm{graph}(F_2) \cup \Gamma$.

(b) The set $\mathrm{ext}(\mathsf{B}_{D(\frac{\pi}{4})})$ *consists of the elements*

$$\pm\left(t, 4 + t + 4\sqrt{1+t}, -2 - 2t - 4\sqrt{1+t}\right) \quad \textit{for} \quad t \in [-1, 1],$$

$$\pm\left(1, s, -2\sqrt{2(1+s)}\right) \quad \textit{for} \quad s \in [1, 5 + 4\sqrt{2}],$$

and

$$\pm(1, 1, 0).$$

Proof. As for the first part of the theorem, notice that $\mathrm{graph}(F_1) \subset \mathsf{S}_{D(\frac{\pi}{4})}$ as seen in the proof of Theorem 2.5.3. By symmetry we also have that $\mathrm{graph}(F_2) \subset \mathsf{S}_{D(\frac{\pi}{4})}$. Finally $\Gamma \subset \mathsf{S}_{D(\frac{\pi}{4})}$ too. Indeed, by symmetry we can focus on the study of the points of Γ of the form $(-1, b, c)$ with $-5 - 4\sqrt{2} \le b \le 3$ and $F_2(-1, b) \le c \le F_1(-1, b)$. Observe that

$$F_1(-1, b) = \begin{cases} 2\sqrt{2(1-b)} & \text{if } -5 - 4\sqrt{2} \le b \le -1, \\ 3 - b & \text{if } -1 \le b \le 3, \end{cases}$$

and

$$F_2(-1,b) = \begin{cases} -1-b & \text{if } -5-4\sqrt{2} \le b \le -1, \\ 0 & \text{if } -1 \le b \le 3. \end{cases}$$

Hence $c \ge F_2(-1,b) \ge 0$ for all $b \in [-5-4\sqrt{2}, 3]$ and $c = 0$ can only be zero when $-3 \le b \le 1$. In that case, since $\frac{1}{2}|b-1| \le 1$, it follows from Theorem 2.5.1 that

$$\|(-1,b,0)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b-1|\right\} = 1.$$

Otherwise $c > 0$. First, if $-5-4\sqrt{2} \le b \le -1$ then $c(-1-b) \ge 0$. Therefore using Theorem 2.5.1 we have

$$\|(-1,b,c)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b+c-1|, \frac{1}{2}|b-1+\sqrt{(b+1)^2+c^2}|\right\}.$$

Since $-5-4\sqrt{2} \le b \le -1$ and $-3-b \le c \le 2\sqrt{2(1-b)} \le 3-b$, from Remark 2.5.4 it follows that $\|(-1,b,c)\|_{D(\frac{\pi}{4})} = 1$. Finally, if $-1 < b \le 3$ then $c(-1-b) < 0$, which implies, from Theorem 2.5.1 that

$$\|(-1,b,c)\|_{D(\frac{\pi}{4})} = \max\left\{1, \frac{1}{2}|b+c-1|\right\}.$$

Since now $-1 \le b \le 3$ and $-3-b \le 0 < c \le -1-b$, from Remark 2.5.4 we have that $\|(-1,b,c)\|_{D(\frac{\pi}{4})} = 1$ too. We have proved so far that $\text{graph}(F_1) \cup \text{graph}(F_2) \cup \Gamma \subset \mathsf{S}_{D(\frac{\pi}{4})}$. On the other hand, suppose $(a,b,c) \notin \text{graph}(F_1) \cup \text{graph}(F_2) \cup \Gamma$. Obviously, $(0,0,0) \notin \mathsf{S}_{D(\frac{\pi}{4})}$, so we can also assume that $(a,b,c) \ne (0,0,0)$. The straight line $\{\lambda(a,b,c) : \lambda \in \mathbb{R}\}$ certainly meets the set $\text{graph}(F_1) \cup \text{graph}(F_2) \cup \Gamma$. Put, $(a,b,c) = \lambda_0(a_0,b_0,c_0)$ with $\lambda_0 \ne 0, 1$ and $(a_0,b_0,c_0) \in \text{graph}(F_1) \cup \text{graph}(F_2) \cup \Gamma$. Then $\|(a,b,c)\|_{D(\frac{\pi}{4})} = \lambda_0\|(a_0,b_0,c_0)\|_{D(\frac{\pi}{4})} = \lambda_0 \ne 1$, and therefore $(a,b,c) \notin \mathsf{S}_{(\frac{\pi}{4})}$. This concludes part (a).

Part (b) is an easy consequence of the fact that the sets $\text{graph}(F_1)$, $\text{graph}(F_2)$ and Γ are ruled surfaces in $\mathbb{R}^3$. Actually Γ is contained in the two planes $a = \pm 1$. Hence the only possible extreme points are contained in the intersections $\text{graph}(F_1) \cap \Gamma$, $\text{graph}(F_2) \cap \Gamma$ and $\text{graph}(F_1) \cap \text{graph}(F_2)$. Once we have removed the interior of all straight lines contained in those intersections, we end up with the following candidates to be extreme points of $\mathsf{B}_{D(\frac{\pi}{4})}$:

$$\pm\left(t, 4+t+4\sqrt{1+t}, -2-2t-4\sqrt{1+t}\right) \quad \text{for} \quad t \in [-1,1],$$

$$\pm\left(1, s, -2\sqrt{2(1+s)}\right) \quad \text{for} \quad s \in [1, 5+4\sqrt{2}],$$

and

$$\pm(1,1,0).$$

Constructing a supporting hyperplane to each of the previous points is a straightforward exercise and the details shall not be taken into consideration. This last observation finishes the proof. □

We reproduce a sketch of $\mathsf{S}_{D(\frac{\pi}{4})}$ in Figure 2.8.

2.5.2 Markov and Bernstein inequalities in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$

Theorem 2.5.6 (G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [1])**.** *(Bernstein inequality) For every $(x,y)\in D\left(\frac{\pi}{4}\right)$ the following inequality is sharp:*

$$\|\nabla P(x,y)\|_2 \leq \Phi_{\pi/4}(x,y)\|P\|_{D(\frac{\pi}{4})},$$

where $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and

$$\Phi_{\pi/4}(x,y)=\begin{cases} F(x,y) & \text{if } 0\leq y\leq \frac{\sqrt{2}-1}{2}x \text{ or } \left(4\sqrt{2}-5\right)x\leq y\leq x,\\ G(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x\leq y\leq \left(\sqrt{2}-1\right)x,\\ H(x,y) & \text{if } \left(\sqrt{2}-1\right)x\leq y\leq\left(4\sqrt{2}-5\right)x,\end{cases}$$

and

$$\begin{aligned} F(x,y)&=2\sqrt{\left(13+8\sqrt{2}\right)x^2+\left(69+48\sqrt{2}\right)y^2-2\left(28+20\sqrt{2}\right)xy},\\ G(x,y)&=\sqrt{\frac{x^4}{y^2}+4(x^2+y^2)},\\ H(x,y)&=\sqrt{\frac{9x^4-12x^3y+22x^2y^2-12xy^3+9y^4}{2(x-y)^2}}.\end{aligned}$$

Proof. In order to calculate $\Phi_{\pi/4}(x,y):=\sup\{\|\nabla P(x,y)\|_2 : \|P\|_{D(\frac{\pi}{4})}\leq 1\}$, by the Krein-Milman approach, it is sufficient to calculate

$$\sup\{\|\nabla P(x,y)\|_2 : P\in \operatorname{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Theorem 2.5.5 with positive sign. Let us start first with $P_t(x,y)=tx^2+\left(4+t+4\sqrt{1+t}\right)y^2-2\left(1+t+2\sqrt{1+t}\right)xy$, $t\in[-1,1]$. Then,

$$\nabla P_t(x,y)=\left(2tx-2\left(1+t+2\sqrt{1+t}\right)y, 2\left(4+t+4\sqrt{1+t}\right)y-2\left(1+t+2\sqrt{1+t}\right)x\right),$$

so that

$$\begin{aligned}\|\nabla P_t(x,y)\|_2^2 =& 4t^2x^2+4\left(1+t+2\sqrt{1+t}\right)^2y^2-8t\left(1+t+2\sqrt{1+t}\right)xy\\ &+4\left(4+t+4\sqrt{1+t}\right)^2y^2+4\left(1+t+2\sqrt{1+t}\right)^2x^2\\ &-8\left(4+t+4\sqrt{1+t}\right)\left(1+t+2\sqrt{1+t}\right)xy\end{aligned}$$

Make now the change $u=\sqrt{1+t}\in\left[0,\sqrt{2}\right]$, so that

$$\begin{aligned}\|\nabla P_u(x,y)\|_2^2 =& 8(x-y)^2u^4+16\left(x^2-4xy+3y^2\right)u^3\\ &+8\left(x^2-10xy+13y^2\right)u^2+32\left(3y^2-xy\right)u+4\left(x^2+9y^2\right).\end{aligned}$$

Since

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2=16\left(2\left(x-y\right)^2u^2+\left(x^2-8xy+7y^2\right)u+2y\left(3y-x\right)\right)\left(u+1\right),$$

it follows that the critical points of $\|DP_u(x,y)\|_2^2$ are $u=\frac{2y}{x-y}$, $u=\frac{3y-x}{2(x-y)}$ and $u=-1$ if $x\neq y$ and $u=4$ and $u=-1$ if $x=y$. Since we need to consider $0\leq u\leq\sqrt{2}$, we can directly omit the case $x=y$.

Therefore, we can write

$$\frac{\partial}{\partial u}\|\nabla P_u(x,y)\|_2^2=32(x-y)^2\left(u-\frac{2y}{x-y}\right)\left(u-\frac{3y-x}{2(x-y)}\right)(u+1).$$

Let $u_1=\frac{2y}{x-y}$ and $u_2=\frac{3y-x}{2(x-y)}$ (Again, since we need to consider $0\leq u\leq\sqrt{2}$, we can omit the solution $u=-1$). Also, we have the extra conditions $u_1\in[0,\sqrt{2}]$ whenever $0\leq y\leq\left(\sqrt{2}-1\right)x$ and $u_2\in[0,\sqrt{2}]$ whenever $\frac{1}{3}x\leq y\leq\left(4\sqrt{2}-5\right)x$. Considering all these facts, we need to compare the quantities

$$\begin{aligned}C_1(x,y):=\|\nabla P_{u_1}(x,y)\|_2^2=\|\nabla P_{t_1}\|_2^2&=4\frac{x^6-4x^5y+7x^4y^2-8x^3y^3+7x^2y^4-4xy^5+y^6}{(x-y)^4}\\&=4\left(x^2+y^2\right),\end{aligned}$$

for $0\leq y\leq\left(\sqrt{2}-1\right)x$ and $t_1=\frac{3y^2+2xy-x^2}{(x-y)^2}$,

$$\begin{aligned}C_2(x,y):=\|\nabla P_{u_2}(x,y)\|_2^2=\|\nabla P_{t_2}\|_2^2&=\frac{9x^6-30x^5y+55x^4y^2-68x^3y^3+55x^2y^4-30xy^5+9y^6}{2(x-y)^4}\\&=\frac{\left(3x^2-2xy+3y^2\right)^2}{2(x-y)^2},\end{aligned}$$

for $\frac{1}{3}x\leq y\leq\left(4\sqrt{2}-5\right)x$ and $t_2=\frac{5y^2+2xy-3x^2}{4(x-y)^2}$,

$$C_3(x,y):=\|\nabla P_{t_3=-1}\|_2^2=4\left(x^2+9y^2\right),$$

and

$$C_4(x,y):=\|\nabla P_{t_4=1}\|_2^2=4\left[\left(13+8\sqrt{2}\right)x^2+\left(69+48\sqrt{2}\right)y^2-2\left(28+20\sqrt{2}\right)xy\right].$$

Let us focus now on $Q_s=\left(1,s,-2\sqrt{2(1+s)}\right)$, $1\leq s\leq 5+4\sqrt{2}$. Then, we have

$$\|\nabla Q_s(x,y)\|_2^2=4x^2+4s^2y^2+8(1+s)(x^2+y^2)-8(1+s)\sqrt{2(1+s)}xy.$$

Making the change $v=\sqrt{2(1+s)}\in\left[2,2+2\sqrt{2}\right]$, we need to study the function

$$\|\nabla Q_v(x,y)\|_2^2=v^2\left(y^2v^2-4xyv+4x^2\right)+4\left(x^2+y^2\right).$$

If $x=y=0$ we have $\|\nabla Q_v(x,y)\|_2^2=0$, so we will assume both $x\neq0$ and $y\neq0$. The critical points of $\|\nabla Q_v(x,y)\|_2^2$ are $v=\frac{x}{y}$, $v=\frac{2x}{y}$ and $v=0$ (but $0\notin[2,2+2\sqrt{2}]$). Observe that $v_1=\frac{x}{y}\in\left[2,2+2\sqrt{2}\right]$ whenever $\frac{\sqrt{2}-1}{2}x\leq y\leq\frac{1}{2}x$ and $v_2=\frac{2x}{y}\in\left[2,2+2\sqrt{2}\right]$ whenever $y\geq\left(\sqrt{2}-1\right)x$. Thus, we also need to compare the quantities

$$C_5(x,y):=\|\nabla Q_{v_1}(x,y)\|_2^2=\|\nabla Q_{s_1}(x,y)\|_2^2=\frac{x^4}{y^2}+4\left(x^2+y^2\right),$$

for $\frac{\sqrt{2}-1}{2}x \le y \le \frac{1}{2}x$ and $s_1 = \frac{x^2-2y^2}{2y^2}$,

$$C_6(x,y) := \|\nabla Q_{v_2}(x,y)\|_2^2 = \|\nabla Q_{s_2}(x,y)\|_2^2 = 4\left(x^2+y^2\right),$$

for $\left(\sqrt{2}-1\right)x \le y \le x$ and $s_2 = \frac{2x^2-y^2}{y^2}$, and also

$$C_7(x,y) := \|\nabla Q_{s_3=1}\|_2^2 = 4\left(x^2+y^2\right) + 16(x-y)^2,$$

and

$$\begin{aligned} C_8(x,y) &:= \|\nabla Q_{s_4=5+4\sqrt{2}}\|_2^2 \\ &= \left(12+8\sqrt{2}\right)\left[4x^2 + \left(12+8\sqrt{2}\right)y^2 - \left(8+8\sqrt{2}\right)xy\right] + 4\left(x^2+y^2\right) \\ &= 4\left[\left(13+8\sqrt{2}\right)x^2 + \left(69+48\sqrt{2}\right)y^2 - 2\left(28+20\sqrt{2}\right)xy\right]. \end{aligned}$$

Note that (the reader can take a look at Figures 2.9, 2.10 and 2.11)

$$\begin{aligned} &C_1(x,y), C_6(x,y) \le C_7(x,y) \le \begin{cases} C_4(x,y) & \text{if } 0 \le y \le \frac{2-\sqrt{2}}{2}x \text{ or } \frac{1}{2}x \le y \le x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \le y \le \frac{1}{2}x, \end{cases} \\ &C_3(x,y) \le \begin{cases} C_2(x,y) & \text{if } \frac{1}{3}x \le y \le \left(4\sqrt{2}-5\right)x, \\ C_4(x,y) & \text{if } 0 \le y \le \frac{1}{3}x \text{ or } (4\sqrt{2}-5)x \le y \le x, \end{cases} \\ &C_8(x,y) = C_4(x,y). \end{aligned}$$

Hence, for $(x,y) \in D\left(\frac{\pi}{4}\right)$,

$$\begin{aligned} \Phi_{\pi/4}(x,y) &= \sup\left\{\|\nabla P(x,y)\|_2 \,:\, P \in \operatorname{ext}\left(B_{D(\frac{\pi}{4})}\right)\right\} \\ &= \begin{cases} C_4(x,y) & \text{if } 0 \le y \le \frac{\sqrt{2}-1}{2}x \text{ or } (4\sqrt{2}-5)x \le y \le x, \\ C_5(x,y) & \text{if } \frac{\sqrt{2}-1}{2}x \le y \le (\sqrt{2}-1)x, \\ C_2(x,y) & \text{if } (\sqrt{2}-1)x \le y \le (4\sqrt{2}-5)x. \end{cases} \end{aligned}$$

In order to illustrate the previous step, the reader can take a look at Figure 2.12. □

With the above Bernstein type estimate now it is easy to derive the following sharp Markov type bound on the gradient of polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

Theorem 2.5.7 (G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [1])**.** *(Markov inequality) For every $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and every $(x,y) \in \mathcal{P}D\left(\frac{\pi}{4}\right)$ we have that*

$$\|\nabla P(x,y)\|_2 \le 2\sqrt{13+8\sqrt{2}}\,\|P\|_{D(\frac{\pi}{4})}.$$

Moreover, equality is achieved for the polynomials $\pm P_1(x,y) = \pm\left(x^2 + \left(5+4\sqrt{2}\right)y^2 - 4\left(1+\sqrt{2}\right)xy\right)$ at $(1,0)$.

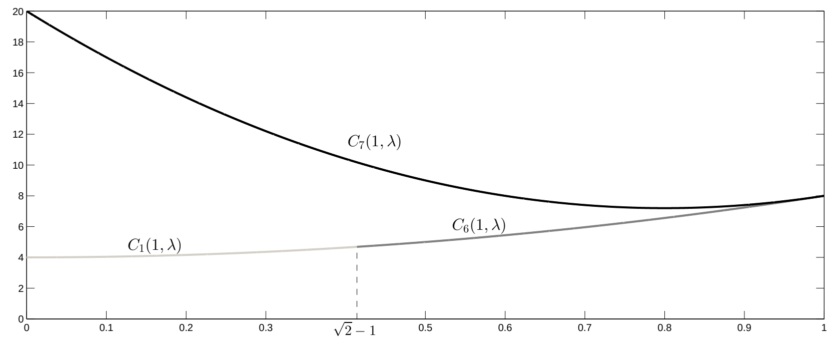


Figure 2.9: Graphs of the mappings $C_1(1,\lambda)$, $C_6(1,\lambda)$, $C_7(1,\lambda)$.

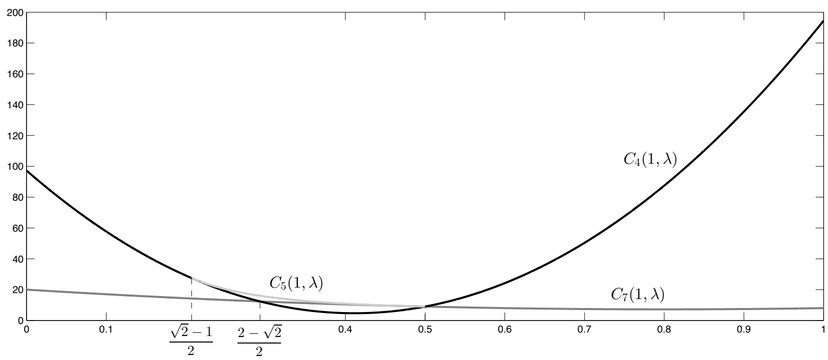


Figure 2.10: Graphs of the mappings $C_4(1,\lambda)$, $C_5(1,\lambda)$, $C_7(1,\lambda)$.

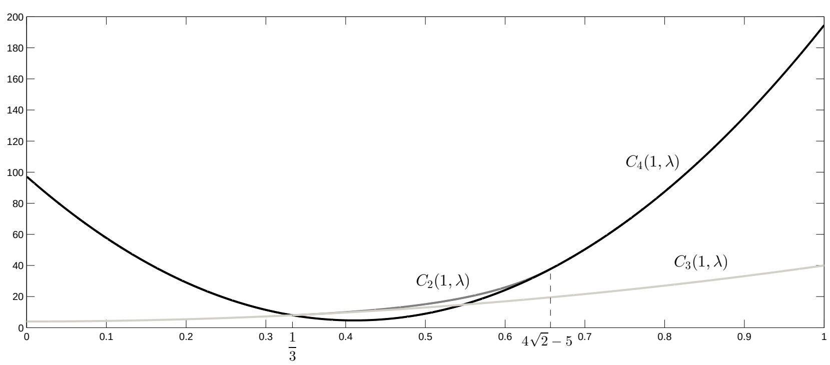


Figure 2.11: Graphs of the mappings $C_2(1,\lambda)$, $C_3(1,\lambda)$, $C_4(1,\lambda)$.

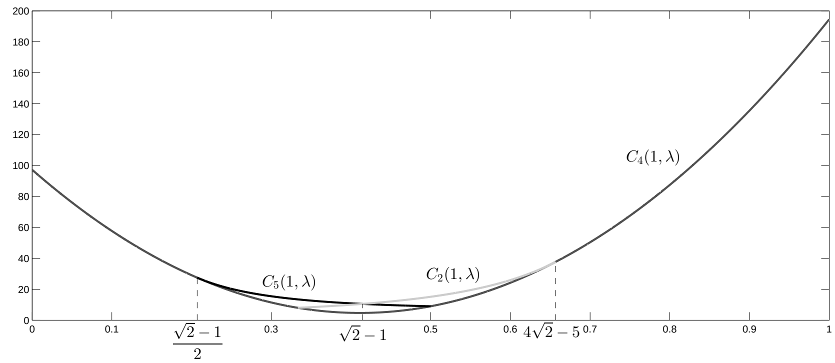


Figure 2.12: Graphs of the mappings $C_2(1,\lambda)$, $C_4(1,\lambda)$, $C_5(1,\lambda)$.

2.5.3 Polarization constant of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$

In order to calculate the polarization constant, we prove a Bernstein type inequality for polynomials in $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Observe that if $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ and $(x,y) \in D\left(\frac{\pi}{4}\right)$ then the differential $DP(x,y)$ of P at (x,y) can be viewed as a linear form. What we shall do is to find the best estimate for $\|DP(x,y)\|_{D(\frac{\pi}{4})}$ in terms of (x,y) and $\|P\|_{D(\frac{\pi}{4})}$.
We shall first state a lemma that will be useful in the future:

Lemma 2.5.8. *Let $a, b \in \mathbb{R}$. Then,*

$$\sup_{\theta\in[0,\frac{\pi}{4}]} |a\cos\theta + b\sin\theta| = \begin{cases} \max\left\{|a|, \frac{\sqrt{2}}{2}|a+b|\right\} & \text{if } \frac{b}{a} > 1 \text{ or } \frac{b}{a} < 0, \\ \sqrt{a^2+b^2} & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \sqrt{a^2+b^2} & \text{if } 0 < \frac{b}{a} < 1, \\ \frac{\sqrt{2}}{2}|a+b| & \text{if } \left(1-\sqrt{2}\right)b < a < b \text{ or } b < a < \left(1-\sqrt{2}\right)b, \\ |a| & \text{if } -\left(1+\sqrt{2}\right)a < b < 0 \text{ or } 0 < b < -\left(1+\sqrt{2}\right)a. \end{cases}$$

Theorem 2.5.9 (G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [1])**.** *For every $(x,y) \in D(\frac{\pi}{4})$ and $P \in \mathcal{P}({}^2D(\frac{\pi}{4}))$ we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{4})} \leq \Psi_{\pi/4}(x,y)\|P\|_{D(\frac{\pi}{4})}, \tag{2.5.3}$$

where

$$\Psi_{\pi/4}(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x - \left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < (\sqrt{2}-1)x, \\ 2\left(x + \frac{y^2}{x-y}\right) & \text{if } (\sqrt{2}-1)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x \end{cases}$$

Moreover, inequality (2.5.3) *is optimal for each $(x,y) \in D(\frac{\pi}{4})$.*

Proof. In order to calculate $\Psi_{\pi/4}(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : \|P\|_{D(\frac{\pi}{4})} \leq 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D(\frac{\pi}{4})} : P \in \mathrm{ext}(B_{D(\frac{\pi}{4})})\}.$$

By symmetry, we may just study the polynomials of Lemma 2.5.5 with positive sign. Let us start first with

$$P_t(x,y) = tx^2 + \left(4+t+4\sqrt{1+t}\right)y^2 - \left(2+2t+4\sqrt{1+t}\right)xy.$$

So we may write

$$\nabla P_t(x,y) = \left(2tx - \left(2+2t+4\sqrt{1+t}\right)y,\, 2\left(4+t+4\sqrt{1+t}\right)y - \left(2+2t+4\sqrt{1+t}\right)x\right),$$

from which

$$\begin{aligned}\|DP_t(x,y)\|_{D(\frac{\pi}{4})} &= \sup_{0\leq\theta\leq\frac{\pi}{4}} \Big|2\left[tx - \left(1+t+2\sqrt{1+t}\right)y\right]\cos\theta \\ &\quad + 2\left[\left(4+t+4\sqrt{1+t}\right)y - \left(1+t+2\sqrt{1+t}\right)x\right]\sin\theta\Big| \\ &= 2x \sup_{0\leq\theta\leq\frac{\pi}{4}} |f_\lambda(t,\theta)|, \\ &\quad \text{for } f_\lambda(t,\theta) = \left[t - \left(1+t+2\sqrt{1+t}\right)\lambda\right]\cos\theta \\ &\qquad + \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right]\sin\theta,\end{aligned}$$

where $\lambda = \frac{y}{x}$, $x \neq 0$ (the case $x = 0$ is trivial, since the only point in $D(\frac{\pi}{4})$ where $x = 0$ is $(0,0)$, in which case $P_t(0,0) = \|DP_t(0,0)\|_{D(\frac{\pi}{4})} = 0$).
We need to calculate

$$\sup_{-1\leq t\leq 1} \|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{\substack{0\leq\theta\leq\frac{\pi}{4}\\ -1\leq t\leq 1}} |f_\lambda(t,\theta)|.$$

Let us define $C^{(1)}_{\frac{\pi}{4}} = [-1,1]\times[0,\frac{\pi}{4}]$. We will analyze 5 cases.

(1) $(t,\theta) \in (-1,1)\times(0,\frac{\pi}{4})$.

We are interested just in critical points. Hence,

$$\begin{aligned}\frac{\partial f_\lambda}{\partial t}(t,\theta) &= \left[\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda - \left(1+\frac{1}{\sqrt{1+t}}\right)\right]\sin\theta \\ &\quad + \left[1 - \left(1+\frac{1}{\sqrt{1+t}}\right)\lambda\right]\cos\theta = 0,\end{aligned} \tag{2.5.4}$$

$$\begin{aligned}\frac{\partial f_\lambda}{\partial\theta}(t,\theta) &= \left[\left(1+t+2\sqrt{1+t}\right)\lambda - t\right]\sin\theta \\ &\quad + \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right]\cos\theta = 0\end{aligned} \tag{2.5.5}$$

Equation (2.5.5) tells us that

$$\sin\theta = \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t - \left(1+t+2\sqrt{1+t}\right)\lambda}\cos\theta. \tag{2.5.6}$$

If we now plug (2.5.6) in equation (2.5.4), we obtain

$$0 = \left\{ \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda \right] + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right) \right] \right.$$
$$\left. \times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t - \left(1+t+2\sqrt{1+t}\right)\lambda} \right\} \cos\theta.$$

Using that $0 < \theta < \frac{\pi}{4}$, we can conclude

$$0 = \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda \right] + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right) \right]$$
$$\times \frac{\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)}{t - \left(1+t+2\sqrt{1+t}\right)\lambda}$$

and thus

$$\begin{aligned}
0 &= \left[1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda \right] \cdot \left[t - \left(1+t+2\sqrt{1+t}\right)\lambda\right] \\
&\quad + \left[\left(1 + \frac{2}{\sqrt{1+t}}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right) \right] \cdot \left[\left(4+t+4\sqrt{1+t}\right)\lambda - \left(1+t+2\sqrt{1+t}\right)\right] \\
&= t - \left(1+t+2\sqrt{1+t}\right)\lambda - t\lambda + \left(1+t+2\sqrt{1+t}\right)\lambda^2 - \frac{\lambda t}{\sqrt{1+t}} \\
&\quad + \frac{\lambda^2}{\sqrt{1+t}}\left(1+t+2\sqrt{1+t}\right) + \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda^2 \\
&\quad - \left(1 + \frac{2}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right)\lambda - \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(4+t+4\sqrt{1+t}\right)\lambda \\
&\quad + \left(1 + \frac{1}{\sqrt{1+t}}\right)\left(1+t+2\sqrt{1+t}\right) \\
&= t\left(1 - 2\lambda + 2\lambda^2 - 2\lambda + 1\right) + \left(-2\lambda + 2\lambda^2 + 4\lambda^2 - 2\lambda - 4\lambda + 2\right)\sqrt{1+t} \\
&\quad + \frac{t}{\sqrt{1+t}}\left(-\lambda + \lambda^2 + 2\lambda^2 - 2\lambda - \lambda + 1\right) + \frac{1}{\sqrt{1+t}}\left(\lambda^2 + 8\lambda^2 - 2\lambda - 4\lambda + 1\right) \\
&\quad + \left(-\lambda + \lambda^2 + 2\lambda^2 + 4\lambda^2 - \lambda - 4\lambda + 1 + 2 + 8\lambda^2 - 8\lambda\right) \\
&= 2t(\lambda-1)^2 + 6\sqrt{1+t}(\lambda-1)\left(\lambda - \frac{1}{3}\right) + 3\frac{t}{\sqrt{1+t}}(\lambda-1)\left(\lambda - \frac{1}{3}\right) \\
&\quad + \frac{1}{\sqrt{1+t}}(3\lambda-1)^2 + 15\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right).
\end{aligned}$$

Working with this last expression, we get

$$\begin{aligned}
0 \quad &= \quad 2t\sqrt{1+t}(\lambda-1)^2 + 6(1+t)(\lambda-1)\left(\lambda - \frac{1}{3}\right) + 3t(\lambda-1)\left(\lambda - \frac{1}{3}\right) \\
&\quad\quad + (3\lambda-1)^2 + 15\sqrt{1+t}\left(\lambda - \frac{1}{3}\right)\left(\lambda - \frac{3}{5}\right)
\end{aligned}$$

and hence, rearranging terms,

$$\sqrt{1+t}\left[15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right)+2t(\lambda-1)^2\right]=-9t(\lambda-1)\left(\lambda-\frac{1}{3}\right)-15\left(\lambda-\frac{1}{3}\right)\left(\lambda-\frac{3}{5}\right). \tag{2.5.7}$$

If $\lambda=1$, we obtain

$$\sqrt{1+t}+1=0$$

and so, in particular, we have $\lambda\neq 1$. Equation (2.5.7) has two solutions,

$$t_1(\lambda)=\frac{-1+2\lambda+3\lambda^2}{(\lambda-1)^2}\quad\text{and}\quad t_2(\lambda)=\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2}.$$

Using equation (2.5.4), we may see

$$\tan\theta=\frac{\left(1+\frac{1}{\sqrt{1+t}}\right)\lambda-1}{\left(1+\frac{2}{\sqrt{1+t}}\right)\lambda-\left(1+\frac{1}{\sqrt{1+t}}\right)}.$$

In particular, evaluating in $t_1(\lambda)$ we obtain

$$\tan\theta_1=\frac{\left(1+\frac{1-\lambda}{2\lambda}\right)\lambda-1}{\left(1+\frac{1-\lambda}{\lambda}\right)\lambda-\left(1+\frac{1-\lambda}{2\lambda}\right)}=\lambda,$$

in which case we have

$$D_{1,1}(\lambda):=|f_\lambda(t_1,\theta_1)|=\left|-\sqrt{1+\lambda^2}\right|=\sqrt{1+\lambda^2}.$$

Regarding $t_2(\lambda)$, we obtain

$$\tan\theta_2=\frac{\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-1}{\left(1+2\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)\lambda-\left(1+\sqrt{\frac{4(\lambda-1)^2}{(3\lambda-1)^2}}\right)}.$$

Since $\theta_2\in\left(0,\frac{\pi}{4}\right)$, we need to guarantee $0<\tan\theta_2<1$, and for this we need $0<\lambda<\frac{1}{5}$. Therefore

$$\tan\theta_2=\frac{5\lambda-1}{7\lambda-3}$$

and in this case,

$$\begin{aligned}
D_{1,2}(\lambda)&:=|f_\lambda(t_2,\theta_2)|\\
&=\left|\left[\frac{5\lambda^2+2\lambda-3}{4(\lambda-1)^2}-\left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{3\lambda-1}{\lambda-1}\right)\lambda\right]\frac{3-7\lambda}{\sqrt{74\lambda^2-52\lambda+10}}\right.\\
&\quad\left.+\left[\left(3+\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{6\lambda-2}{\lambda-1}\right)\lambda-\left(\frac{9\lambda^2-6\lambda+1}{4(\lambda-1)^2}+\frac{3\lambda-1}{\lambda-1}\right)\right]\frac{1-5\lambda}{\sqrt{74\lambda^2-52\lambda+10}}\right|\\
&=\left|-\frac{78\lambda^4-208\lambda^3+196\lambda^2-80\lambda+14}{4(\lambda-1)^2\sqrt{74\lambda^2-52\lambda+10}}\right|\\
&=\left|-\frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}\right|\\
&=\frac{39\lambda^2-26\lambda+7}{2\sqrt{74\lambda^2-52\lambda+10}}.
\end{aligned}$$

(2) $\theta = 0, -1 \leq t \leq 1$.

We have

$$f_\lambda(t,0) = t - \left(1 + t + 2\sqrt{1+t}\right)\lambda.$$

Then,

$$f_\lambda(-1,0) = -1,$$
$$f_\lambda(1,0) = 1 - 2\left(1+\sqrt{2}\right)\lambda,$$

and hence

$$|f_\lambda(1,0)| = \begin{cases} 1 - 2(1+\sqrt{2})\lambda & \text{if } 0 \leq \lambda < \frac{\sqrt{2}-1}{2}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{\sqrt{2}-1}{2} \leq \lambda \leq 1. \end{cases}$$

Working now on $(-1,1)$, since

$$f'_\lambda(t,0) = 1 - \left(1 + \frac{1}{\sqrt{1+t}}\right)\lambda,$$

the critical point of $f_\lambda(t,0)$ is

$$t = \frac{\lambda^2}{(1-\lambda)^2} - 1.$$

Recall that we need to make sure that $-1 < t < 1$. Therefore, in this case we also need to ask

$$\lambda < \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2}.$$

Plugging the critical point of $f_\lambda(t,0)$ into $f_\lambda(t,0)$, we obtain

$$f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2} - 1, 0\right) = \frac{\lambda^2}{(\lambda-1)^2} - 1 - \left[\frac{\lambda^2}{(\lambda-1)^2} + \frac{2\lambda}{1-\lambda}\right]\lambda = \frac{\lambda^2}{\lambda-1} - 1,$$

and hence

$$\left|f_\lambda\left(\frac{\lambda^2}{(\lambda-1)^2} - 1, 0\right)\right| = 1 + \frac{\lambda^2}{1-\lambda}.$$

- Assume first $0 \leq \lambda < \frac{\sqrt{2}-1}{2}$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda(t,0)| = \max\left\{1,\, 1 - 2\left(1+\sqrt{2}\right)\lambda,\, 1 + \frac{\lambda^2}{1-\lambda}\right\} = 1 + \frac{\lambda^2}{1-\lambda}.$$

- Assume now $\frac{\sqrt{2}-1}{2} \leq \lambda < 2 - \sqrt{2}$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda(t,0)| = \max\left\{1,\, 2\left(1+\sqrt{2}\right)\lambda - 1,\, 1 + \frac{\lambda^2}{1-\lambda}\right\} = 1 + \frac{\lambda^2}{1-\lambda}.$$

- Assume finally $2 - \sqrt{2} \leq \lambda \leq 1$. Then,

$$\sup_{-1\leq t\leq 1} |f_\lambda(t,0)| = \max\left\{1,\, 2\left(1+\sqrt{2}\right)\lambda - 1\right\} = 2\left(1+\sqrt{2}\right)\lambda - 1.$$

So, in conclusion,

$$\begin{aligned}\sup_{-1\le t\le 1}|f_\lambda(t,0)| &= \begin{cases} 1+\frac{\lambda^2}{1-\lambda} & \text{if } 0\le\lambda<2-\sqrt{2},\\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1,\end{cases}\\ &=: \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2},\\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1.\end{cases}\end{aligned}$$

(3) $\theta=\frac{\pi}{4}$ and $-1\le t\le 1$.

We have

$$\begin{aligned} f_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[t-\left(1+t+2\sqrt{1+t}\right)\lambda+\left(4+t+4\sqrt{1+t}\right)\lambda-\left(1+t+2\sqrt{1+t}\right)\right]\\ &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{1+t}\right)\lambda-\left(1+2\sqrt{1+t}\right)\right].\end{aligned}$$

Again, we have

$$\begin{aligned} f_\lambda\left(-1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left(3\lambda-1\right),\\ f_\lambda\left(1,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right],\\ f'_\lambda\left(t,\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\frac{\lambda}{\sqrt{1+t}}-\frac{1}{\sqrt{1+t}}\right].\end{aligned}$$

and $f'_\lambda(t,\frac{\pi}{4})=0$ implies $\lambda=1$ (in which case $f_\lambda(t,\frac{\pi}{4})=\sqrt{2}$ for every t).

- Assume first $0\le\lambda<\frac{1}{3}$. Then,

$$\begin{aligned}\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\ 1-3\lambda\right\}\\ &= \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right]\end{aligned}$$

- Assume now $\frac{1}{3}\le\lambda<4\sqrt{2}-5$. Then,

$$\begin{aligned}\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| &= \frac{\sqrt{2}}{2}\max\left\{\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda,\ 3\lambda-1\right\}\\ &= \begin{cases}\frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } \frac{1}{3}\le\lambda<\frac{2\sqrt{2}+1}{7},\\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda<4\sqrt{2}-5.\end{cases}\end{aligned}$$

- Assume finally $4\sqrt{2}-5\le\lambda\le 1$. Then,

$$\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\frac{\pi}{4}\right)\right| = \frac{\sqrt{2}}{2}\max\left\{3\lambda-1,\ \left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right\} = \frac{\sqrt{2}}{2}(3\lambda-1).$$

Hence, we can say that

$$\sup_{-1\le t\le 1}\left|f_\lambda\left(t,\tfrac{\pi}{4}\right)\right| = \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7} \\ \frac{\sqrt{2}}{2}\left(3\lambda-1\right) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases}$$
$$=: \begin{cases} D_{3,1}(\lambda) & \text{if } 0\le\lambda<\frac{2\sqrt{2}+1}{7} \\ D_{3,2}(\lambda) & \text{if } \frac{2\sqrt{2}+1}{7}\le\lambda\le 1. \end{cases}$$

(4) $t=-1$, $0\le\theta\le\frac{\pi}{4}$.

Applying lemma 2.5.8, we obtain

$$\sup_{0\le\theta\le\frac{\pi}{4}} f_\lambda(-1,\theta) = \begin{cases} 1 & \text{if } 0\le\lambda<\frac{1+\sqrt{2}}{3}, \\ \frac{\sqrt{2}}{2}(3\lambda-1) & \text{if } \frac{1+\sqrt{2}}{3}\le\lambda\le 1. \end{cases}$$
$$=: \begin{cases} D_{4,1}(\lambda) & \text{if } 0\le\lambda<\frac{1+\sqrt{2}}{3}, \\ D_{4,2}(\lambda) & \text{if } \frac{1+\sqrt{2}}{3}\le\lambda\le 1. \end{cases}$$

(5) $t=1$, $0\le\theta\le\frac{\pi}{4}$.

We use again lemma 2.5.8, with $a=1-\left(2+2\sqrt{2}\right)\lambda$ and $b=\left(5+4\sqrt{2}\right)\lambda-\left(2+2\sqrt{2}\right)$. Through standard calculations, we see that $\frac{b}{a}<0$ if and only if $\lambda\in\left[0,\frac{\sqrt{2}-1}{2}\right)\cup\left(\frac{6-2\sqrt{2}}{7},1\right]$ and $\frac{b}{a}>1$ if and only if $\frac{\sqrt{2}-1}{2}<\lambda<\frac{3+4\sqrt{2}}{23}$. Therefore,

$$\sup_{0\le\theta\le\frac{\pi}{4}}\left|f_\lambda(1,\theta)\right|$$
$$= \begin{cases} \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23}, \\ \sqrt{\left(1-\left(2+2\sqrt{2}\right)\lambda\right)^2+\left(\left(5+4\sqrt{2}\right)\lambda-\left(2+2\sqrt{2}\right)\right)^2} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7}, \\ \max\left\{\left|1-\left(2+2\sqrt{2}\right)\lambda\right|, \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|\right\} & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}$$

Since $0 \le \lambda < \sqrt{2}-1$ implies $\left|1-\left(2+2\sqrt{2}\right)\lambda\right| < \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right|$, it follows that

$$
\begin{aligned}
&\sup_{0\le\theta\le\frac{\pi}{4}} |f_\lambda(1,\theta)| \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left|\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right| & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left|1-\left(2+2\sqrt{2}\right)\lambda\right| & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1 \end{cases} \\
&= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ \sqrt{48\sqrt{2}\lambda^2-56\lambda+69\lambda^2-40\sqrt{2}\lambda+8\sqrt{2}+13} & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases} \\
&=: \begin{cases} D_{5,1}(\lambda) & \text{if } 0\le\lambda<\frac{3+4\sqrt{2}}{23} \\ D_{5,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23}\le\lambda<\frac{6-2\sqrt{2}}{7} \\ D_{5,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\le\lambda\le 1. \end{cases}
\end{aligned}
$$

Since (see Figures 2.13 and 2.14)

$$
\begin{aligned}
&D_{1,1}(\lambda) \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1, \end{cases} \\
&D_{1,2}(\lambda) \le D_{3,1}(\lambda) \text{ for } 0<\lambda<\tfrac{1}{5},
\end{aligned}
$$

we can rule out case (1). Since

$$
\begin{aligned}
D_{3,1}(\lambda) &= D_{5,1}(\lambda) \quad \text{for } 0\le\lambda\le\tfrac{3+4\sqrt{2}}{23}, \\
D_{3,2}(\lambda) &= D_{4,2}(\lambda) \quad \text{for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,
\end{aligned}
$$

we can directly rule out case (3). Since (see Figures 2.13 and 2.15)

$$
\begin{aligned}
&D_{4,1}(\lambda) = 1 \le \begin{cases} D_{2,1}(\lambda) & \text{if } 0\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda<\frac{1+\sqrt{2}}{3}, \end{cases} \\
&D_{4,2}(\lambda) \le D_{2,2} \text{ for } \tfrac{1+\sqrt{2}}{3}\le\lambda\le 1,
\end{aligned}
$$

we can rule out case (4). Finally, since (see Figure 2.16)

$$
\begin{aligned}
D_{5,2}(\lambda) &\le D_{2,1}(\lambda) \quad \text{for } \tfrac{3+4\sqrt{2}}{23}\le\lambda<\tfrac{6-2\sqrt{2}}{7}, \\
D_{5,3}(\lambda) &= D_{2,2}(\lambda) \quad \text{for } 2-\sqrt{2}\le\lambda\le 1,
\end{aligned}
$$

we can rule out the expressions $D_{5,2}(\lambda)$ and $D_{5,3}(\lambda)$ of case (5).

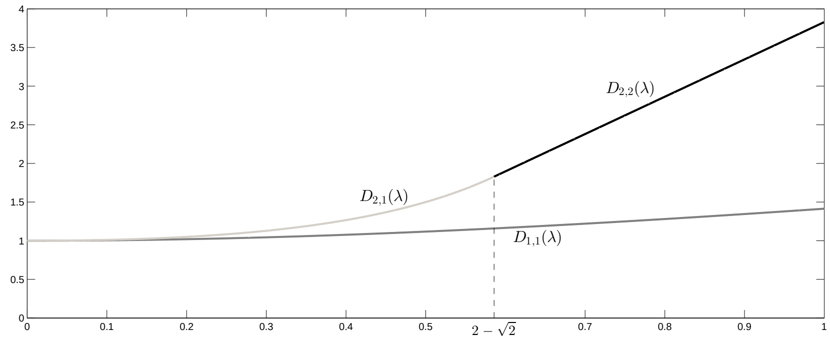


Figure 2.13: Graphs of the mappings $D_{1,1}(\lambda)$, $D_{2,1}(\lambda)$ and $D_{2,2}(\lambda)$.

Thus, putting all the above cases together, we may reach the conclusion

$$\begin{aligned}\sup_{(t,\theta)\in C^{(1)}_{\frac{\pi}{4}}} |f_\lambda(t,\theta)| &= \begin{cases} D_{5,1}(\lambda) & \text{if } 0\le\lambda<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ D_{2,1}(\lambda) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}\le\lambda<2-\sqrt{2}, \\ D_{2,2}(\lambda) & \text{if } 2-\sqrt{2}\le\lambda\le 1, \end{cases} \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\le\lambda<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}, \\ 1+\frac{\lambda^2}{1-\lambda} & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}\le\lambda<2-\sqrt{2}, \\ \left(2+2\sqrt{2}\right)\lambda-1 & \text{if } 2-\sqrt{2}\le\lambda\le 1, \end{cases}\end{aligned}$$

and hence

$$\begin{aligned}&\sup_{-1\le t\le 1}\|DP_t(x,y)\|_{D(\frac{\pi}{4})} = 2x\sup_{(t,\theta)\in C^{(1)}_{\frac{\pi}{4}}}|f_\lambda(t,\theta)| \\ &= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0\le y<\frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \frac{(2-3\sqrt{2})\sqrt{4\sqrt{2}+7}+5\sqrt{2}+6}{14}x\le y<\left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x\le y\le x, \end{cases}\end{aligned}$$

assuming in every moment $x\neq 0$ (in order to illustrate the previous step, the reader can take a look at Figure 2.17).

Let us deal now with the polynomials

$$Q_s(x,y)=x^2+sy^2-2\sqrt{2(1+s)}xy,\quad 1\le s\le 5+4\sqrt{2}.$$

Then,

$$\begin{aligned}\nabla Q_s(x,y) &= \left(2x-2\sqrt{2(1+s)}y,\, 2sy-2\sqrt{2(1+s)}x\right), \\ \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} &= \sup_{0\le\theta\le\frac{\pi}{4}}\left|2x\left[\left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta+\left(s\lambda-\sqrt{2(1+s)}\right)\sin\theta\right]\right|,\end{aligned}$$

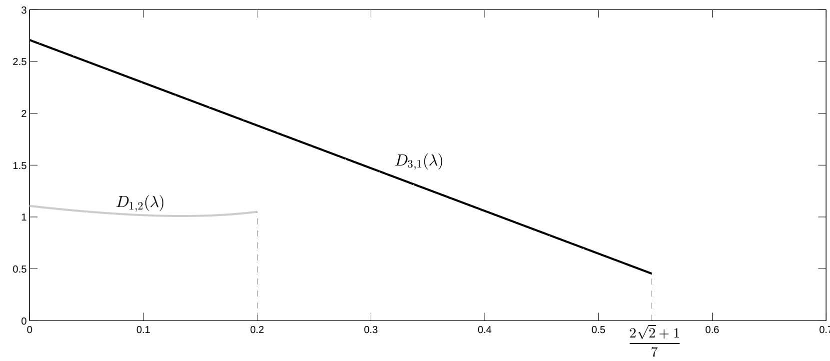


Figure 2.14: Graphs of the mappings $D_{1,2}(\lambda)$ and $D_{3,1}(\lambda)$.

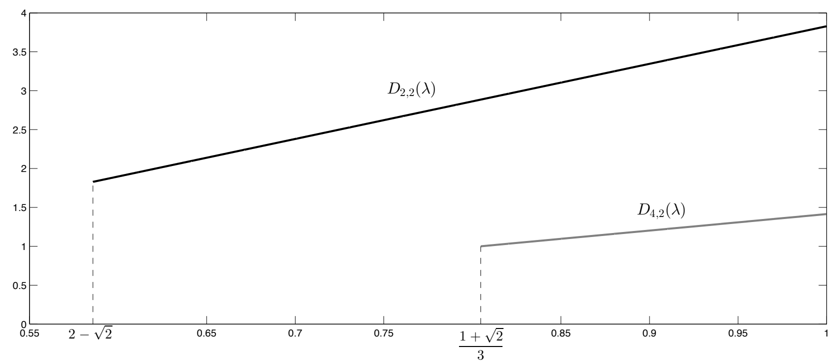


Figure 2.15: Graphs of the mappings $D_{2,2}(\lambda)$ and $D_{4,2}(\lambda)$.

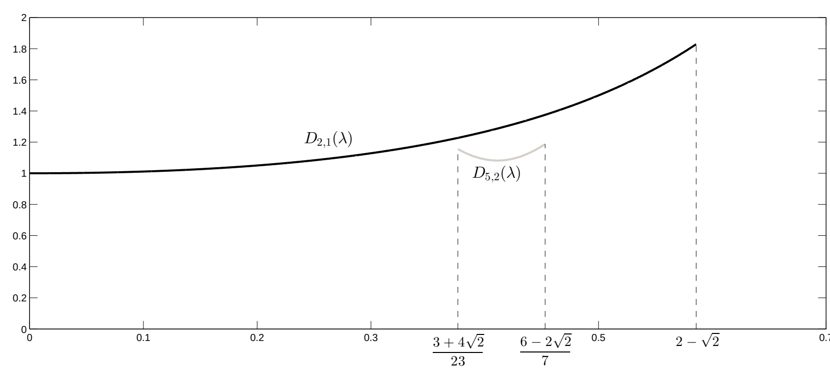


Figure 2.16: Graphs of the mappings $D_{2,1}(\lambda)$ and $D_{5,2}(\lambda)$.

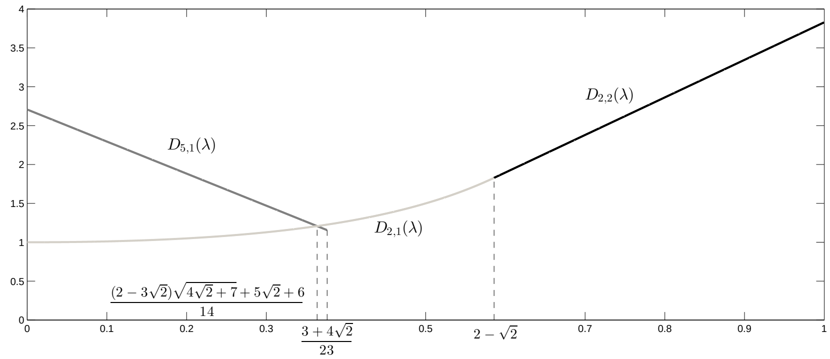


Figure 2.17: Graphs of the mappings $D_{2,1}(\lambda)$, $D_{2,2}(\lambda)$ and $D_{5,1}(\lambda)$.

and thus

$$\sup_{1\le s\le 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} = 2x \sup_{(s,\theta)\in C^{(2)}_{\frac{\pi}{4}}} |g_\lambda(s,\theta)|,$$

with

$$g_\lambda(s,\theta) = \left(1-\sqrt{2(1+s)}\lambda\right)\cos\theta + \left(s\lambda - \sqrt{2(1+s)}\right)\sin\theta$$

and $C^{(2)}_{\frac{\pi}{4}} = [1, 5+4\sqrt{2}] \times [0, \frac{\pi}{4}]$. Again, we have several cases:

(6) $(s,\theta) \in (1, 5+4\sqrt{2}) \times (0, \frac{\pi}{4})$.

Let us first calculate the critical points of g_λ over $C^{(2)}_{\frac{\pi}{4}}$.

$$\frac{\partial g_\lambda}{\partial s}(s_0,\theta_0) = \frac{-\lambda}{\sqrt{2(1+s_0)}}\cos\theta_0 + \left(\lambda - \frac{1}{\sqrt{2(1+s_0)}}\right)\sin\theta_0,$$
$$\frac{\partial g_\lambda}{\partial \theta}(s_0,\theta_0) = \left(s_0\lambda - \sqrt{2(1+s_0)}\right)\cos\theta_0 - \left(1-\sqrt{2(1+s_0)}\lambda\right)\sin\theta_0,$$

so, if $Dg_\lambda(s_0,\theta_0) = 0$, using the first expression, we obtain $\tan\theta_0 = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda-1}$, and, using the second one, we obtain $\tan\theta_0 = \frac{s_0\lambda-\sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda}$.
Hence, we may say

$$\frac{s_0\lambda - \sqrt{2(1+s_0)}}{1-\sqrt{2(1+s_0)}\lambda} = \frac{\lambda}{\sqrt{2(1+s_0)}\lambda - 1}$$

and thus

$$s_0 = \frac{2-\lambda^2}{\lambda^2}.$$

Then, $\tan\theta_0 = \lambda$ and also, if we want to guarantee that $1 < s_0 < 5+4\sqrt{2}$, we need $\sqrt{2}-1 < \lambda < 1$. In that case, $\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}}$ and $\cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}$, and then

$$g_\lambda(s_0,\theta_0) = \frac{-1}{\sqrt{1+\lambda^2}} + \frac{-\lambda^2}{\sqrt{1+\lambda^2}} = -\sqrt{1+\lambda^2},$$

so

$$D_6(\lambda) := |g_\lambda(s_0, \theta_0)| = \sqrt{1+\lambda^2}.$$

(7) $s = 1$, $0 \le \theta \le \frac{\pi}{4}$.

Apply lemma 2.5.8 with $a = 1 - 2\lambda$ and $b = \lambda - 2$. Using $0 \le \lambda \le 1$, observe that we always have $b < 0$ and $b \le a$. Also, $a < \left(1 - \sqrt{2}\right) b$ if and only if $\lambda > \frac{5-3\sqrt{2}}{7}$.
Putting everything together, we can say

$$\begin{aligned} \sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(1,\theta)| &= \begin{cases} 1-2\lambda & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ \frac{\sqrt{2}}{2}(1+\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1, \end{cases} \\ &=: \begin{cases} D_{7,1}(\lambda) & \text{if } 0 \le \lambda < \frac{5-3\sqrt{2}}{7}, \\ D_{7,2}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7} \le \lambda \le 1. \end{cases} \end{aligned}$$

(8) $s = 5 + 4\sqrt{2}$, $0 \le \theta \le \frac{\pi}{4}$.

Apply again lemma 2.5.8, this time to $a = 1 - 2\left(1+\sqrt{2}\right)\lambda$ and $b = \left(5+4\sqrt{2}\right)\lambda - 2\left(1+\sqrt{2}\right)$. As usual, we notice that $a < 0$ if and only if $\lambda > \frac{\sqrt{2}-1}{2}$, $b < 0$ if and only if $\lambda < \frac{6-2\sqrt{2}}{7}$ and $a < b$ if and only if $\lambda > \frac{3+4\sqrt{2}}{23}$. All together, we can say that, for $\frac{3+4\sqrt{2}}{23} < \lambda < \frac{6-2\sqrt{2}}{7}$, we have

$$\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| = \sqrt{a^2+b^2} = \sqrt{13+8\sqrt{2} - \left(56+40\sqrt{2}\right)\lambda + \left(69+48\sqrt{2}\right)\lambda^2}.$$

Also, notice that, for any $\lambda \in [0,1]$, we are going to have $b < -\left(1+\sqrt{2}\right)a$ and $a < \left(1-\sqrt{2}\right)b$. Hence,

$$\begin{aligned} &\sup_{0\le\theta\le\frac{\pi}{4}} |g_\lambda(5+4\sqrt{2},\theta)| \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right) - \left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ \sqrt{13+8\sqrt{2} - \left(56+40\sqrt{2}\right)\lambda + \left(69+48\sqrt{2}\right)\lambda^2} & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1, \end{cases} \\ &=: \begin{cases} D_{8,1}(\lambda) & \text{if } 0 \le \lambda < \frac{3+4\sqrt{2}}{23}, \\ D_{8,2}(\lambda) & \text{if } \frac{3+4\sqrt{2}}{23} \le \lambda < \frac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7} \le \lambda \le 1. \end{cases} \end{aligned}$$

(9) $\theta = 0$, $1 \le s \le 5 + 4\sqrt{2}$.

We have

$$\begin{aligned} g_\lambda(s,0) &= 1 - \sqrt{2(1+s)}\lambda, \\ g_\lambda(1,0) &= 1 - 2\lambda, \\ g_\lambda(5+4\sqrt{2},0) &= 1 - 2\left(1+\sqrt{2}\right)\lambda, \\ g'_\lambda(s,0) &= -\frac{\lambda}{\sqrt{2(1+s)}} \ne 0 \text{ for } \lambda \ne 0. \end{aligned}$$

Then,

$$\begin{aligned}\sup_{1\leq s\leq 5+4\sqrt{2}}|g_\lambda(s,0)| &= \max\left\{|1-2\lambda|,\, |1-2(1+\sqrt{2})\lambda|\right\}\\ &= \begin{cases} 1-2\lambda & \text{if } 0\leq\lambda<\frac{2-\sqrt{2}}{2},\\ 2\left(1+\sqrt{2}\right)\lambda-1 & \text{if } \frac{2-\sqrt{2}}{2}\leq\lambda\leq 1,\end{cases}\\ &=: \begin{cases} D_{9,1}(\lambda) & \text{if } 0\leq\lambda<\frac{2-\sqrt{2}}{2},\\ D_{9,2}(\lambda) & \text{if } \frac{2-\sqrt{2}}{2}\leq\lambda\leq 1.\end{cases}\end{aligned}$$

(10) $\theta=\frac{\pi}{4}$, $1\leq s\leq 5+4\sqrt{2}$.

We have

$$g_\lambda\left(s,\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\left[1+s\lambda-\sqrt{2(1+s)}(1+\lambda)\right].$$

Then

$$\begin{aligned} g_\lambda\left(1,\frac{\pi}{4}\right) &= -\frac{\sqrt{2}}{2}(1+\lambda),\\ g_\lambda\left(5+4\sqrt{2},\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}\left[\left(3+2\sqrt{2}\right)\lambda-\left(1+2\sqrt{2}\right)\right],\\ g'_\lambda\left(s_0,\frac{\pi}{4}\right) &= 0 \text{ if and only if } s_0=\frac{(1+\lambda)^2}{2\lambda^2}-1\end{aligned}$$

and since we need to ensure that $1<s_0<5+4\sqrt{2}$, we need $\frac{2\sqrt{2}-1}{7}<\lambda<1$. In that case,

$$g_\lambda\left(s_0,\frac{\pi}{4}\right)=-\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda}.$$

Hence,

$$\begin{aligned}\sup_{1\leq s\leq 5+4\sqrt{2}}\left|g_\lambda\left(s.\frac{\pi}{4}\right)\right| &= \begin{cases}\frac{\sqrt{2}}{2}\left[\left(1+2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0\leq\lambda<\frac{2\sqrt{2}-1}{7},\\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7}\leq\lambda\leq 1,\end{cases}\\ &=: \begin{cases} D_{10,1}(\lambda) & \text{if } 0\leq\lambda<\frac{2\sqrt{2}-1}{7},\\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7}\leq\lambda\leq 1.\end{cases}\end{aligned}$$

Since (the reader can take a look at Figure 2.18)

$$D_6(\lambda)\leq\begin{cases} D_{8,2}(\lambda) & \text{if } \sqrt{2}-1<\lambda<\frac{6-2\sqrt{2}}{7},\\ D_{8,3}(\lambda) & \text{if } \frac{6-2\sqrt{2}}{7}\leq\lambda<1,\end{cases}$$

we can rule out case (6). Since (see Figures 2.19 and 2.20)

$$\begin{aligned} &D_{7,1}(\lambda)\leq D_{10,1}(\lambda) \text{ for } 0\leq\lambda<\frac{5-3\sqrt{2}}{7}\\ &D_{7,2}(\lambda)\leq\begin{cases} D_{10,1}(\lambda) & \text{if } \frac{5-3\sqrt{2}}{7}\leq\lambda<\frac{2\sqrt{2}-1}{7},\\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7}\leq\lambda\leq 1,\end{cases}\end{aligned}$$

we can rule out case (7). Since

$$D_{8,1}(\lambda) = D_{10,1}(\lambda) \text{ for } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}$$

we can rule out the expression $D_{8,1}(\lambda)$ of case (8). Since

$$\begin{aligned} D_{9,1}(\lambda) &= D_{7,1}(\lambda) \quad \text{for } 0 \leq \lambda < \tfrac{5-3\sqrt{2}}{7}, \\ D_{9,2}(\lambda) &= D_{8,3}(\lambda) \quad \text{for } \tfrac{6-2\sqrt{2}}{7} \leq \lambda \leq 1, \end{aligned}$$

we can directly rule out case (9). Furthermore, since (see Figure 2.21)

$$\begin{aligned} D_{8,2}(\lambda) &\leq D_{10,2}(\lambda) \text{ for } \tfrac{3+4\sqrt{2}}{23} \leq \lambda < \tfrac{6-2\sqrt{2}}{7}, \\ D_{8,3}(\lambda) &\leq D_{10,2}(\lambda) \text{ for } \tfrac{6-2\sqrt{2}}{7} \leq \lambda \leq \tfrac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \end{aligned}$$

we can conclude that

$$\begin{aligned} \sup_{(s,\theta)\in C^{(2)}_{\frac{\pi}{4}}} |g_\lambda(s,\theta)| &= \begin{cases} D_{10,1}(\lambda) & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ D_{10,2}(\lambda) & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ D_{8,3}(\lambda) & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1. \end{cases} \\ &= \begin{cases} \frac{\sqrt{2}}{2}\left[1+2\sqrt{2}-\left(3+2\sqrt{2}\right)\lambda\right] & \text{if } 0 \leq \lambda < \frac{2\sqrt{2}-1}{7}, \\ \frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} & \text{if } \frac{2\sqrt{2}-1}{7} \leq \lambda < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}, \\ 2\left(1+\sqrt{2}\right)\lambda - 1 & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7} \leq \lambda \leq 1, \end{cases} \end{aligned}$$

and hence

$$\begin{aligned} &\sup_{1\leq s\leq 5+4\sqrt{2}} \|DQ_s(x,y)\|_{D(\frac{\pi}{4})} \\ &= \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x, \\ 4\left(1+\sqrt{2}\right)y - 2x & \text{if } \frac{(4\sqrt{2}-5)\sqrt{4\sqrt{2}+7}+8-5\sqrt{2}}{7}x \leq y \leq x. \end{cases} \end{aligned}$$

Finally, if we compare the results obtained with P_t and Q_s, since $\frac{\sqrt{2}(1+3\lambda^2)}{4\lambda} \geq 1+\frac{\lambda^2}{1-\lambda}$ whenever $\lambda \leq \sqrt{2}-1$, we obtain

$$\Phi_{\pi/4}(x,y) = \begin{cases} \sqrt{2}\left[\left(1+2\sqrt{2}\right)x-\left(3+2\sqrt{2}\right)y\right] & \text{if } 0 \leq y < \frac{2\sqrt{2}-1}{7}x, \\ \frac{\sqrt{2}(x^2+3y^2)}{2y} & \text{if } \frac{2\sqrt{2}-1}{7}x \leq y < \left(\sqrt{2}-1\right)x, \\ 2\left(x+\frac{y^2}{x-y}\right) & \text{if } \left(\sqrt{2}-1\right)x \leq y < \left(2-\sqrt{2}\right)x, \\ 4\left(1+\sqrt{2}\right)y-2x & \text{if } \left(2-\sqrt{2}\right)x \leq y \leq x. \end{cases}$$

□

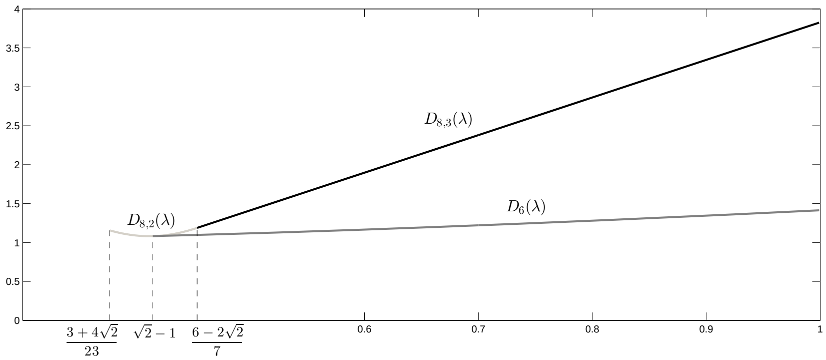


Figure 2.18: Graphs of the mappings $D_6(\lambda)$, $D_{8,2}(\lambda)$ and $D_{8,3}(\lambda)$.

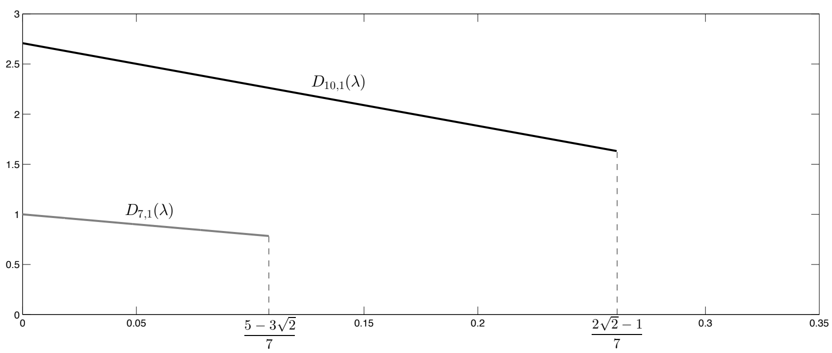


Figure 2.19: Graphs of the mappings $D_{7,1}(\lambda)$ and $D_{10,1}(\lambda)$.

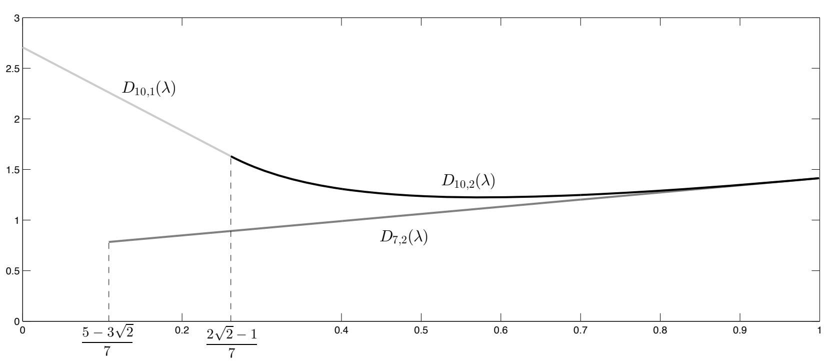


Figure 2.20: Graphs of the mappings $D_{7,2}(\lambda)$, $D_{10,1}(\lambda)$ and $D_{10,2}(\lambda)$.

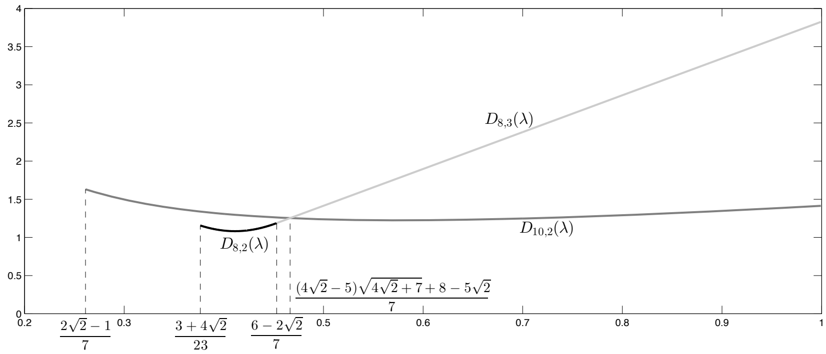


Figure 2.21: Graphs of the mappings $D_{8,2}(\lambda)$, $D_{8,3}(\lambda)$ and $D_{10,2}(\lambda)$.

We can see that $\Phi_{\pi/4}(x,y) \leq 4+\sqrt{2}$, for all $(x,y) \in D\left(\frac{\pi}{4}\right)$. Furthermore, the maximum is attained by the polynomials

$$P_1(x,y) = x^2 + \left(5+4\sqrt{2}\right)y^2 - \left(4+4\sqrt{2}\right)xy = Q_{5+4\sqrt{2}}(x,y).$$

Corollary 2.5.10. *Let $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Then*

$$\|\check{P}\|_{D\left(\frac{\pi}{4}\right)} \leq \left(2+\frac{\sqrt{2}}{2}\right)\|P\|_{D\left(\frac{\pi}{4}\right)}.$$

Moreover, equality is achieved for $P_1(x,y) = Q_{5+4\sqrt{2}}(x,y) = x^2 + \left(5+4\sqrt{2}\right)y^2 - \left(4+4\sqrt{2}\right)xy$. Hence, the polarization constant of the polynomial space $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $2+\frac{\sqrt{2}}{2}$.

2.5.4 Unconditional constant of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$.

In this case, we obtain, again following the Krein-Milman Theorem,

Theorem 2.5.11 (G. Araújo, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, J.B. Seoane-Sepúlveda, [1]). *The unconditional constant of the canonical basis of $\mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$ is $5+4\sqrt{2}$. In other words, the inequality*

$$\||P|\|_{D\left(\frac{\pi}{4}\right)} \leq (5+4\sqrt{2})\|P\|_{D\left(\frac{\pi}{4}\right)},$$

for all $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{4}\right)\right)$. Furthermore, the previous inequality is sharp and equality is attained for the polynomials $\pm P_1(x,y) = \pm Q_{5+4\sqrt{2}}(x,y) = \pm\left[x^2+(5+4\sqrt{2})y^2-(4+4\sqrt{2})xy\right]$.

Proof. We just need to calculate

$$\sup\left\{\||P|\|_{D\left(\frac{\pi}{4}\right)} : \ P \in \text{ext}\left(B_{D\left(\frac{\pi}{4}\right)}\right)\right\}.$$

In order to calculate the above supremum we use the extreme polynomials described in Lemma 2.5.5. If we consider first the polynomials P_t, then $|P_t| = \left(|t|, 4+t+4\sqrt{1+t}, 2+2t+4\sqrt{1+t}\right)$.

Now, using Lemma 2.5.5 we have

$$\begin{aligned}\sup_{-1\leq t\leq 1}\||P_t|\|_{D(\frac{\pi}{4})} &= \sup_{-1\leq t\leq 1}\max\left\{|t|, \frac{1}{2}\left(|t+4+t+4\sqrt{1+t}+2+2t+4\sqrt{1+t}\right)\right\}\\ &= \sup_{-1\leq t\leq 1}\frac{1}{2}\left(|t|+6+3t+8\sqrt{1+t}\right) = 5+4\sqrt{2}.\end{aligned}$$

Notice that the above supremum is attained at $t=1$. On the other hand, if we consider the polynomials Q_s, we have $|Q_s| = \left(1, s, 2\sqrt{2(1+s)}\right)$. Now, using Lemma 2.5.5 we have

$$\begin{aligned}\sup_{1\leq s\leq 5+4\sqrt{2}}\||Q_s|\|_{D(\frac{\pi}{4})} &= \sup_{1\leq s\leq 5+4\sqrt{2}}\max\left\{1, \frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right)\right\}\\ &= \sup_{1\leq s\leq 5+4\sqrt{2}}\frac{1}{2}\left(1+s+2\sqrt{2(1+s)}\right) = 5+4\sqrt{2}.\end{aligned}$$

Observe that the last supremum is now attained at $s = 5+4\sqrt{2}$. □

2.6 The case of $\mathsf{B}_{D(\frac{\pi}{2})}$

We shall study now the case of homogeneous polynomials of degree 2, but with the norm considered over vectors on $D\left(\frac{\pi}{2}\right)$. As with the case of $D\left(\frac{\pi}{4}\right)$, we shall first study the geometry of $B_{D(\frac{\pi}{2})}$.

2.6.1 The geometry of $B_{D(\frac{\pi}{2})}$

Theorem 2.6.1 (G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [77])**.** *The projection of $\mathsf{S}_{D(\frac{\pi}{2})}$ over the ab-plane is $[-1,1]^2$.*

Proof. Using the second formula in Theorem 2.5.1 we have that

$$\|(a,b,c)\|_{D(\frac{\pi}{2})} \geq \max\{|a|,|b|\},$$

for all $(a,b,c)\in\mathbb{R}^3$. This shows right away that $\pi_{ab}(\mathsf{S}_{D(\frac{\pi}{2})}) \subset [-1,1]^2$. In order to prove that $[-1,1]^2 \subset \pi_{ab}(\mathsf{S}_{D(\frac{\pi}{2})})$ take $(a,b)\in\mathbb{R}^2$ so that $|a|\leq 1$ and $|b|\leq 1$.

Then, using again the same formula, one shows easily that

$$\begin{aligned}&\left\|\left(a,b,2\sqrt{(1-a)(1-b)}\right)\right\|_{D(\frac{\pi}{2})}\\ &\quad= \max\left\{|a|,|b|,\frac{1}{2}\left|a+b+\sqrt{(a-b)^2+4(1-a)(1-b)}\right|\right\}\\ &\quad= \max\left\{|a|,|b|,\frac{1}{2}\left|a+b+\sqrt{(2-a-b)^2}\right|\right\}\\ &\quad= \max\{|a|,|b|,1\} = 1.\end{aligned}$$

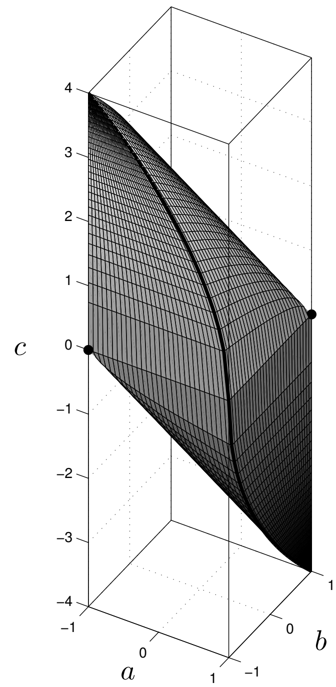


Figure 2.22: $\mathsf{S}_{D(\frac{\pi}{2})}$. The extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ are drawn with a thick line and dots.

Observe that, similarly,

$$\left\|\left(a, b, -2\sqrt{(1+a)(1+b)}\right)\right\|_{D(\frac{\pi}{2})} = 1.$$

This concludes the proof. □

Theorem 2.6.2 (G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [77]). *If we define the mappings*

$$G_1(a, b) = 2\sqrt{(1-a)(1-b)}$$

and

$$G_2(a, b) = -f_+(-a, -b) = -2\sqrt{(1+a)(1+b)},$$

for every $(a, b) \in [-1, 1]^2$ *and the set*

$$\Upsilon = \{(a, b, c) \in \mathbb{R}^3 : (a, b) \in \partial[-1, 1]^2 \text{ and } G_2(a, b) \le c \le G_1(a, b)\},$$

where $\partial[-1, 1]^2$ *is the boundary of* $[-1, 1]^2$*, then* *(a)* $\mathsf{S}_{D(\frac{\pi}{2})} = \mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon$.

(b) The set $\mathrm{ext}(\mathsf{B}_{D(\frac{\pi}{2})})$ *consists of the elements*

$$\pm\left(1, t, -2\sqrt{2(1+t)}\right),\ \pm\left(t, 1, -2\sqrt{2(1+t)}\right) \quad \textit{and} \quad \pm(1,1,0),$$

with $t \in [-1, 1]$.

Proof. As for part (a), we have already seen in the proof of Theorem 2.6.1 that

$$\mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \subset \mathsf{S}_{(\frac{\pi}{2})}.$$

Also, if $(a, b, c) \in \Upsilon$, then $G_2(a, b) \le c \le G_1(a, b)$ with $\max\{|a|, |b|\} = 1$. Suppose $a = -1$. Then $0 \le c \le 2\sqrt{2(1-b)}$ with $|b| \le 1$ and

$$\begin{aligned}\frac{1}{2}\left|a + b + \mathrm{sign}(c)\sqrt{(a-b)^2 + c^2}\right| &\le \frac{1}{2}\left|-1 + b + \sqrt{(1+b)^2 + 8(1-b)}\right| \\ &= \frac{1}{2}\left|-1 + b + \sqrt{(3-b)^2}\right| = 1.\end{aligned}$$

Hence using the second formula in Theorem 2.5.1 we see that $\|(-1, b, c)\|_{D(\frac{\pi}{2})} = 1$. With similar considerations, we can see that the same holds if $a = 1$ or $|b| = 1$. Therefore

$$\mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon \subset \mathsf{S}_{(\frac{\pi}{2})}.$$

On the other hand, suppose $(a, b, c) \notin \mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon$. Obviously, $(0, 0, 0) \notin \mathsf{S}_{D(\frac{\pi}{2})}$, so we can also assume that $(a, b, c) \neq (0, 0, 0)$. The straight line $\{\lambda(a, b, c) : \lambda \in \mathbb{R}\}$ certainly meets the set $\mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon$ as natural observations may entail. Put, $(a, b, c) = \lambda_0(a_0, b_0, c_0)$ with $\lambda_0 \neq 0, 1$ and $(a_0, b_0, c_0) \in \mathrm{graph}(G_1) \cup \mathrm{graph}(G_2) \cup \Upsilon$. Then $\|(a, b, c)\|_{D(\frac{\pi}{2})} = \lambda_0 \|(a_0, b_0, c_0)\|_{D(\frac{\pi}{2})} = \lambda_0 \neq 1$, and therefore $(a, b, c) \notin \mathsf{S}_{(\frac{\pi}{2})}$. This concludes part (a).

Part (b) is an easy consequence of the fact that the sets $\mathrm{graph}(G_1)$, $\mathrm{graph}(G_2)$ and Υ are ruled surfaces in $\mathbb{R}^3$. Actually Υ is contained in the four planes $a = \pm 1$ and $b = \pm 1$. Using this with the help of Figure 2.22 the reader will soon arrive at the conclusion that the only possible extreme points are contained in the intersections $\mathrm{graph}(G_1) \cap \Upsilon$, $\mathrm{graph}(G_2) \cap \Upsilon$ and $\mathrm{graph}(G_1) \cap \mathrm{graph}(G_2)$. The union of these three sets consists of the elements

$$\pm\left(1, t, -2\sqrt{2(1+t)}\right),\ \pm\left(t, 1, -2\sqrt{2(1+t)}\right),\ (\pm 1, t, 0), \quad \text{and} \quad (t, \pm 1, 0),$$

with $t \in [-1, 1]$. Observe that $(\pm 1, t, 0)$ and $(t, \pm 1, 0)$ with $|t| < 1$ cannot obviously be extreme and the point $\pm(1, -1, 0)$ are already considered in the curves $\left(1, t, -2\sqrt{2(1+t)}\right), \pm\left(t, 1, -2\sqrt{2(1+t)}\right)$ with $t \in [-1, 1]$. Hence the candidates to be extreme points of $\mathsf{B}_{D(\frac{\pi}{2})}$ reduce to

$$\pm\left(1, t, -2\sqrt{2(1+t)}\right),\ \pm\left(t, 1, -2\sqrt{2(1+t)}\right) \quad \text{and} \quad \pm(1,1,0),$$

with $t \in [-1, 1]$. Constructing a supporting hyperplane to each of the previous points is a straightforward exercise and it finishes the proof. □

We shall also include a sketch of $\mathsf{S}_{D(\frac{\pi}{2})}$ in Figure 2.22.

2.6.2 Markov and Bernstein inequalities in $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$.

As before, we shall start first with a Bernstein type estimate:

Theorem 2.6.3 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [64])**.** *. For every $(x,y)\in D\left(\frac{\pi}{2}\right)$ and $P\in\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$ we have that*

$$\|\nabla P(x,y)\|_2\leq \Phi_{\pi/2}(x,y)\|P\|_{D\left(\frac{\pi}{2}\right)},$$

with

$$\Phi_{\pi/2}(x,y)=\begin{cases}\sqrt{16(x-y)^2+4(x^2+y^2)} & \text{if } 0\leq y\leq \frac{x}{2},\\ \sqrt{\frac{x^4}{y^2}+4(x^2+y^2)} & \text{if } 0<\frac{x}{2}<y\leq x,\\ \sqrt{\frac{y^4}{x^2}+4(x^2+y^2)} & \text{if } 0<x<y\leq 2x,\\ \sqrt{16(y-x)^2+4(x^2+y^2)} & \text{if } 2x<y\leq 1.\end{cases}$$

Proof. We want to calculate

$$\sup\left\{\|\nabla P(x,y)\|_2\,:\,P\in \operatorname{ext}\left(B_{D\left(\frac{\pi}{2}\right)}\right)\right\}.$$

For $P=(1,1,0)$, we have $\|\nabla P(x,y)\|_2^2=4(x^2+y^2)$.
For the rest of polynomials, the case $xy=0$ is trivial, so assume that both $x\neq 0$ and $y\neq 0$. First, consider $P_t=(t,1,-2\sqrt{2(1+t)})$. Then,

$$\|\nabla P_t(x,y)\|_2^2=4t^2x^2+8(1+t)y^2-8t\sqrt{2(1+t)}xy+4y^2+8(1+t)x^2-8\sqrt{2(1+t)}xy.$$

Make now the change $u=\sqrt{2(1+t)}$ (so $u\in[0,2]$) to have

$$g_{x,y}(u):=\|\nabla P_u(x,y)\|_2^2=x^2u^4-4xyu^3+4y^2u^2+4(x^2+y^2).$$

The critical points for $g_{x,y}$ are $u=0$, $u=\frac{2y}{x}$ and $u=\frac{y}{x}$. Notice $g''_{x,y}(\frac{2y}{x})>0$, so we are in a relative minimum and therefore this point shall not be taken into consideration. Also,

$$\begin{aligned}g_{x,y}(0)&=4(x^2+y^2),\\ g_{x,y}\left(\frac{y}{x}\right)&=\frac{y^4}{x^2}+4(x^2+y^2),\\ g_{x,y}(2)&=16(x-y)^2+4(x^2+y^2).\end{aligned}$$

Hence,

$$\begin{aligned}\sup\left\{\|\nabla P_t(x,y)\|_2^2\,:\,-1\leq t\leq 1\right\}&=\begin{cases}\max\left\{g_{x,y}(0),g_{x,y}\left(\frac{y}{x}\right),g_{x,y}(2)\right\} & \text{if } 0\leq\frac{y}{x}\leq 2\ ,\\ \max\left\{g_{x,y}(0),g_{x,y}(2)\right\} & \text{otherwise.}\end{cases}\\ &=\begin{cases}\max\left\{g_{x,y}\left(\frac{y}{x}\right),g_{x,y}(2)\right\} & \text{if } 0\leq\frac{y}{x}\leq 2\ ,\\ g_{x,y}(2) & \text{otherwise.}\end{cases}\end{aligned}$$

Since $g_{x,y}(2)\leq g_{x,y}\left(\frac{y}{x}\right)$ if $0\leq\frac{y}{x}\leq 2$, we conclude that

$$\sup\left\{\|\nabla P_t(x,y)\|_2^2\,:\,-1\leq t\leq 1\right\}=\begin{cases}\frac{y^4}{x^2}+4(x^2+y^2) & \text{if } 0\leq\frac{y}{x}\leq 2\ ,\\ 16(x-y)^2+4(x^2+y^2) & \text{otherwise.}\end{cases}$$

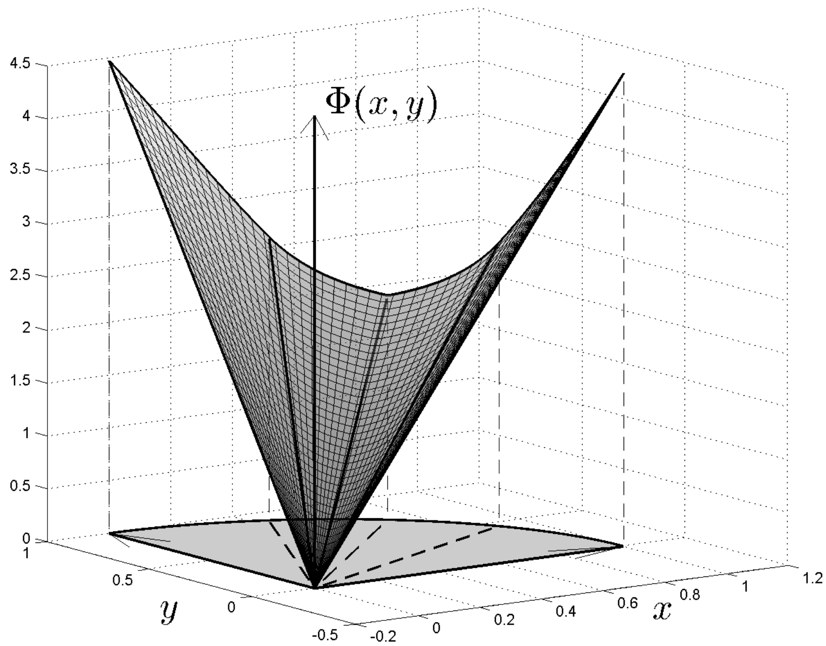


Figure 2.23: Graph of the mapping $\Phi_{\pi/2}(x,y)$

Since $Q_t(x,y) = P_t(y,x)$, by symmetry, we obtain

$$\sup\left\{\|\nabla Q_t(x,y)\|_2^2 \,:\, -1\leq t\leq 1\right\} = \begin{cases} \frac{x^4}{y^2}+4(x^2+y^2) & \text{if } 0\leq \frac{x}{y}\leq 2\ , \\ 16(x-y)^2+4(x^2+y^2) & \text{otherwise.}\end{cases}$$

Putting all together we have that $\sup\limits_{-1\leq t\leq 1}\left\{\|\nabla P_t(x,y)\|_2^2, \|\nabla Q_t(x,y)\|_2^2\right\}$ is given by

$$\begin{cases} \max\left\{\frac{y^4}{x^2}+4(x^2+y^2), 16(x-y)^2+4(x^2+y^2)\right\} & \text{if } 0\leq \frac{y}{x}\leq\frac{1}{2}\ , \\ \max\left\{\frac{y^4}{x^2}+4(x^2+y^2), \frac{x^4}{y^2}+4(x^2+y^2)\right\} & \text{if } \frac{1}{2}\leq\frac{y}{x}\leq 2, \\ \max\left\{\frac{x^4}{y^2}+4(x^2+y^2), 16(x-y)^2+4(x^2+y^2)\right\} & \text{if } 2\leq\frac{y}{x}.\end{cases}$$

Taking the latter into account, we can conclude that

$$\|\nabla P(x,y)\|_2 \leq \Phi_{\pi/2}(x,y)\|P\|_{D(\frac{\pi}{2})},$$

for any $P\in D(\frac{\pi}{2})$, where

$$\Phi_{\pi/2}(x,y) = \begin{cases} \sqrt{16(x-y)^2+4(x^2+y^2)} & \text{if } 0\leq y\leq\frac{x}{2}, \\ \sqrt{\frac{x^4}{y^2}+4(x^2+y^2)} & \text{if } 0<\frac{x}{2}<y\leq x, \\ \sqrt{\frac{y^4}{x^2}+4(x^2+y^2)} & \text{if } 0<x<y\leq 2x, \\ \sqrt{16(y-x)^2+4(x^2+y^2)} & \text{if } 2x<y\leq 1.\end{cases}$$

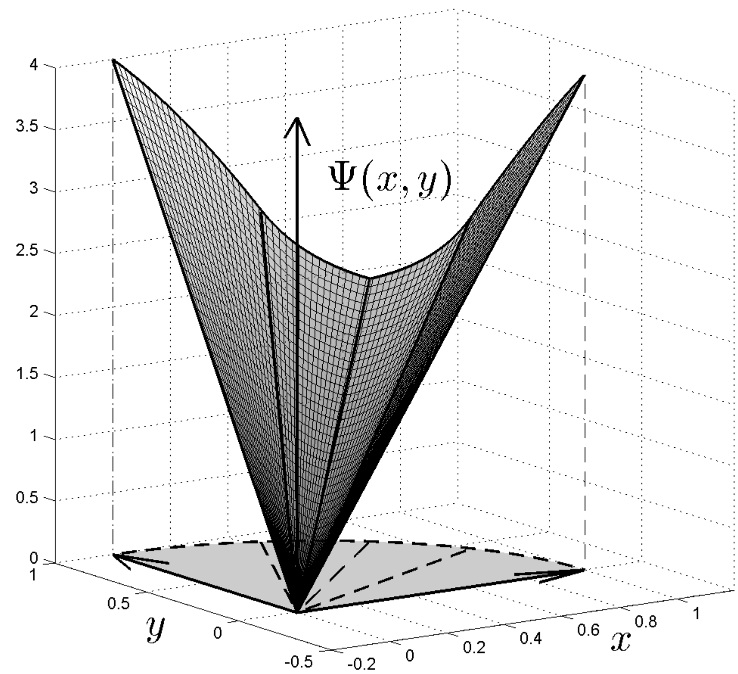


Figure 2.24: Graph of the mapping $\Psi_{\pi/2}(x,y)$

It may be interesting for the reader to take a look to the graph of the function $\Phi_{\pi/2}$ in Figure 2.23. The different parts described by the different formulae are indicated by a thicker line. □

Using the previous inequality we can derive the following Markov type estimate:

Corollary 2.6.4. *For every $(x,y) \in D\left(\frac{\pi}{2}\right)$, we have*

$$\|DP(x,y)\|_2 \leq 2\sqrt{5}\|P\|_{D\left(\frac{\pi}{2}\right)}, \tag{2.6.1}$$

with equality attained for $\pm P_1(x,y) = \pm Q_1(x,y) = \pm(x^2+y^2-4xy)$.

Proof. It suffices to check that

$$\max_{(x,y)\in D\left(\frac{\pi}{2}\right)} \Phi_{\pi/2}(x,y) = 2\sqrt{5},$$

being the maximum attained at the points $(1,0)$ and $(0,1)$. An inspection of the proof of Proposition 2.6.3 reveals that equality in (2.6.1) holds for the extreme polynomials $\pm P_1 = \pm Q_1$, or in other words $\pm(x^2+y^2-4xy)$. □

2.6.3 Polarization constant of $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$.

To conclude the result on this issue, we shall first derive a Bernstein-type inequality. Same thing as we did in the case of $D\left(\frac{\pi}{4}\right)$, for which we had to state first lemma 2.5.8, we state first the following result that shall be useful for the proof of Theorem 2.6.6:

Lemma 2.6.5. *Let $f(t) = a\cos t + b\sin t$ be defined for $0 \le t \le \frac{\pi}{2}$ and with $ab \neq 0$. Then,*

$$\max_{0\le\theta\le\frac{\pi}{2}} |f(t)| = \begin{cases} \max\{|a|,|b|\} & \text{if } ab < 0, \\ \sqrt{a^2+b^2} & \text{otherwise.} \end{cases}$$

Theorem 2.6.6 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [64])**.** *For every $(x,y) \in D(\frac{\pi}{2})$ and $P \in \mathcal{P}(^2D(\frac{\pi}{2}))$ we have that*

$$\|DP(x,y)\|_{D(\frac{\pi}{2})} \le \Psi_{\pi/2}(x,y)\|P\|_{D(\frac{\pi}{2})}, \tag{2.6.2}$$

where

$$\Psi_{\pi/2}(x,y) = \begin{cases} 2(2x-y) & \text{if } 0 \le y < \frac{x}{2}, \\ 2\left(y+\frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \le y < x, \\ 2\left(x+\frac{y^2}{2x}\right) & \text{if } x \le y < 2x, \\ 2(2y-x) & \text{if } y \ge 2x. \end{cases}$$

Moreover, inequality (2.6.2) is optimal for each $(x,y) \in D(\frac{\pi}{2})$.

Proof. In order to calculate $\Psi_{\pi/2}(x,y) := \sup\{\|DP(x,y)\|_{D(\frac{\pi}{2})} : \ \|P\|_{D(\frac{\pi}{2})}) \le 1\}$, by the Krein-Milman approach, it suffices to calculate

$$\sup\{\|DP(x,y)\|_{D\left(\frac{\pi}{2}\right)} : P \in \mathrm{ext}(B_{D(\frac{\pi}{2})})\}.$$

Again, The easy case $xy = 0$ is trivial and shall not be taken into consideration, so assume that both $x \neq 0$ and $y \neq 0$. Let us deal first with the polynomials $Q_t = (1, t, -2\sqrt{2(1+t)})$, with $t \in [-1,1]$.

Since $Q_t(x,y) = x^2 + ty^2 - 2\sqrt{2(1+t)}xy$, then

$$\nabla Q_t(x,y) = \left(2x - 2\sqrt{2(1+t)}y, 2ty - 2\sqrt{2(1+t)}x\right).$$

Therefore,

$$\|DQ_t(x,y)\|_{D\left(\frac{\pi}{2}\right)} = \sup_{(h,k)\in D\left(\frac{\pi}{2}\right)} \left|\left(2x - 2\sqrt{2(1+t)}y\right)h + \left(2ty - 2\sqrt{2(1+t)}x\right)k\right|.$$

In order to calculate the above supremum we can restrict attention to the extreme points of $D\left(\frac{\pi}{2}\right)$ (except for the point $(h,k) = (0,0)$ that does not contribute anything to the supremum). Thus, putting $(h,k) = (\cos\theta, \sin\theta)$ with $0 \le \theta \le \frac{\pi}{2}$ and $\lambda = \frac{y}{x}$,

$$\sup_{-1\le t\le 1} \|\nabla Q_t(x,y)\|_{D(\frac{\pi}{2})} = 2x \sup_{(t,\theta)\in C_{\frac{\pi}{2}}} \left|\left(1 - \sqrt{2(1+t)}\lambda\right)\cos\theta + \left(t\lambda - \sqrt{2(1+t)}\right)\sin\theta\right|,$$

where $C_{\frac{\pi}{2}} = [-1,1] \times [0, \frac{\pi}{2}]$.

Define

$$f_\lambda(t,\theta) = \left(1 - \sqrt{2(1+t)}\lambda\right)\cos\theta + \left(t\lambda - \sqrt{2(1+t)}\right)\sin\theta,$$

and consider the following cases:

- $0 < \theta < \frac{\pi}{2}$, $-1 < t < 1$.

The critical points of f_λ in the interior of $C_{\frac{\pi}{2}}$ are the solutions of the equations:

$$\frac{\partial f_\lambda}{\partial t}(t_0,\theta_0) = \frac{-\sqrt{2}\lambda}{2\sqrt{1+t_0}}\cos\theta_0 + \left(\lambda - \frac{\sqrt{2}}{2\sqrt{1+t_0}}\right)\sin\theta_0 = 0, \tag{2.6.3}$$

$$\frac{\partial f_\lambda}{\partial \theta}(t_0,\theta_0) = -\left(1-\sqrt{2(1+t_0)}\lambda\right)\sin\theta_0 + \left(t_0\lambda - \sqrt{2(1+t_0)}\right)\cos\theta_0 = 0. \tag{2.6.4}$$

Working with equation (2.6.3), we get to the next expression:

$$\sin\theta_0 = \frac{\frac{\sqrt{2}}{2\sqrt{1+t_0}}\lambda}{\lambda - \frac{\sqrt{2}}{2\sqrt{1+t_0}}}\cos\theta_0 = \frac{\sqrt{2}\lambda}{2\lambda\sqrt{1+t_0}-\sqrt{2}}\cos\theta_0 \tag{2.6.5}$$

and, plugging the expression in (2.6.5) into equation (2.6.4) we obtain

$$\left[\left(t_0\lambda - \sqrt{2(1+t_0)}\right) - \frac{\sqrt{2}\lambda\left(1-\sqrt{2(1+t_0)}\lambda\right)}{2\lambda\sqrt{1+t_0}-\sqrt{2}}\right]\cos\theta_0 = \left[t_0\lambda - \sqrt{2(1+t_0)}+\lambda\right]\cos\theta_0 = 0.$$

Now, since $0<\theta_0<\frac{\pi}{2}$ we can have $\cos\theta_0 \neq 0$ and hence,

$$\lambda(1+t_0) = \sqrt{2(1+t_0)},$$

from which $t_0 = \frac{2}{\lambda^2}-1$.

If we now apply the condition $-1<t_0<1$ we get the restriction $\lambda>1$, that is, we will only have critical points in the interior of $C_{\frac{\pi}{2}}$ when $y>x$.

Now, plugging t_0 in (2.6.5), we obtain $\tan\theta_0=\lambda$, from which

$$\sin\theta_0 = \frac{\lambda}{\sqrt{1+\lambda^2}} \quad \text{and} \quad \cos\theta_0 = \frac{1}{\sqrt{1+\lambda^2}}.$$

Then,

$$\begin{aligned} 2x|f_\lambda(t_0,\theta_0)| &= 2x\left|\left(1-\sqrt{\frac{4}{\lambda^2}}\lambda\right)\frac{1}{\sqrt{1+\lambda^2}} + \left[\left(\frac{2}{\lambda^2}-1\right)\lambda - \sqrt{\frac{4}{\lambda^2}}\right]\frac{\lambda}{\sqrt{1+\lambda^2}}\right| \\ &= 2x\left|\frac{-1}{\sqrt{1+\lambda^2}} - \frac{\lambda^2}{\sqrt{1+\lambda^2}}\right| = 2x\sqrt{1+\lambda^2} \end{aligned} \tag{2.6.6}$$

- $t=-1$ and $0\le\theta\le\frac{\pi}{2}$.

Using lemma 2.6.5, we may conclude that

$$2x \max_{0\le\theta\le\frac{\pi}{2}} |f_\lambda(-1,\theta)| = 2x\max\{1,\lambda\}. \tag{2.6.7}$$

- $t=1$, $0\le\theta\le\frac{\pi}{2}$.

In this case we shall study the expression $2x|f_\lambda(1,\theta)| = 2x|(1-2\lambda)\cos\theta + (\lambda-2)\sin\theta|$. Again, by lemma 2.6.5, we have the following:

$$\max_{0\le\theta\le\frac{\pi}{2}} 2x|f_\lambda(1,\theta)| = \begin{cases} 2x\max\{|1-2\lambda|,|\lambda-2|\} & \text{if } 0\le\lambda<\frac{1}{2} \text{ or } \lambda>2,\\ 2x\sqrt{(2\lambda-1)^2+(2-\lambda)^2} & \text{if } \frac{1}{2}\le\lambda\le 2. \end{cases} \tag{2.6.8}$$

It can be easily checked that the expression we have arrived at in (2.6.8) is greater than (2.6.6).

- $\theta = 0$, $-1\le t\le 1$.

We need to calculate

$$\begin{aligned} 2x\max_{-1\le t\le 1}|f_\lambda(t,0)| &= 2x\max_{-1\le t\le 1}\left|1-\sqrt{2(1+t)}\lambda\right| = 2x\max\{1,|1-2\lambda|\}\\ &= \begin{cases} 2x & \text{if } 0\le\lambda<1,\\ 2x(2\lambda-1) & \text{if } \lambda\ge 1. \end{cases} \end{aligned} \tag{2.6.9}$$

Observe that the latter is always greater that (2.6.7).

- $\theta = \frac{\pi}{2}$, $-1\le t\le 1$.

We have to calculate

$$2x\max_{-1\le t\le 1}|t\lambda-\sqrt{2(1+t)}|,$$

for which we define $h(t) = t\lambda - \sqrt{2(1+t)}$, for $t\in[-1,1]$. The critical points of h satisfy

$$h'(t) = \lambda - \frac{\sqrt{2}}{2\sqrt{1+t}} = 0,$$

from which

$$t_1 = \frac{1}{2\lambda^2} - 1.$$

Observe that $t_0\in[-1,1]$ if and only if $|\lambda|\ge\frac{1}{2}$.

To summarize we have equation

$$\begin{aligned} 2x\max_{-1\le t\le 1}|t\lambda-\sqrt{2(1+t)}| &= \begin{cases} 2x\max\{|h(-1)|,|h(1)|,|h(t_1)|\} & \text{if } \lambda\ge\frac{1}{2},\\ 2x\max\{|h(-1)|,|h(1)|\} & \text{if } 0\le\lambda<\frac{1}{2}, \end{cases}\\ &= \begin{cases} 2x\left(\lambda+\frac{1}{2\lambda}\right) & \text{if } \lambda\ge\frac{1}{2},\\ 2x(2-\lambda) & \text{if } 0\le\lambda<\frac{1}{2}. \end{cases} \end{aligned} \tag{2.6.10}$$

Since we have already discarded (2.6.6) and (2.6.7), putting together (2.6.8), (2.6.9) and (2.6.10) we arrive at

$$2x\sup_{(t,\theta)\in C_{\frac{\pi}{2}}}|f_\lambda(t,\theta)| = 2x\begin{cases} \max\{|1-2\lambda|,|2-\lambda|,1\} & \text{if } 0\le\lambda\le\frac{1}{2},\\ \max\left\{\sqrt{(2\lambda-1)^2+(2-\lambda)^2},1,\left(\lambda+\frac{1}{2\lambda}\right)\right\} & \text{if } \frac{1}{2}\le\lambda\le 1,\\ \max\left\{\sqrt{(2\lambda-1)^2+(2-\lambda)^2},2\lambda-1,\left(\lambda+\frac{1}{2\lambda}\right)\right\} & \text{if } 1\le\lambda\le 2,\\ \max\left\{\left(\lambda+\frac{1}{2\lambda}\right),|1-2\lambda|,|2-\lambda|\right\} & \text{if } \lambda\ge 2. \end{cases}$$

The only comparisons in the previous expression with some difficulty are

$$\lambda + \frac{1}{2\lambda} \quad \text{and} \quad \sqrt{(2\lambda-1)^2 + (\lambda-2)^2}.$$

Through standard calculations, we deduce that $\lambda + \frac{1}{2\lambda} \geq \sqrt{(2\lambda-1)^2 + (\lambda-2)^2}$ whenever $\lambda \leq \frac{1+\sqrt{2}}{2}$. Since we shall consider this situation when $\lambda \geq \frac{1}{2}$, we conclude

$$\sup_{-1\leq t\leq 1} \|\nabla Q_t(x,y)\|_{D(\frac{\pi}{2})} = \Psi_1(x,y),$$

where

$$\Psi_1(x,y) = \begin{cases} 2(2x-y) & \text{if } 0 \leq y < \frac{x}{2}, \\ 2\left(y + \frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \leq y < \frac{1+\sqrt{2}}{2}x, \\ 2\sqrt{(2x-y)^2 + (2y-x)^2} & \text{if } \frac{1+\sqrt{2}}{2}x \leq y < 2x, \\ 2(2y-x) & \text{if } y \geq 2x. \end{cases}$$

Using the symmetry $P_t(x,y) = Q_t(y,x)$, we may see

$$\sup_{-1\leq t\leq 1} \|\nabla P_t(x,y)\|_{D(\frac{\pi}{2})} = \Psi_2(x,y),$$

where

$$\Psi_2(x,y) = \Psi_1(y,x) = \begin{cases} 2(2x-y) & \text{if } 0 \leq y < \frac{x}{2}, \\ 2\sqrt{(2x-y)^2 + (2y-x)^2} & \text{if } \frac{x}{2} \leq y < \left(2\sqrt{2}-2\right)x, \\ 2\left(x + \frac{y^2}{2x}\right) & \text{if } \left(2\sqrt{2}-2\right)x \leq y < 2x, \\ 2(2y-x) & \text{if } y \geq 2x. \end{cases}$$

Therefore, we conclude

$$\begin{aligned} \Psi_{\pi/2}(x,y) &= \max\{\Psi_1(x,y), \Psi_2(x,y)\} \\ &= \begin{cases} 2(2x-y) & \text{if } 0 \leq y < \frac{x}{2}, \\ \max\left\{2\sqrt{(2x-y)^2 + (2y-x)^2}, 2\left(y + \frac{x^2}{2y}\right)\right\} & \text{if } \frac{x}{2} \leq y \leq \left(2\sqrt{2}-2\right)x, \\ \max\left\{2\left(y + \frac{x^2}{2y}\right), 2\left(x + \frac{y^2}{2x}\right)\right\} & \text{if } \left(2\sqrt{2}-2\right)x \leq y \leq \frac{1+\sqrt{2}}{2}x, \\ \max\left\{2\left(x + \frac{y^2}{2x}\right), 2\sqrt{(2x-y)^2 + (2y-x)^2}\right\} & \text{if } \frac{1+\sqrt{2}}{2}x \leq y \leq 2x, \\ 2(2y-x) & \text{if } y \geq 2x. \end{cases} \\ &= \begin{cases} 2(2x-y) & \text{if } 0 \leq y < \frac{x}{2}, \\ 2\left(y + \frac{x^2}{2y}\right) & \text{if } \frac{x}{2} \leq y < x, \\ 2\left(x + \frac{y^2}{2x}\right) & \text{if } x \leq y < 2x, \\ 2(2y-x) & \text{if } y \geq 2x. \end{cases} \end{aligned}$$

As with the case of $\Phi_{\pi/2}$, we can take a look to the graph of $\Psi_{\pi/2}$ in Figure 2.24, where we have used a thicker line to indicate the different regions defined by the different formulae.

□

Taking the maximum of $\Psi_{\pi/2}(x,y)$ with $(x,y) \in D\left(\frac{\pi}{2}\right)$ we can obtain the polarization constant of $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$:

Corollary 2.6.7. *Let $P \in \mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$. Then*

$$\|\check{P}\|_{D(\frac{\pi}{2})} \leq 2\|P\|_{D(\frac{\pi}{2})}.$$

Moreover, equality is achieved for $\pm P_1(x,y) = \pm(x^2 + y^2 - 4xy)$.

2.6.4 Unconditional constant of $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$

Let us conclude finally with the unconditional constant for $\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$.

Theorem 2.6.8 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [64])**.** *The unconditional constant of the canonical basis of the space $\mathcal{P}({}^2D(\frac{\pi}{2}))$ is 3. In other words, the inequality*

$$\||ax^2 + by^2 + cxy|\|_{D(\frac{\pi}{2})} \leq 3\|ax^2 + by^2 + cxy\|_{D(\frac{\pi}{2})},$$

holds for all $a, b, c \in \mathbb{R}$ and 3 is optimal since equality is achieved for the polynomials $\pm(x^2 + y^2 - 4xy)$.

Proof. Observe that the extreme polynomials in the unit ball of $\mathcal{P}({}^2D(\frac{\pi}{2}))$ are

$$P_t(x) = tx^2 + y^2 - 2\sqrt{2(1+t)}xt,$$
$$Q_t(x) = x^2 + ty^2 - 2\sqrt{2(1+t)}xt,$$

with $t \in [-1,1]$. If we plug these polynomials in Theorem 2.5.1, due to the symmetry of the problem we end up with the maximum of

$$\max\left\{|t|, 1, \frac{1}{2}\left||t| + 1 + \sqrt{(|t|-1)^2 + 8(1+t)}\right|\right\}.$$

The latter function attains its maximum at 1 and turns out to be 3. □

.

2.7 Conclusions

If we put together all the constants that came out during all our calculations, we can derive the following table:

	$\mathcal{P}({}^2\Delta)$	$\mathcal{P}\left({}^2D\left(\frac{\pi}{2}\right)\right)$	$\mathcal{P}({}^2\square)$
Markov constants	$2\sqrt{10}$	$2\sqrt{5}$	$\sqrt{13}$
Polarization constants	3	2	$\frac{3}{2}$
Unconditional Constants	2	3	5

Furthermore, all the constants appearing in the previous table are sharp. Actually, the extreme polynomials where the constants are attained are the following:

1. $\pm(x^2 + y^2 - 6xy)$ for the simplex.
2. $\pm(x^2 + y^2 - 4xy)$ for the sector $D\left(\frac{\pi}{2}\right)$.
3. $\pm(x^2 + y^2 - 3xy)$ for the unit square.

Remark also that Markov and Polarization constants (where the comparison between the norms of a polynomial and its differential lies beneath) seem to happen decreasingly with the factor p in the space ℓ_p^2, while the unconditional constant seems to happen increasingly. The question that arises now, and that may constitute an interesting way for research, is whether this behavior translates to the general case when the unit ball is the intersection of the unit ball of ℓ_p^2 (for $1 < p < \infty$) with the first quadrant. Furthermore, we shall wonder whether all the three constants, Markov's, Polarization and Unconditional, happens to turn into equality for the same polynomial.

Chapter 3

The Bohnenblust-Hille inequality

While in the previous chapter we stressed the existence of bounds between the operator norm (or the euclidean length) of the differential of a polynomial and the operator norm of the polynomial itself, in this chapter we shall study how a polynomial behaves when applied different norms. More concretely, if $P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$ is a homogeneous polynomial of degree m defined over $\mathbb{K}^n$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) we define

$$|P|_p = \left(\sum_{|\alpha|=m} |a_\alpha|^p \right)^{\frac{1}{p}}.$$

We know that, working over $\mathbb{R}^n$ (endowed with the infinite norm), the polynomial norm $\|\cdot\|$ (still defined as supremum over the unit ball) and the ℓ_p norm $|\cdot|_p$ ($p \geq 1$) are equivalent, and therefore there exist constants $k(m,n), K(m,n) > 0$ such that

$$k(m,n)|P|_p \leq \|P\| \leq K(m,n)|P|_p, \tag{3.0.1}$$

for all $P \in \mathcal{P}(^m\mathbb{R}^n)$. The latter inequalities may provide a good estimate on $\|P\|$ (which usually is hard to compute) as long as we know the exact value of the best possible constants $k(m,n)$ and $K(m,n)$ appearing in (3.0.1).
The problem presented above is an extension of the the well known polynomial Bohnenblust-Hille inequality (polynomial BH inequality for short). It was proved in [22] that there exists a constant $D_m \geq 1$ such that for every $P \in \mathcal{P}(^m\ell_\infty^n)$ we have

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|. \tag{3.0.2}$$

Observe that (3.0.2) coincides with the first inequality in (3.0.1) for $p = \frac{2m}{m+1}$ except for the fact that D_m in (3.0.2) can be chosen in such a way that it is independent from the dimension n. Actually Bohnenblust and Hille showed that $\frac{2m}{m+1}$ is optimal in (3.0.2) in the sense that for $p < \frac{2m}{m+1}$, any constant D fitting in the inequality

$$|P|_p \leq D\|P\|,$$

for all $P \in \mathcal{P}(^m\ell_\infty^n)$ depends necessarily on n.
The best constants in (3.0.2) may depend on whether we consider the real or the complex version

of ℓ_∞^n, which motivates the following definition

$$D_{\mathbb{K},m} := \inf\left\{D > 0 \,:\, |P|_{\frac{2m}{m+1}} \leq D\|P\|, \text{ for all } n \in \mathbb{N} \text{ and } P \in \mathcal{P}(^m\ell_\infty^n)\right\}$$

If we restrict attention to a certain subset E of $\mathcal{P}(^m\ell_\infty^n)$ for some $n \in \mathbb{N}$, then we define

$$D_{\mathbb{K},m}(E) := \inf\left\{D > 0 : |P|_{\frac{2m}{m+1}} \leq D\|P\| \text{ for all } P \in E\right\}.$$

For simplicity we will often use the notation $D_{\mathbb{K},m}(n)$ instead of $D_{\mathbb{K},m}(\mathcal{P}(^m\ell_\infty^n))$. Note that $D_{\mathbb{K},m}(n) \leq D_{\mathbb{K},m}$ for all $n \in \mathbb{N}$.

Definition 3.0.1. *The asymptotic hypercontractivity constant of the real polynomial BH inequality is*

$$H_{\infty,\mathbb{R}} := \limsup_m \sqrt[m]{D_{\mathbb{R},m}}.$$

Similarly, if we restrict attention to polynomials in n variables then we define

$$H_{\infty,\mathbb{R}}(n) := \limsup_m \sqrt[m]{D_{\mathbb{R},m}(n)}.$$

Of course $H_{\infty,\mathbb{R}}(n) \leq H_{\infty,\mathbb{R}}$, for all $n \in \mathbb{N}$.

In Section 3.1 we show that $H_{\infty,\mathbb{R}}$ is finite (in fact later we will show that its precise value is 2); observe that for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $D_{\mathbb{R},m} \leq (H_{\infty,\mathbb{R}} + \epsilon)^m$ for all $m \geq N_\epsilon$ and $H_{\infty,\mathbb{R}}$ is optimal having this property. Hence $H_{\infty,\mathbb{R}}$ is a sharp measure of the asymptotic growth of the constants $D_{\mathbb{R},m}$. Exactly the same thing happens with the constants $H_{\infty,\mathbb{R}}(n)$.

It was recently shown in [35] that the complex polynomial Bohnenblust–Hille inequality is, at most, hypercontractive, that is, we can take $C > 1$ so that $D_{\mathbb{C},m} \leq C^m$ for every natural number m.

The multilinear Bohnenblust–Hille inequality asserts that there is a constant $C_{\mathbb{K},m}$ (depending only on m and $\mathbb{K}$) such that

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} |L(e_{i_1},\ldots,e_{i_m})|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C\,\|L\|,$$

for all positive integers n and all $L \in \mathcal{L}(^m\ell_\infty^n)$. Observe that the left hand side of the previous inequality is the $\ell_{\frac{2m}{m+1}}$-norm of the vector $(L(e_{i_1},\ldots,e_{i_m}))_{i_1,\ldots,i_m=1}^n$ of the coefficients of L, which we denote by $|L|_{\frac{2m}{m+1}}$. Thus, the multilinear Bohnenblust-Hille constant $C_{\mathbb{K},m}$ (multilinear BH constant for short) is defined as

$$C_{\mathbb{K},m} := \inf\left\{C > 0 : |L|_{\frac{2m}{m+1}} \leq C\|L\|, \text{ for all } n \in \mathbb{N} \text{ and } L \in \mathcal{L}(^m\ell_\infty^n)\right\}.$$

The polynomial and multilinear Bohnenblust–Hille inequalities have important applications in different fields of Mathematics and Physics, such as Operator Theory, Fourier and Harmonic Analysis, Complex Analysis, Analytic Number Theory and Quantum Information Theory. Since its appearance in 1931, in the *Annals of Mathematics* (intended to solve Bohr's famous absolute convergence problem within the theory of Dirichlet series) it has proved to be a field of study that has provided interesting results (see [22, 21]), but it has been in the last few years, with works of A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes, D. Popa, U. Schwarting, K. Seip, among others (see, e.g., [42, 45, 34, 75, 85, 92, 83, 46, 76, 84, 86]), where its most fruitful facet has been shown. Remark that the study of the applications of the Bohnenblust-Hille inequality for solving Bohr's problem has being recently explored by several authors (see [17, 27, 36, 38, 39, 40, 42] and references therein).

The main motivation of this chapter are the following open problems:

(I) Is the *real* polynomial BH inequality hypercontractive?

(II) What is the optimal growth of the real polynomial BH inequality?

(III) Can we make some links between what happens with the real case and what happens in the complex case?

The chapter is arranged as follows. In Sections 3.1 and 3.2 we give an answer to questions (I) and (II), showing that the real polynomial BH inequality is hypercontractive and the hypercontractivity is actually optimal (in fact exponential). More concretely, after the results shown in those first sections we will be able to conclude

$$\limsup_m D_{\mathbb{R},m}^{1/m} = 2.$$

In the final part of the chapter we will deal with the connections between real and complex Bohnenblust-Hille inequalities. First, we will stress the existing differences between both problems, and how results obtained in the real case may not work in the complex setting. After that, we shall employ some results on the geometry of spaces of polynomials in order to provide the exact value for $D_{\mathbb{C},2}(2)$. To finish, we will use a similar technique to find the exact value of $D_{\mathbb{R},2}(2)$ and we shall also provide lower estimates for $D_{\mathbb{R},m}(2)$ for higher values of m and $H_{\mathbb{R},\infty}(2)$ by means of numerical calculus.

3.1 The upper estimate

The proof of the subexponentiality of the complex BH inequality given in [14] lies heavily in arguments restricted to complex scalars (it uses, for instance estimates from [13] for complex scalars); so a simple adaptation for the real case does not work. The calculation of the upper estimate of the BH inequality is quite simplified by the use of complexifications of polynomials. In particular we are interested in the following deep result due to Visser [97], which generalizes and old result of Chebyshev:

Theorem 3.1.1 (Visser, [97]). *Let*

$$P(y_1, \ldots, y_n) = \sum_{|\alpha| \leq m} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n},$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, be a polynomial of total degree at most $m \in \mathbb{N}$ in the variables $y_1, \ldots, y_n$ and with real coefficients a_α. Suppose $0 \leq k \leq m$ and P_k is the homogeneous polynomial of degree k defined by

$$P_k(y_1, \ldots, y_n) = \sum_{|\alpha|=k} a_\alpha y_1^{\alpha_1} \cdots y_n^{\alpha_n}.$$

Then we have

$$\max_{z_1,\ldots,z_n \in \mathbb{D}} |P_m(z_1, \ldots, z_n)| \leq 2^{m-1} \cdot \max_{x_1,\ldots,x_n \in [-1,1]} |P(x_1, \ldots, x_n)|,$$

where $\mathbb{D}$ stands for the closed unit disk in $\mathbb{C}$. In particular, if P is homogeneous, then

$$\max_{z_1,\ldots,z_n \in \mathbb{D}} |P(z_1, \ldots, z_n)| \leq 2^{m-1} \cdot \max_{x_1,\ldots,x_n \in [-1,1]} |P(x_1, \ldots, x_n)|.$$

Moreover, the constant 2^{m-1} cannot be replaced by any smaller one.

Let $P : \ell_\infty^n(\mathbb{R}) \to \mathbb{R}$ be an m-homogeneous polynomial

$$P(x) = \sum_{|\alpha|=m} a_\alpha x^\alpha$$

and consider the complexification $P_{\mathbb{C}} : \ell_\infty^n(\mathbb{C}) \to \mathbb{C}$ of P given by

$$P_{\mathbb{C}}(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha.$$

From Theorem 3.1.1 above we know that

$$\|P_{\mathbb{C}}\| \leq 2^{m-1} \|P\|. \tag{3.1.1}$$

Thus, since the complex polynomial Bohnenblust–Hille inequality is subexponential (see [14]), for all $\varepsilon > 0$ there exists $C_\varepsilon > 1$ such that

$$|P|_{\frac{2m}{m+1}} = |P_{\mathbb{C}}|_{\frac{2m}{m+1}} \leq C_\varepsilon (1+\varepsilon)^m \|P_{\mathbb{C}}\| \tag{3.1.2}$$

and combining (3.1.1) and (3.1.2) we conclude that

$$\limsup_m D_{\mathbb{R},m}^{1/m} \leq 2.$$

As we mentioned earlier, Bayart et al. proved, recently, that the complex polynomial Bohnenblust–Hille inequality is subexponential (see [14]). The following result shows that the exponential growth of the real polynomial BH inequality is sharp in a very strong way: the exponential bound can not be reduced in any sense, i.e., there is an exponential lower bound for $D_{\mathbb{R},m}$ which holds for every $m \in \mathbb{N}$.

Theorem 3.1.2 (J.R. Campos, P. Jiménez-Rodríguez, G-A Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [28])**.**

$$D_{\mathbb{R},m} > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^m > (1.17)^m$$

for all positive integers $m > 1$.

Proof. Let m be an even integer. Consider the m-homogeneous polynomial

$$R_m(x_1,\dots,x_m) = \left(x_1^2 - x_2^2 + x_1x_2\right)\left(x_3^2 - x_4^2 + x_3x_4\right)\cdots\left(x_{m-1}^2 - x_m^2 + x_{m-1}x_m\right).$$

Since $\|R_2\| = 5/4$, it is simple to see that

$$\|R_m\| = (5/4)^{m/2}.$$

From the BH inequality for R_m we have

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq D_{\mathbb{R},m}\,\|R_m\|,$$

that is,

$$D_{\mathbb{R},m} \geq \frac{\left(3^{\frac{m}{2}}\right)^{\frac{m+1}{2m}}}{\left(\frac{5}{4}\right)^{\frac{m}{2}}} \geq \frac{\left(\sqrt{3}\right)^{\frac{m+1}{2}}}{\left(\frac{5}{4}\right)^{\frac{m}{2}}} > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^m.$$

Now let us suppose that m is odd. Keeping the previous notation, consider the m homogeneous polynomial

$$R_m\left(x_1,...,x_{2m}\right) = \left(x_{2m} + x_{2m-1}\right)R_{m-1}\left(x_1,...,x_{m-1}\right) + \left(x_{2m} - x_{2m-1}\right)R_{m-1}\left(x_m,...,x_{2m-2}\right).$$

So we have

$$D_{\mathbb{R},m} \geq \frac{\left(4\cdot 3^{\frac{m-1}{2}}\right)^{\frac{m+1}{2m}}}{2\cdot\left(\frac{5}{4}\right)^{\frac{m-1}{2}}} > 2^{m-1+\frac{1}{m}}\left(\frac{\sqrt[4]{3}}{\sqrt{5}}\right)^{m-1} > \left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^{m-1}.$$

□

3.2 The lower estimate

Using our previous results, in order to show that $\limsup_m D_{\mathbb{R},m}^{1/m} = 2$, we just need the following theorem:

Theorem 3.2.1 (J.R. Campos, P. Jiménez-Rodríguez, G-A Muñoz-Fernández, D. Pellegrino, J.B. Seoane-Sepúlveda, [28])**.** *If $k \in \mathbb{N}$ is fixed, then*

$$\limsup_m D_{\mathbb{R},m}^{1/m}(2^k) \geq 2^{1-2^{-k}}.$$

Therefore,

$$\limsup_m D_{\mathbb{R},m}^{1/m} \geq 2.$$

Proof. Consider the sequence of polynomials (with norm 1) defined recursively by

$$Q_2(x_1, x_2) = x_1^2 - x_2^2,$$
$$Q_{2^m}(x_1, \ldots, x_{2^k}) = Q_{2^{m-1}}(x_1, \ldots, x_{2^{m-1}})^2 - Q_{2^{m-1}}(x_{2^{m-1}+1}, \ldots, x_{2^m})^2.$$

Let us show (by induction on m) that

$$|Q_{2^m}^n|_\infty \geq \left(\frac{2^n}{n+1}\right)^{2^m-1} \tag{3.2.1}$$

for every natural number n. The case $m = 1$ comes from the fact that, since

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \leq (n+1) \max_{0 \leq k \leq n} \binom{n}{k},$$

the $2n$-homogeneous polynomial Q_2^n admits the following estimate:

$$|Q_2^n|_{\frac{4n}{2n+1}} \geq |Q_2^n|_\infty = \max_{0 \leq k \leq n} \binom{n}{k} \geq \frac{2^n}{n+1}. \tag{3.2.2}$$

Let us now suppose that equation (3.2.1) holds for some m, and notice that

$$Q_{2^m+1}^n(x_1, x_2, \ldots, x_{2^{m+1}}) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} Q_{2^m}^{2k}(x_1, \ldots, x_{2^m}) Q_{2^m}^{2(n-k)}(x_{2^m+1}, \ldots, x_{2^{m+1}}). \tag{3.2.3}$$

The coefficient of maximal absolute value in a product of polynomials in disjoint sets of variables is the product of the respective maximal coefficients, thus

$$|Q_{2^m+1}^n|_\infty = \max_{0 \leq k \leq n} \binom{n}{k} |Q_{2^m}^{2k}|_\infty |Q_{2^m}^{2(n-k)}|_\infty \geq \max_{0 \leq k \leq n} \binom{n}{k} \left(\frac{2^{2n}}{(2k+1)(2n-2k+1)}\right)^{2^m-1}$$

by the induction hypothesis. However, $(2k+1)(2n-2k+1) \leq (n+1)^2$ when $0 \leq k \leq n$; thus

$$|Q_{2^{m+1}}^n|_\infty \geq \left(\frac{2^n}{n+1}\right)^{2^{m+1}-2} \max_{0 \leq k \leq n} \binom{n}{k} \geq \left(\frac{2^n}{n+1}\right)^{2^{m+1}-1},$$

by equation (3.2.2). Therefore, the formula given in (3.2.1) holds for every positive integer m. Next, every n-homogeneous polynomial P admits the clear estimate given by

$$|P|_{\frac{2n}{n+1}} \geq |P|_\infty,$$

from which equation (3.2.1) yields that

$$D_{\mathbb{R},n2^m}(2^m) \geq \left(\frac{2^n}{n+1}\right)^{2^m-1},$$

and the proof follows straightforwardly. □

3.3 Contractivity in finite dimensions: complex versus real scalars

Let us first give the following result for the complex polynomial Bohnenblust-Hille constants for polynomials on $\mathbb{C}^n$, with $n \in \mathbb{N}$ fixed.

Proposition 3.3.1. *For all $n \geq 2$ the complex polynomial BH inequality is contractive in $\mathcal{P}(^m\ell_\infty^n)$,that is, for all fixed $n \in \mathbb{N}$, there are constants D_m, with $\lim_{m\to\infty} D_m = 1$, so that*

$$|P|_{\frac{2m}{m+1}} \leq D_m \|P\|$$

for all $P \in \mathcal{P}(^m\ell_\infty^n)$.

Proof. Let $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ and $f(t) = P(e^{it_1}, \ldots, e^{it_n}) = \sum_{|\alpha|=m} c_\alpha e^{i\alpha t}$, where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ $\alpha \in (\mathbb{N} \cup \{0\})^n$ and $\alpha t = \alpha_1 t_1 + \cdots + \alpha_n t_n$. . Observe that if $\|f\|$ denotes the sup norm of f on $[-\pi, \pi]$, by the Maximum Modulus Principle $\|f\| = \|P\|$. Also, due to the orthogonality of the system $\{e^{iks} : k \in \mathbb{Z}\}$ in $L^2([-\pi, \pi])$ we have

$$\|P\|^2 = \|f\|^2 \geq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2 dt = \sum_{|\alpha|=m} |c_\alpha|^2 = |P|_2^2,$$

from which $|P|_2 \leq \|P\|$. On the other hand it is well known that in $\mathbb{K}^d$ we have

$$|\cdot|_q \leq |\cdot|_p \leq d^{\frac{1}{p}-\frac{1}{q}} |\cdot|_q, \tag{3.3.1}$$

for all $1 \leq p \leq q$. Since the dimension of $\mathcal{P}(^m\ell_\infty^n)$ is $\binom{m+n-1}{n-1}$, the result follows from $|P|_2 \leq \|P\|$ by setting in the equation (3.3.1) $p = \frac{2m}{m+1}$, $q = 2$ and $d = \binom{m+n-1}{n-1}$. So $D_m = \binom{m+n-1}{n-1}^{\frac{1}{2m}}$ and since

$$\lim_{m\to\infty} \binom{m+n-1}{n-1}^{\frac{1}{2m}} = 1,$$

the proof is done. ☐

The next result shows that the *real version* of Proposition 3.3.1 is not valid; we stress that Theorem 3.1.2 cannot be used here since it uses polynomials in a growing number of variables.

Proposition 3.3.2. *For all fixed positive integer $N \geq 2$, the exponentiality of the real polynomial Bohnenblust-Hille inequality in $\mathcal{P}(^m\ell_\infty^N)$ cannot be improved. More precisely,*

$$\limsup_m D_{\mathbb{R},m}^{1/m}(N) \geq \sqrt[8]{27} \approx 1.5098$$

for all $N \geq 2$.

Proof. If suffices to set $N = 2$ and prove that, for $m = 4n$,

$$D_{\mathbb{R},4n}(2) \geq \sqrt[4]{\frac{4}{m\pi}} \left(\sqrt[8]{27}\right)^m.$$

Consider the 4-homogeneous polynomial given by

$$P_4(x,y) = x^3y - xy^3 = xy(x^2 - y^2).$$

A straightforward calculation shows that P_4 attains its norm at $\pm(\pm\frac{1}{\sqrt{3}}, 1)$ and $\pm(1, \pm\frac{1}{\sqrt{3}})$ and that $\|P_4\| = \frac{2\sqrt{3}}{9}$. On the other hand $\|P_4^n\| = \left(\frac{2\sqrt{3}}{9}\right)^n$ and

$$P_4(x,y)^n = x^n y^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2k} y^{2(n-k)}.$$

Hence, if $\mathbf{a}$ is the vector of the coefficients of P_4, using the fact that $|\cdot|_{\frac{8n}{4n+1}} \geq |\cdot|_2$ (notice that here $|\cdot|_2$ is the Euclidean norm), we have

$$\begin{aligned} D_{\mathbb{R},4n}(2) &\geq \frac{|\mathbf{a}|_{\frac{8n}{4n+1}}}{\|P_4\|^n} = \frac{\left[\sum_{k=0}^{n} \binom{n}{k}^{\frac{8n}{4n+1}}\right]^{\frac{4n+1}{8n}}}{\left(\frac{2\sqrt{3}}{9}\right)^n} \\ &\geq \frac{\left[\sum_{k=0}^{n} \binom{n}{k}^2\right]^{\frac{1}{2}}}{\left(\frac{2\sqrt{3}}{9}\right)^n} = \frac{\sqrt{\binom{2n}{n}}}{\left(\frac{2\sqrt{3}}{9}\right)^n} = \frac{\sqrt{(2n)!}}{\left(\frac{2\sqrt{3}}{9}\right)^n n!}. \end{aligned} \tag{3.3.2}$$

Above we have used the well known formula

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Using Stirling's approximation formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

in (3.3.2) we have, for $m = 4n$,

$$D_{\mathbb{R},m}(2) = D_{\mathbb{R},4n}(2) \geq \frac{\sqrt{(2n)!}}{\left(\frac{2\sqrt{3}}{9}\right)^n n!} \sim \frac{\sqrt{2\sqrt{n\pi}\left(\frac{2n}{e}\right)^{2n}}}{\left(\frac{2\sqrt{3}}{9}\right)^n \sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = \sqrt[4]{\frac{4}{m\pi}}\left(\sqrt[8]{27}\right)^m.$$

□

3.4 The exact value of $D_{\mathbb{C},2}(2)$

Throughout this section we will often identify, similarly as we did in chapter 2, any two-variable polynomial $az^2 + bwz + cw^2$ or any one-variable polynomial $a\lambda^2 + b\lambda + c$, for $a, b, c \in \mathbb{K}$, with the vector $(a, b, c) \in \mathbb{K}^3$. Also, we will consider the norm $\|az^2 + bwz + cw^2\|_{\mathbb{D}}$ for the supremum of $|az^2 + bwz + cw^2|$ for z, w in the unit disk $\mathbb{D}$ of $\mathbb{C}$. In a similar fashion, $\|a\lambda^2 + b\lambda + c\|_{\mathbb{D}}$ stands for the supremum of $|a\lambda^2 + b\lambda + c|$ for $\lambda \in \mathbb{D}$. Observe that

$$\|az^2 + bwz + cw^2\|_{\mathbb{D}} = \|a\lambda^2 + b\lambda + c\|_{\mathbb{D}} = \max_{|\lambda|=1} |a\lambda^2 + b\lambda + c|,$$

being the last of the latter equalities due to the Maximum Modulus Principle.

The main result of this section depends upon the following lemma, which is of independent interest.

Lemma 3.4.1. *Let $a, b, c \in \mathbb{C}$. There exist $a', b', c' \in \mathbb{R}$ such that*

$$\|az^2 + bwz + cw^2\|_{\mathbb{D}} \geq \|a'z^2 + b'zw + c'w^2\|_{\mathbb{D}} \quad \text{and} \quad \|(a,b,c)\|_{\frac{4}{3}} = \|(a',b',c')\|_{\frac{4}{3}}.$$

Proof. If we perform the change of variables

$$z \mapsto ze^{-\frac{i\arg(a)}{2}} \quad \text{and} \quad w \mapsto we^{-\frac{i\arg(c)}{2}},$$

in $\|az^2+bzw+cw^2\|_{\mathbb{D}}$, we can assume (without loss of generality) that $a, c \geq 0$. We can also assume that $a \geq c$ by swapping z and w. We have:

$$\begin{aligned}
|a\lambda^2 + b\lambda + c|^2 &= \left(a\lambda^2 + b\lambda + c\right)\left(a\overline{\lambda^2} + \overline{b\lambda} + c\right) \\
&= a^2 + ac\lambda^2 + a\bar{b}\lambda + ac\overline{\lambda^2} + c^2 + c\overline{b\lambda} + ab\overline{\lambda} + bc\lambda + |b|^2 \\
&= a^2 + c^2 + |b|^2 + 2\left[ac\mathrm{Re}(\lambda^2) + a\mathrm{Re}(\bar{b}\lambda) + c\mathrm{Re}(b\lambda)\right] \\
&= a^2 + c^2 + |b|^2 + 2\left[ac\mathrm{Re}(\lambda^2) + (a+c)\mathrm{Re}(b)\mathrm{Re}(\lambda) + (a-c)\mathrm{Im}(b)\mathrm{Im}(\lambda)\right].
\end{aligned}$$

Similarly, if a', b', c' are real numbers, then:

$$|a'\lambda^2 + b'\lambda + c'|^2 = a'^2 + c'^2 + |b'|^2 + 2\left[a'c'\mathrm{Re}(\lambda^2) + (a'+c')b'\mathrm{Re}(\lambda)\right].$$

1. Assume first $a \geq c \geq |b|$. Then, choose

$$a' = \frac{(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{4}}}{2^{\frac{3}{4}}},\; c' = -a',\; b' = a.$$

Then, $\|(a,b,c)\|_{\frac{4}{3}} = \|(a',b',c')\|_{\frac{4}{3}}$. On the other hand,

$$a'^2 + c'^2 + |b'|^2 + 2\left[a'c'\mathrm{Re}(\lambda^2) + (a'+c')\mathrm{Re}(\lambda)\right] = 2a'^2 + a^2 - 2a'^2\mathrm{Re}(\lambda^2),$$

so that it is easy to see

$$\|(a',b',c')\|_{\mathbb{D}}^2 = 4a'^2 + a^2 = \sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}} + a^2.$$

Also, giving the value $\lambda = 1$ (if $\mathrm{Re}(b) \geq 0$) or $\lambda = -1$ (if $\mathrm{Re}(b) \leq 0$), we can see that

$$\|az^2 + cw^2 + bzw\|_{\mathbb{D}}^2 \geq a^2 + c^2 + |b|^2 + 2ac.$$

Now, we want

$$\sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}} + a^2 \leq a^2 + c^2 + |b|^2 + 2ac,$$

that is,

$$\frac{c^2 + |b|^2 + 2ac}{\sqrt{2}(c^{\frac{4}{3}} + |b|^{\frac{4}{3}})^{\frac{3}{2}}} \geq 1.$$

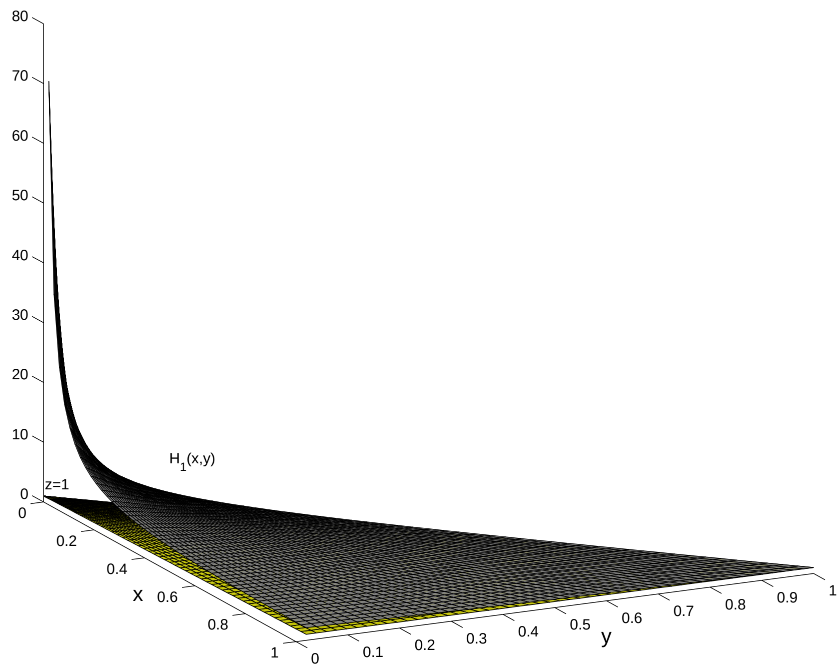


Figure 3.1: Graph of the mapping $H_1(x, y)$ on $0 \leq y \leq x \leq 1$

Divide both, numerator and denominator, by a^2 in order to convert the problem in having to achieve

$$H_1(x, y) := \frac{x^2 + y^2 + 2x}{\sqrt{2}(x^{\frac{4}{3}} + y^{\frac{4}{3}})^{\frac{3}{2}}} \geq 1,$$

for $0 \leq y \leq x \leq 1$.
We can find a sketch of the function H_1 over the set $0 \leq y \leq x \leq 1$ in Figure 3.1.

2. Assume next $a \geq |b| \geq c$. In this second part of the proof we shall need to employ a couple of real valued functions that will come in handy to achieve our purpose. Let us first focus our attention on the choice of the constants a', b', c', as before,

$$a' = \frac{\left(|c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{2^{\frac{3}{4}}}, \; c' = -a', \; b' = a.$$

In this case, choose

$$\lambda = \text{sign}(\text{Re}(b))\sqrt{\frac{1}{2}} + i\text{sign}(\text{Im}(b))\sqrt{\frac{1}{2}}.$$

Then,

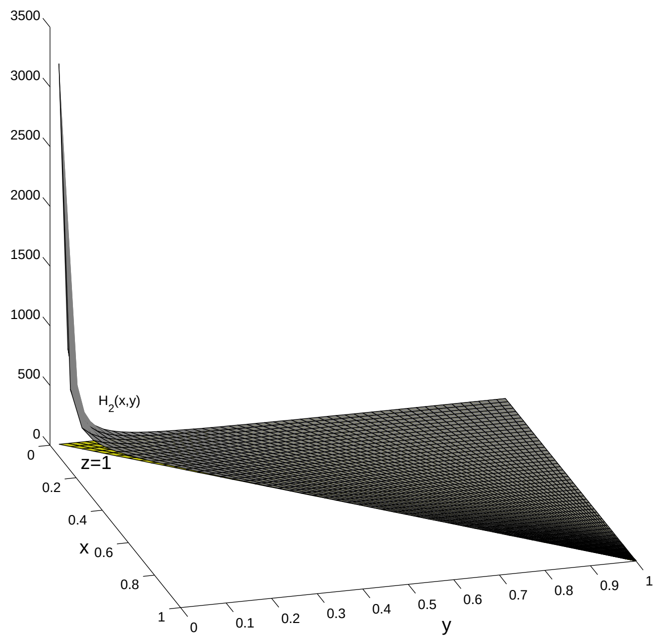


Figure 3.2: Graph of the mapping $H_2(x, y)$ on $0 \le x \le y \le 1$

$$\begin{aligned}\|(a,b,c)\|_{\mathbb{D}} &\ge a^2 + c^2 + |b|^2 + 2\left[\sqrt{\frac{1}{2}}(a+c)|\mathrm{Re}(b)| + \sqrt{\frac{1}{2}}(a-c)|\mathrm{Im}(b)|\right] \\ &\ge a^2 + c^2 + |b|^2 + \sqrt{2}|b|(a-c).\end{aligned}$$

Hence, we will achieve the desired result if we can guarantee

$$\sqrt{2}\left(|c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{2}} + a^2 \le a^2 + c^2 + |b|^2 + \sqrt{2}(a-c)|b|,$$

in other words,

$$1 \le \Phi_1(x, y) := \frac{x^2 + y^2 + \sqrt{2}(1-x)y}{\sqrt{2}\left(x^{\frac{4}{3}} + y^{\frac{4}{3}}\right)^{\frac{3}{2}}},$$

where $0 \le x \le y \le 1$.

Let us focus now in another choice of constants a', b', c':

$$a' = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{(2 + k^{4/3})^{\frac{3}{4}}},\ c' = -a',\ b' = ka,$$

where k has been chosen so that $\|(a,b,c)\|_{\frac{4}{3}} = \|(a',b',c')\|_{\frac{4}{3}}$. It can be proved that $k \approx 2.828$. Still giving the value $\lambda = \mathrm{sign}(\mathrm{Re}(b))$

$$\|(a,b,c)\|_{\mathbb{D}} \geq a^2 + c^2 + |b|^2 + 2ac,$$

and again we guarantee that we achieve what we are searching for if we get

$$4a'^2 + b'^2 = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{2}}}{(2+k^{4/3})^{\frac{3}{2}}}(4+k^2) \leq a^2 + c^2 + |b|^2 + 2ac,$$

in other words,

$$1 \leq \Psi_1(x,y) := \frac{\left(1 + x^2 + y^2 + 2yx\right)(2+k^{4/3})^{\frac{3}{2}}}{(4+k^2)\left(y^{\frac{4}{3}} + x^{\frac{4}{3}} + 1\right)^{\frac{3}{2}}},$$

with $0 \leq x \leq y \leq 1$.

It is easy to check using elementary calculus (take a look to Figure 3.2) that, if

$$H_2(x,y) := \max\{\Phi_1(x,y),\ \Psi_1(x,y)\},$$

then

$$1 \leq H_2(x,y) \text{ for every } \quad 0 \leq x \leq y \leq 1.$$

3. Assume finally $|b| \geq a \geq c$. Then, we may choose

$$a' = \frac{\left(|a|^{\frac{4}{3}} + |c|^{\frac{4}{3}} + |b|^{\frac{4}{3}}\right)^{\frac{3}{4}}}{(2+k^{4/3})^{\frac{3}{4}}},\ c' = -a',\ b' = ka,$$

where k is chosen as in the previous case. For $\lambda = \mathrm{sign}(\mathrm{Re}(b))$, we still need to make sure that

$$1 \leq \Phi_2(x,y) := \frac{(1 + x^2 + y^2 + 2xy)(2+k^{4/3})^{\frac{3}{2}}}{(4+k^2)(x^{4/3} + y^{4/3} + 1)^{\frac{3}{2}}}.$$

For $\lambda = \mathrm{sign}(\mathrm{Re}(b))\sqrt{\frac{1}{2}} + i\mathrm{sign}(\mathrm{Im}(b))\sqrt{\frac{1}{2}}$, we need to make sure that

$$1 \leq \Psi_2(x,y) := \frac{(1 + x^2 + y^2 + \sqrt{2}(y-x)(2+k^{4/3})^{\frac{3}{2}}}{(4+k^2)(x^{4/3} + y^{4/3} + 1)^{\frac{3}{2}}}.$$

Next, choose

$$a' = \frac{(|a|^{4/3} + |c|^{4/3})^{3/4}}{2^{3/4}}, \quad c' = -a' \quad \text{and} \quad b' = |b|,$$

such that

$$\|(a',b',c')\|_{\mathbb{D}}^2 = 4\frac{(|a|^{4/3} + |c|^{4/3})^{3/2}}{2^{3/2}} + |b|^2,$$

and

$$\lambda = \operatorname{sign}(\operatorname{Re}(b))\sqrt{1-\left(\frac{a-c}{|b|}\right)^2} + i\operatorname{sign}(\operatorname{Im}(b))\frac{a-c}{|b|}.$$

In that case,

$$\begin{aligned}\|(a,b,c)\|_{\mathbb{D}} &\geq a^2+c^2+|b|^2+2\left[ac\operatorname{Re}(\lambda^2)+(a+c)\operatorname{Re}(b)\operatorname{Re}(\lambda)+(a-c)\operatorname{Im}(b)\operatorname{Im}(\lambda)\right]\\ &= a^2+c^2+|b|^2+2\left[ac\left(1-2\left(\frac{a-c}{|b|}\right)^2\right)+(a+c)|\operatorname{Re}(b)|\sqrt{1-\left(\frac{a-c}{|b|}\right)^2}\right.\\ &\quad\left.+(a-c)|\operatorname{Im}(b)|\frac{a-c}{|b|}\right].\end{aligned}$$

Assume first $|\operatorname{Im}(b)| \geq \frac{\sqrt{2}}{2}$. Then,

$$\begin{aligned}\|(a,b,c)\|_{\mathbb{D}} &\geq a^2+c^2+|b|^2\\ &\quad+2\left[ac\left(1-2\left(\frac{a-c}{|b|}\right)^2\right)+(a-c)\frac{\sqrt{2}}{2}|b|\frac{a-c}{|b|}\right].\end{aligned}$$

Hence, we will achieve what we are searching for if we can assure that

$$1 \leq \Omega_2^{(1)}(x,y) := \frac{x^2+y^2+2xy(1-2(y-x)^2)+\sqrt{2}(y-x)^2}{\sqrt{2}(x^{4/3}+y^{4/3})^{\frac{3}{2}}}.$$

On in on, we need to prove

$$1 \leq H_3(x,y) := \max\{\Phi_2(x,y),\ \Psi_2(x,y),\ \Omega_2^{(1)}(x,y)\},$$

for $0 \leq x \leq y \leq 1$. This can be done by means of elementary calculus (take a look to the first sketch considered in Figure 3.3).

On the other hand if, instead, we have $|\operatorname{Re}(b)| \geq \frac{\sqrt{2}}{2}$, then

$$\begin{aligned}\|(a,b,c)\|_{\mathbb{D}} &\geq a^2+c^2+|b|^2\\ &\quad+2\left[ac\left(1-2\left(\frac{a-c}{|b|}\right)^2\right)+(a+c)\frac{\sqrt{2}}{2}|b|\sqrt{1-\left(\frac{a-c}{|b|}\right)^2}\right],\end{aligned}$$

and (in this case) we will be working with the condition

$$1 \leq \Omega_2^{(2)}(x,y) := \frac{x^2+y^2+2xy(1-2(y-x)^2)+\sqrt{2}(y+x)\sqrt{1-(y-x)^2}}{\sqrt{2}(x^{4/3}+y^{4/3})^{\frac{3}{2}}},$$

and, in conclusion, we shall need to guarantee that

$$1 \leq H_4(x,y) := \max\{\Phi_2(x,y),\ \Psi_2(x,y),\ \Omega_2^{(2)}(x,y)\},$$

for $0 \leq x \leq y \leq 1$ (again, we gather a representation for the function $H_4(x,y)$ over $0 \leq x \leq y \leq 1$ in Figure 3.3).

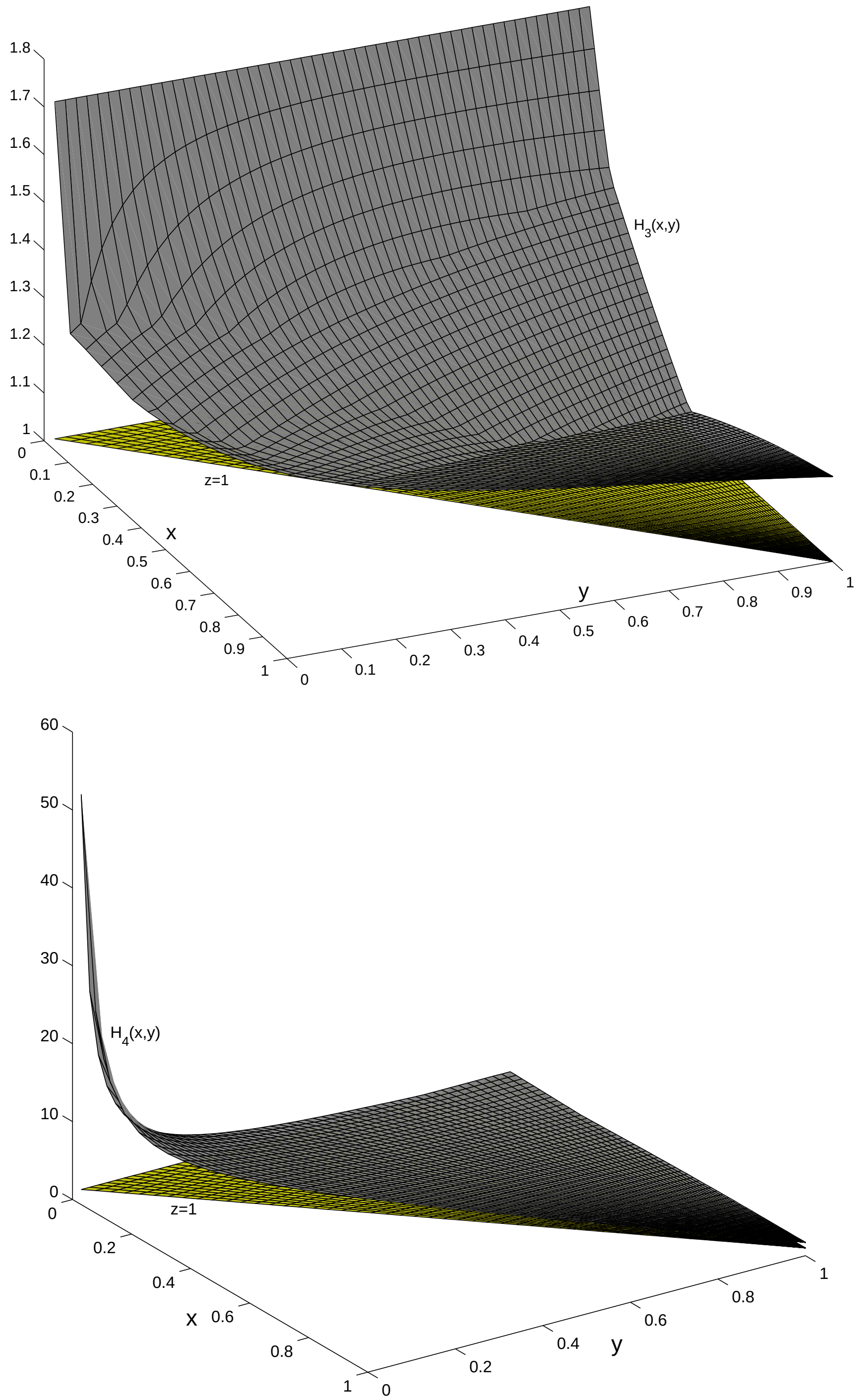


Figure 3.3: Graphs of the mappings $H_3(x, y)$ and $H_4(x, y)$ on $0 \leq x \leq y \leq 1$

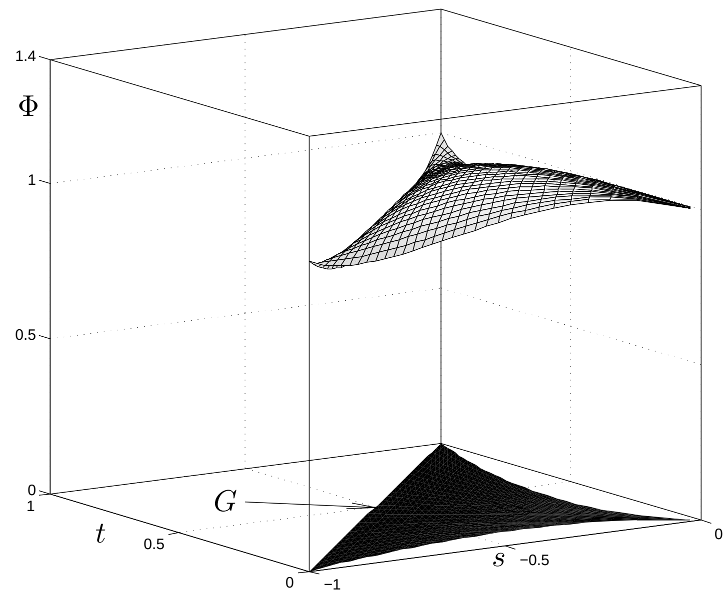


Figure 3.4: Graph of the mapping $\Phi(x, y)$

And, with this last case, the proof is complete.

□

In order to prove that $D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}}$ we will also need the following description of the extreme points of the unit ball of $\mathbb{R}^3$ endowed with the norm

$$\|(a,b,c)\|_{\mathbb{D}} := \sup\{|az^2 + bz + c| : |z| \le 1\}$$

for $a, b, c \in \mathbb{R}$. This norm has been studied by Aron and klimek in [8], where they denote it by $\|\cdot\|_{\mathbb{C}}$. Observe, once again that

$$\|(a,b,c)\|_{\mathbb{D}} = \|az^2 + bwz + cz^2\|_{\mathbb{D}}.$$

Theorem 3.4.2 (Aron and Klimek, [8]). *Let $E_{\mathbb{R}}$ be the real subspace of $\mathcal{P}(^2\ell_\infty^2(\mathbb{C}))$ given by $\{az^2 + bwz + cw^2 : (a,b,c) \in \mathbb{R}^3\}$. Then*

$$\mathrm{ext}(\mathsf{B}_{E_{\mathbb{R}}}) = \left\{ \left(s, \sqrt{4|s||t| \left(\frac{1}{(|s|+|t|)^2} - 1 \right)}, t \right) : (s,t) \in G \right\},$$

where $\mathrm{ext}(\mathsf{B}_{E_{\mathbb{R}}})$ *is the set of extreme points of the unit ball of $E_{\mathbb{R}}$, namely $\mathsf{B}_{E_{\mathbb{R}}}$ and $G = \{(s,t) \in \mathbb{R}^2 : |s| + |t| < 1$ and $|s+t| \le (s+t)^2\} \cup \{\pm(1,0), \pm(0,1)\}$.*

Theorem 3.4.3 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, M. Murillo-Arcila, J.B. Seoane-Sepúlveda, [63]). *The optimal complex polynomial Bohnenblust-Hille constant for polynomials in $E_{\mathbb{R}}$, which we denote by $D_{\mathbb{C},2}(E_{\mathbb{R}})$, is given by $D_{\mathbb{C},2}(E_{\mathbb{R}}) = \sqrt[4]{\frac{3}{2}}$. Moreover,*

$$D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}} \approx 1.1066.$$

Proof. Using convexity we have

$$\begin{aligned} D_{\mathbb{C},2}(E_{\mathbb{R}}) &= \sup\{\|(a,b,c)\|_{\frac{4}{3}} : \|az_1^2 + bz_1z_2 + cz_2^2\| \leq 1\} \\ &= \sup\{\|(a,b,c)\|_{\frac{4}{3}} : \|(a,b,c)\|_{\mathbb{C}} \leq 1\} \\ &= \sup\{\|(a,b,c)\|_{\frac{4}{3}} : (a,b,c) \in \operatorname{ext}(\mathsf{B}_{E_{\mathbb{R}}})\}, \end{aligned}$$

Hence

$$D_{\mathbb{C},2}(E_{\mathbb{R}}) = \sup\left\{\left(|s|^{\frac{4}{3}} + |t|^{\frac{4}{3}} + \left[4|s||t|\left(\frac{1}{(|s|+|t|)^2} - 1\right)\right]^{\frac{2}{3}}\right)^{\frac{3}{4}} : (s,t) \in G\right\}.$$

If $\Phi(s,t) = \left(|s|^{\frac{4}{3}} + |t|^{\frac{4}{3}} + \left[4|s||t|\left(\frac{1}{(|s|+|t|)^2} - 1\right)\right]^{\frac{2}{3}}\right)^{\frac{3}{4}}$ for $(s,t) \in G$, one can prove using elementary calculus that Φ attains its maximum on G at $\pm\left(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}\right)$ and $\Phi\left(\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}\right) = \sqrt[4]{\frac{3}{2}}$. Finally, from Lemma 3.4.1 we also obtain that $D_{\mathbb{C},2}(2) = \sqrt[4]{\frac{3}{2}} \approx 1.1066$. □

To help the intuition behind the elementary calculus that should rise in this last result, we can find a sketh of the graph of Φ on the part of G containued in the second quadrant in Figure 3.4.

3.5 The exact value of $D_{\mathbb{R},2}(2)$ and lower bounds for $D_{\mathbb{R},m}(2)$

In [28] it is proved that the asymptotic hypercontractivity constant of the real polynomial BH inequality is exactly 2. Is it true that $H_{\mathbb{R},\infty}(2) = 2$? The results presented here suggest that, perhaps $H_{\mathbb{R},\infty}(2) < 2$. In this section, as we did in the previous ones, we will also identify polynomials with the vector of its coefficients.

Remark 3.5.1. *Throughout this section we will compute several times norms of polynomials on the real line numerically. This is done by using Matlab. In particular, if $P(x)$ is a real polynomial on $\mathbb{R}$, we apply the predefined Matlab function* `roots.m` *to P' in order to obtain an approximation of all the critical points of P. If $x_1, \ldots, x_k$ are all the roots of P' in $[-1,1]$, then we approach the norm of P as*

$$\|P\| := \max\{|P(x)| : x \in [-1,1]\} = \max\{|P(x_i)|, |P(\pm 1)| : i = 1, \ldots, k\}.$$

Another Matlab predefined function, namely `conv.m`*, is used in order to to multiply polynomials. This is done to obtain Figure 3.8.*

3.5.1 The exact calculation of $D_{\mathbb{R},2}(2)$

The value of the constant $D_{\mathbb{R},2}(2)$ can be obtained using the geometry of the unit ball of $\mathcal{P}(\ell_\infty^2(\mathbb{R}))$ described in [30]. We state the result we need for completeness:

Theorem 3.5.2. *[Choi, Kim [30]] The set* $\text{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))})$ *of extreme points of the unit ball of* $\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))$ *is given by*

$$\text{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))}) = \{\pm x^2,\ \pm y^2,\ \pm(tx^2 - ty^2 \pm 2\sqrt{t(1-t)}xy) :\ t \in [1/2, 1]\}.$$

As a consequence of the previous result, we obtain the following:

Theorem 3.5.3 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, M. Murillo-Arcila, J.B. Seoane-Sepúlveda, [63])**.** *Let f be the real valued function given by*

$$f(t) = \left[2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

We have that $D_{\mathbb{R},2}(2) = f(t_0) \approx 1.837373$*, where*

$$t_0 = \frac{1}{36}\left(2\sqrt[3]{107 + 9\sqrt{129}} + \sqrt[3]{856 - 72\sqrt{129}} + 16\right) \approx 0.867835.$$

The exact value of $f(t_0)$ is given by

$$\left(\frac{\left(2\sqrt[3]{107+9\sqrt{129}}+\sqrt[3]{856-72\sqrt{129}}+16\right)^{4/3}}{18\ 6^{2/3}} + \frac{1}{9\left(-\frac{3}{-2\sqrt[3]{107+9\sqrt{129}}+(107+9\sqrt{129})^{2/3}-2\sqrt[3]{107-9\sqrt{129}}+(107-9\sqrt{129})^{2/3}-60}\right)^{2/3}}\right)^{3/4},$$

Moreover, the following normalized polynomials are extreme for this problem:

$$P_2(x, y) = \pm(t_0x^2 - t_0y^2 \pm 2\sqrt{t_0(1-t_0)}xy).$$

Proof. Let

$$f(t) = \left[2t^{\frac{4}{3}} + (2\sqrt{t(1-t)})^{\frac{4}{3}}\right]^{\frac{3}{4}}.$$

We just have to notice that due to the convexity of the ℓ_p-norms and Theorem 3.5.2 we have

$$\begin{aligned} D_{\mathbb{R},2}(2) &= \sup\{|\mathbf{a}|_{\frac{4}{3}} : \mathbf{a} \in \mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})}\} \\ &= \sup\{|\mathbf{a}|_{\frac{4}{3}} : \mathbf{a} \in \text{ext}(\mathsf{B}_{\mathcal{P}(^2\ell_\infty^2\mathbb{R})})\} = \sup_{t\in[1/2,1]} f(t). \end{aligned}$$

Some calculations will show that the last supremum is attained at $t = t_0$, concluding the proof. □

Now, if $\mathbf{a}_n$ is the vector of the coefficients of P_2^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},2n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{4n}{2n+1}}}{\|P_2\|^n}. \qquad (3.5.1)$$

Since $\|P_2\| = 1$, then (3.5.1) with $n = 300$ (see also Figure 3.8) proves that

$$D_{\mathbb{R},600}(2) \geq (1.36117)^{600},$$

providing numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.36117.$$

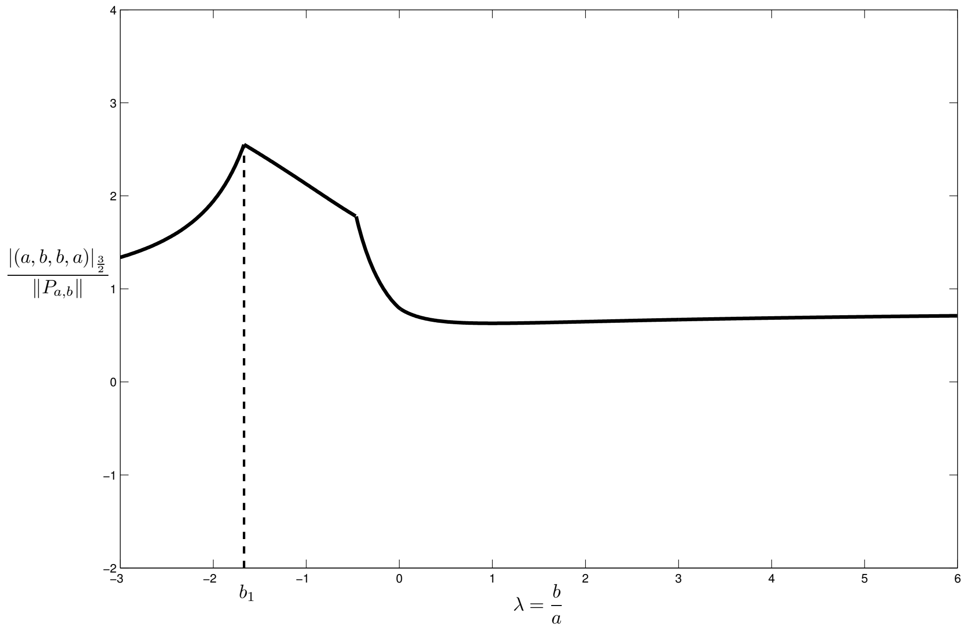


Figure 3.5: Graph of the quotient $\frac{|(a,b,b,a)|_{\frac{3}{2}}}{\|P_{a,b}\|}$ as a function of $\lambda = \frac{a}{b}$.

3.5.2 Educated guess for the exact calculation of $D_{\mathbb{R},3}(2)$

To my knowledge, the calculation of $\|P\|$ is, in general, far from being easy. However there is a way to compute $\|P\|$ for specific cases. For instance Grecu, Muñoz and Seoane prove in [52, Lemma 3.12] the following formula:

Lemma 3.5.4. *If for every $a, b \in \mathbb{R}$ we define $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ then*

$$\|P_{a,b}\| = \begin{cases} \left| a - \frac{b^2}{3a} + \frac{2b^3}{27a^2} + \frac{2a}{27}\left(-\frac{3b}{a} + \frac{b^2}{a^2}\right)^{\frac{3}{2}} \right| & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3 - 2\sqrt{3}, \\ |2a + 2b| & \text{otherwise,} \end{cases}$$

where

$$b_1 = \frac{3}{7}\left(3 - \frac{2\sqrt[3]{9}}{\sqrt[3]{-12 + 7\sqrt{3}}} + 2\sqrt[3]{-36 + 21\sqrt{3}}\right) \approx -1.6692.$$

From Lemma 3.5.4 we have the following sharp polynomial Bohnenblust-Hille type constant:

Theorem 3.5.5 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, M. Murillo-Arcila, J.B. Seoane-Sepúlveda, [63])**.** *Let $P_{a,b}(x, y) = ax^3 + bx^2y + bxy^2 + ay^3$ for $a, b \in \mathbb{R}$ and consider the subset of $\mathcal{P}({}^3\ell_\infty^2(\mathbb{R}))$*

given by $E = \{P_{a,b} : a, b \in \mathbb{R}\}$. *Then*

$$\frac{|(a,b,b,a)|_{\frac{3}{2}}}{\|P_{a,b}\|} = \begin{cases} \frac{27a^2\left(2|a|^{\frac{3}{2}}+2|b|^{\frac{3}{2}}\right)^{\frac{2}{3}}}{\left|27a^3-9ab^2+2b^3+2\,\mathrm{sign}(a)(-3ab+b^2)^{\frac{3}{2}}\right|}, & \text{if } a \neq 0 \text{ and } b_1 < \frac{b}{a} < 3-2\sqrt{3}, \\ \frac{\left(2|a|^{\frac{3}{2}}+2|b|^{\frac{3}{2}}\right)^{\frac{2}{3}}}{2|a+b|}, & \text{otherwise} \end{cases}$$

where b_1 *was defined in Lemma 3.5.4. Moreover, the above function attains its maximum when* $\frac{b}{a} = b_1$, *which implies that*

$$D_{\mathbb{R},3}(E) = \frac{\left(2+2|b_1|^{\frac{3}{2}}\right)^{\frac{2}{3}}}{2|1+b_1|} \approx 2.5525$$

There is numerical evidence to state that

$$D_{\mathbb{R},3}(2) = D_{\mathbb{R},3}(E).$$

Moreover, one polynomial for which $D_{\mathbb{R},3}(2)$ would be attained is

$$P_3(x,y) = x^3 + b_1x^2y + b_1xy^2 + y^3,$$

where $b_1 \approx -1.6692$ is as in Lemma 3.5.4. It can be proved from Lemma 3.5.4 that

$$\|P_3\| \approx 1.33848,$$

up to 5 decimal places. If $\mathbf{a_n}$ is the vector of the coefficients of $P_3(x,y)^n$ and we use the fact that

$$D_{\mathbb{R},3n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{6n}{3n+1}}}{\|P_3\|^n}, \tag{3.5.2}$$

then putting $n = 200$ in (3.5.2) we obtain, for instance,

$$D_{\mathbb{R},600}(2) \geq (1.42234)^{600},$$

which provides numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.42234.$$

3.5.3 Educated guess for the exact calculation of $D_{\mathbb{R},4}(2)$

We have numerical evidence that an extremal polynomial for the Bohnenblust-Hille inequality for $\mathcal{P}(^2\ell_\infty^2(\mathbb{R}))$ may be of the form

$$R_{a,b}(x,y) = ax^3y + bxy^3$$

(remark the similarities with the polynomial $P_4 = R_{1,-1}$ defined in proposition 3.3.2). More concretely, we are going to have the following:

Proposition 3.5.6. *Let*

$$R_{a,b}(x,y) = ax^3y + bxy^3.$$

Then,

$$\max\left\{\frac{|R_{a,b}|_{8/5}}{\|R_{a,b}\|} : a, b \in \mathbb{R}\right\} = \frac{|R_{1,-1}|_{8/5}}{\|R_{1,-1}\|}$$

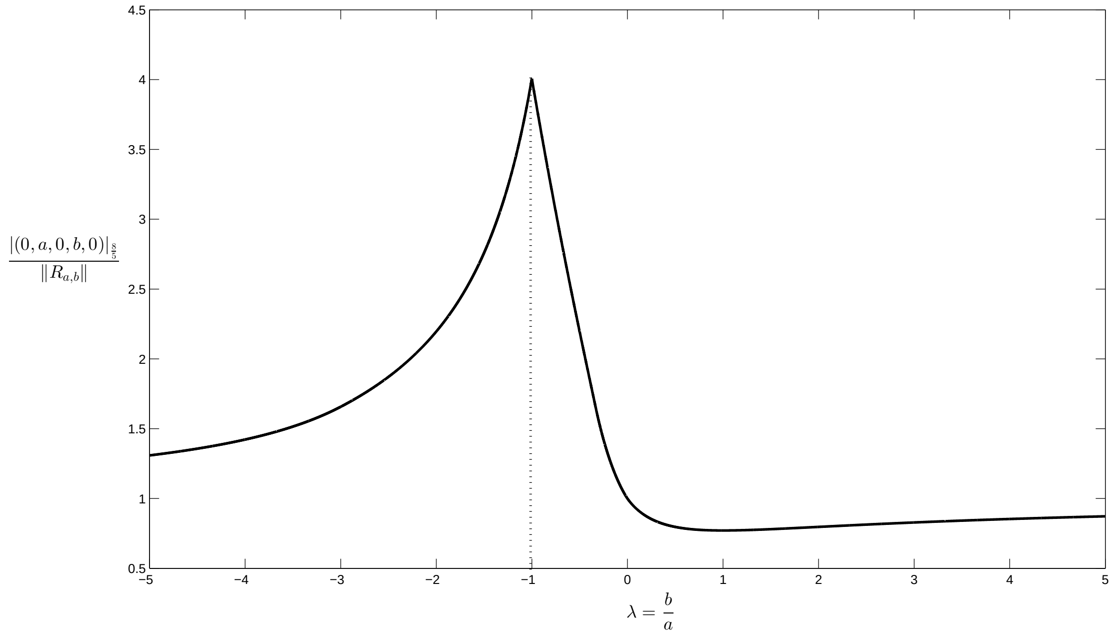


Figure 3.6: Graph of the quotient $\frac{|R_{a,b}|_{8/5}}{\|R_{a,b}\|}$ as a function of $\lambda = \frac{b}{a}$.

Proof. First of all, we need to come out with a formula for $\|R_{a,b}\|$. Some elementary calculations provide

$$\|R_{a,b}\| = \max_{(x,y)\in\partial[-1,1]^2} |ax^3y + bxy^3| = \begin{cases} \frac{2|b|}{3\sqrt{3}}\sqrt{\left|\frac{b}{a}\right|} & \text{if } \frac{b}{a} \in (-3,-1), \\ \frac{2|a|}{3\sqrt{3}}\sqrt{\left|\frac{a}{b}\right|} & \text{if } \frac{b}{a} \in [-1,-1/3), \\ |a+b| & \text{otherwise.} \end{cases}$$

After that, we only need to study the function

$$\frac{|R_{a,b}|_{8/5}}{\|R_{a,b}\|} = \begin{cases} \frac{3\sqrt{3}(|a|^{8/5}+|b|^{8/5})^{5/8}}{2|b|}\sqrt{\left|\frac{a}{b}\right|} & \text{if } \frac{b}{a} \in (-3,-1), \\ \frac{3\sqrt{3}(|a|^{8/5}+|b|^{8/5})^{5/8}}{2|a|}\sqrt{\left|\frac{b}{a}\right|} & \text{if } \frac{b}{a} \in [-1,-1/3), \\ \frac{(|a|^{8/5}+|b|^{8/5})^{5/8}}{|a+b|} & \text{otherwise.} \end{cases}$$

A look to figure 3.6 may hint us that the polynomial $P_4 = R_{1,-1}$ from Proposition 3.3.2 is indeed extremal for the Bohnenblust-Hille inequality, and result follows. □

With the observation we made at the beginning of this subsection, we are confident to guess that, going beyond from the estimate obtained in 3.3.2

$$D_{\mathbb{R},4}(2) = \frac{|R_{1,-1}|_{8/5}}{\|R_{a,b}\|} \approx 4.00678.$$

3.5.4 Numerical calculation of $D_{\mathbb{R},5}(2)$

Let us define the polynomial

$$P_5(x,y) = ax^5 - bx^4y - cx^3y^2 + cx^2y^3 + bxy^4 - ay^5,$$

with

$$a = 0.19462,$$
$$b = 0.66008,$$
$$c = 0.97833.$$

The norm of P_5 can be calculated numerically (using Remark 3.5.1), and it turns out to be

$$\|P_5\| = 0.28617,$$

up to 5 decimal places. We have numerical evidence showing that

$$D_{\mathbb{R},5}(2) \approx 6.83591.$$

In any case we have

$$D_{\mathbb{R},5}(2) \geq \frac{|(a,-b,-c,c,b,-a)|_{\frac{5}{3}}}{\|P_5\|} \approx 6.83591.$$

It is interesting to observe that we can improve numerically the estimate $H_{\infty,\mathbb{R}}(2) \geq \sqrt[8]{27} \approx 1.50980$ (see [28, Theorem 4.2]) by considering polynomials of the form P_5^n. Indeed, if $\mathbf{a}_n$ is the vector of the coefficients of P_5^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},5n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{10n}{5n+1}}}{\|P_5\|^n}, \tag{3.5.3}$$

Using (3.5.3) with $n = 120$ we obtain, in particular (see also Figure 3.8)

$$D_{\mathbb{R},600}(2) \geq (1.54987)^{600},$$

providing numerical evidence showing that

$$H_{\mathbb{R},\infty}(2) \geq 1.54987.$$

3.5.5 Educated guess for the exact calculation of $D_{\mathbb{R},6}(2)$

We have numerical evidence pointing to the fact that an extreme polynomial in the Bohnenblust-Hille inequality for polynomials in $\mathcal{P}(^6\ell_\infty^2(\mathbb{R}))$ may be of the form

$$Q_{a,b}(x,y) = ax^5y + bx^3y^3 + axy^5.$$

This motivates a deeper study of this type of polynomials, which we do in the following result.

Theorem 3.5.7 (P. Jiménez-Rodríguez, G.A. Muñoz-Fernández, M. Murillo-Arcila, J.B. Seoane-Sepúlveda, [63]). *Let $Q_{a,b}(x,y) = ax^5y + bx^3y^3 + axy^5$ for $a, b \in \mathbb{R}$ and consider the subspace of $\mathcal{P}(^6\ell_\infty^2(\mathbb{R}))$ given by $F = \{Q_{a,b} : a, b \in \mathbb{R}\}$. Suppose $\lambda_0 < \lambda_1$ are the only two roots of the equation*

$$\frac{|3\lambda^2 - 20 + \lambda\sqrt{9\lambda^2 - 20}|}{25}\sqrt{\frac{-3\lambda - \sqrt{9\lambda^2 - 20}}{10}} = |2 + \lambda|.$$

Then if $\lambda = \frac{b}{a}$ we have

$$\frac{|(0,a,0,b,0,a,0)|_{\frac{12}{7}}}{\|Q_{a,b}\|} = \begin{cases} \frac{25\sqrt{10}\left(2+|\lambda|^{\frac{12}{7}}\right)^{\frac{12}{7}}}{|3\lambda^2-20+\lambda\sqrt{9\lambda^2-20}|\sqrt{-3\lambda-\sqrt{9\lambda^2-20}}}, & \text{if } a \neq 0 \text{ and } \lambda_0 < \frac{b}{a} < \lambda_1, \\ \frac{\left(2+|\lambda|^{\frac{12}{7}}\right)^{\frac{7}{12}}}{|2+\lambda|}, & \text{otherwise.} \end{cases}$$

Observe that $\lambda_0 \approx -2.2654$, $\lambda_1 \approx -1.6779$ and the above function attains its maximum when $\frac{b}{a} = \lambda_0$ (see Figure 3.7), which implies that

$$D_{\mathbb{R},6}(F) = \frac{\left(2 + |\lambda_0|^{\frac{12}{7}}\right)^{\frac{12}{7}}}{|2 + \lambda_0|} \approx 10.7809.$$

Proof. We do not lose generality by considering only polynomials of the form $Q_{1,\lambda}$, in which case

$$\|Q_{1,\lambda}\| = \sup\{|x^5 + \lambda x^3 + x| : x \in [0,1]\}.$$

The polynomial $q_\lambda(x) := x^5 + \lambda x^3 + x$ has no critical points if $\lambda > -\frac{2\sqrt{5}}{3}$, otherwise it has the following critical points in $[0,1]$:

$$x_0 := \sqrt{\frac{-3\lambda - \sqrt{9\lambda^2 - 20}}{10}} \quad \text{and} \quad x_1 := \sqrt{\frac{-3\lambda + \sqrt{9\lambda^2 - 20}}{10}} \quad \text{if } -2 \leq \lambda \leq -\frac{2\sqrt{5}}{3},$$

and x_0 if $\lambda \leq -2$. Notice that

$$q_\lambda(x_0) = \frac{-3\lambda^2 + 20 - \lambda\sqrt{9\lambda^2 - 20}}{20}x_0$$
$$q_\lambda(x_1) = \frac{-3\lambda^2 + 20 + \lambda\sqrt{9\lambda^2 - 20}}{20}x_1.$$

It is easy to check that $|q_\lambda(x_0)| \geq |q_\lambda(x_1)|$ for $-2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}$, which implies that

$$\|Q_{1,\lambda}\| = \begin{cases} \max\{|2+\lambda|, |q_\lambda(x_0)|\} & \text{if } -2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}, \\ |2+\lambda| & \text{otherwise.} \end{cases}$$

The equation $|2 + \lambda| = |q_\lambda(x_0)|$ turns out to have only two roots, namely $\lambda_0 \approx -2.2654$ and $\lambda_1 \approx -1.6779$. By continuity, it is easy to prove that $|2 + \lambda| \leq |q_\lambda(x_0)|$ only if $-2 \leq \lambda \leq -\frac{2\sqrt{5}}{3}$, which concludes the proof. □

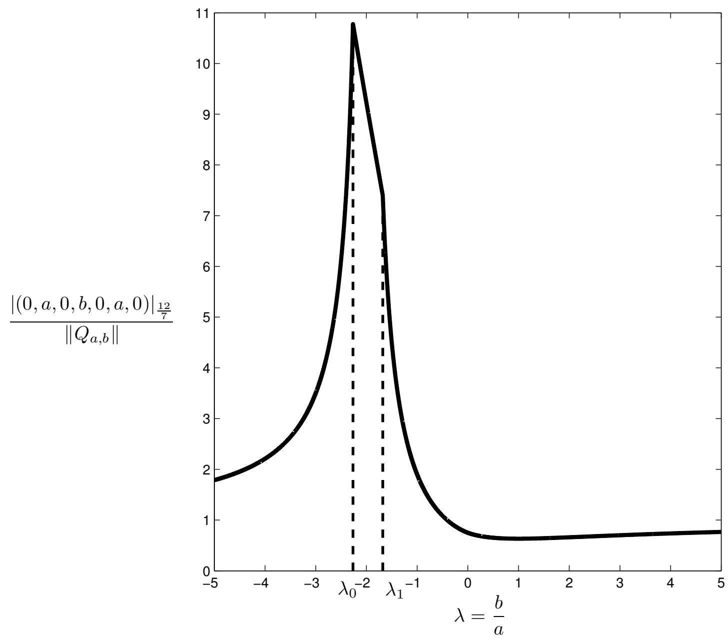


Figure 3.7: Graph of the quotient $\frac{|(0,a,0,b,0,a,0)|_{\frac{12}{7}}}{\|Q_{a,b}\|}$ as a function of $\lambda = \frac{b}{a}$.

As mentioned above, we have numerical evidence showing that

$$D_{\mathbb{R},6}(2) = D_{\mathbb{R},6}(F) = \frac{\left(2 + |\lambda_0|^{\frac{12}{7}}\right)^{\frac{12}{7}}}{|2+\lambda_0|} \approx 10.7809.$$

In any case we do have that

$$D_{\mathbb{R},6}(2) \geq 10.7809.$$

As we did in the previous cases, it would be interesting to know if we can improve numerically our best lower bound on $H_{\mathbb{R},\infty}$ by considering powers of

$$P_6(x,y) = Q_{1,\lambda_0}(x,y) = x^5y + \lambda_0 x^3y^3 + xy^5,$$

with λ_0 as in Theorem 3.5.7 ($\lambda_0 \approx -2.2654$). If $\mathbf{a}_n$ is the vector of the coefficients of P_6^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},6n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{12n}{6n+1}}}{\|P_6\|^n}. \tag{3.5.4}$$

Using (3.5.4) with $n = 100$ and estimating $\|P_6\|$ according to Remark 3.5.1 we obtain

$$D_{\mathbb{R},600}(2) \geq (1.58432)^{600},$$

which suggests that (see Figure 3.8)

$$H_{\infty,\mathbb{R}}(2) \geq 1.58432.$$

3.5.6 Numerical calculation of $D_{\mathbb{R},7}(2)$

Let us define the polynomial

$$P_7(x,y) = -ax^7 + bx^6y + cx^5y^2 - dx^4y^3 - dx^3y^4 + cx^2y^5 + bxy^6 - ay^7,$$

with

$$a = 0.05126,$$
$$b = 0.22070,$$
$$c = 0.50537,$$
$$d = 0.71044.$$

It can be proved numerically (using Remark 3.5.1) that

$$\|P_7\| \approx 0.07138,$$

up to 5 decimal places. We have numerical evidence showing that

$$D_{\mathbb{R},7}(2) \approx \frac{|(-a,b,c,-d,-d,c,b,-a)|_{\frac{7}{4}}}{\|P_7\|} \approx 19.96308.$$

If $\mathbf{a}_n$ is the vector of the coefficients of P_7^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},7n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{14n}{7n+1}}}{\|P_7\|^n}. \tag{3.5.5}$$

Moreover, if we put $n = 86$ in (3.5.5) we obtain

$$D_{\mathbb{R},602}(2) \geq (1.61725)^{602},$$

suggesting that

$$H_{\mathbb{R},\infty}(2) \geq 1.61725.$$

3.5.7 Numerical calculation of $D_{\mathbb{R},8}(2)$

Let us define the polynomial

$$P_8(x,y) = -ax^7y + bx^5y^3 - bx^3y^5 + axy^7,$$

with

$$a = 0.15258,$$
$$b = 0.64697.$$

It can be established numerically (see Remark 3.5.1) that

$$\|P_8\| \approx 0.02985,$$

up to 5 decimal places. We have numerical evidence showing that

$$D_{\mathbb{R},8}(2) \approx \frac{|(0,-a,0,b,0,-b,0,a,0)|_{\frac{16}{9}}}{\|P_8\|} \approx 33.36323.$$

If $\mathbf{a}_n$ is the vector of the coefficients of P_8^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},8n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{16n}{8n+1}}}{\|P_8\|^n}. \tag{3.5.6}$$

Moreover, using (3.5.6) with $n = 75$ we obtain

$$D_{\mathbb{R},600}(2) \geq (1.64042)^{600},$$

which suggests that

$$H_{\mathbb{R},\infty}(2) \geq 1.64042.$$

3.5.8 Numerical calculation of $D_{\mathbb{R},10}(2)$

In this case our numerical estimates show that there exists an extreme polynomial in the Bohnenblust-Hille polynomial inequality in $\mathcal{P}(^{10}\ell_\infty^2(\mathbb{R}))$ of the form

$$P_{10}(x,y) = ax^9y + bx^7y^3 + x^5y^5 + bx^3y^7 + axy^9,$$

with

$$a = 0.0938,$$
$$b = -0.5938.$$

It can be computed numerically (see Remark 3.5.1) that

$$\|P_{10}\| \approx 0.01530,$$

up to 5 decimal places. We have numerical evidence showing that

$$D_{\mathbb{R},10}(2) \approx \frac{|(0,a,0,b,0,1,0,b,0,a,0)|_{\frac{20}{11}}}{\|P_{10}\|} \approx 90.35556.$$

If $\mathbf{a}_n$ is the vector of the coefficients of P_{10}^n for each $n \in \mathbb{N}$, then we know that

$$D_{\mathbb{R},10n}(2) \geq \frac{|\mathbf{a}_n|_{\frac{20n}{10n+1}}}{\|P_{10}\|^n}. \tag{3.5.7}$$

If we set $n = 60$ in (3.5.7) then we obtain

$$D_{\mathbb{R},600}(2) \geq (1.65171)^{600},$$

which suggests that

$$H_{\mathbb{R},\infty}(2) \geq 1.65171.$$

We have sketched in Figure 3.8 a summary of the numerical results obtained in this section.

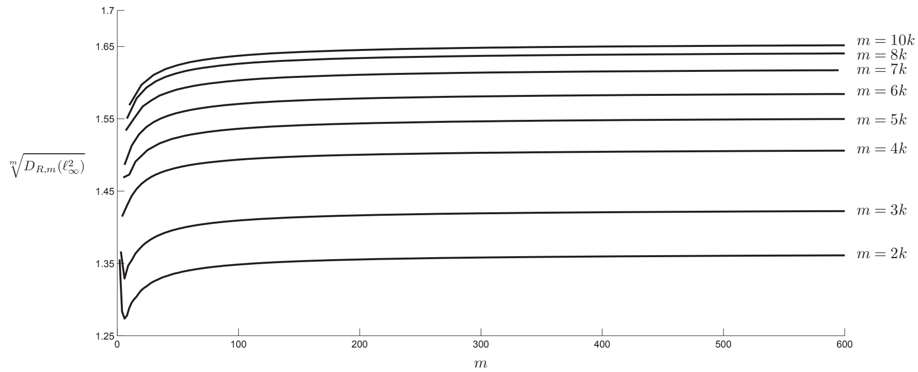


Figure 3.8: Graphs of the estimates on $\sqrt[m]{D_{\mathbb{R},m}(2)}$ obtained by using (3.5.1) through (3.5.7).

Let us notice that in these final results, there are some symbolical and numerical calculations performed. We shall give in the following lines some more details about how we obtained the modeled graphs above:

We would use the program Matlab to perform those symbolic calculations and to have an approximate graph to obtain the extremal polynomial for the lower estimate. To this end, we will use one auxiliar function:

```
function resultado=bh_pol_6_intervalo(a1,a2,b1,b2,c1,c2,d1,d2,e1,e2,f1,f2,g1,g2,n)
resultado=[0,0,0,0,0,0,0,0];
ha=(a2-a1)/n;
hb=(b2-b1)/n;
hc=(c2-c1)/n;
hd=(d2-d1)/n;
he=(e2-e1)/n;
hf=(f2-f1)/n;
hg=(g2-g1)/n;
for a=a1:ha:a2
for b=b1:hb:b2
for c=c1:hc:c2
for d=d1:hd:d2
for e=e1:he:e2
for f=f1:hf:f2
for g=g1:hg:g2
norma_polynomial=max([pol_norm([a,b,c,d,e,f,g]),pol_norm([g,f,e,d,c,b,a])]);
norma_127=normp([a,b,c,d,e,f,g],12/7);
if (norma_polynomial  = 0)
cociente= norma_127/norma_polynomial;
if cociente>=resultado(8)
resultado=[a,b,c,d,e,f,g,cociente];
end
```

```
end
end
end
end
end
end
end
end
```

and we will use it in the code to search for the extremal polynomial of degree n (which, without losing of generality after normalization, we may assume has sup norm less than or equal to 1). For it, we will consider some lattice (not necessarily very fine) and once we have obtained our candidate for extremal polynomial, we will consider a finer lattice, but centred in the candidate to extremal polynomial (so that we do not need to consider the finer lattice in the whole cube $[-1,1]^{n+1}$). We will iterate the process several times, considering in each step a finer lattice:

```
function resultado=bh_pol_6(m,n)
ti=cputime;
intervalos=bh_pol_6_intervalo(-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,-1,1,n);
disp(['El tiempo invertido en el paso 1 es: ',num2str(cputime-ti)])
if m==1
disp(m);
resultado=intervalos;
else
for k=2:m
h=1/2^k;
ti=cputime;
a1=max([-1,intervalos(1)-h]);
a2=min([1,intervalos(1)+h]);
b1=max([-1,intervalos(2)-h]);
b2=min([1,intervalos(2)+h]);
c1=max([-1,intervalos(3)-h]);
c2=min([1,intervalos(3)+h]);
d1=max([-1,intervalos(4)-h]);
d2=min([1,intervalos(4)+h]);
e1=max([-1,intervalos(5)-h]);
e2=min([1,intervalos(5)+h]);
f1=max([-1,intervalos(6)-h]);
f2=min([1,intervalos(6)+h]);
g1=max([-1,intervalos(7)-h]);
g2=min([1,intervalos(7)+h]);
intervalos=bh_pol_6_intervalo(a1,a2,b1,b2,c1,c2,d1,d2,e1,e2,f1,f2,g1,g2,n);
disp(['El tiempo invertido en el paso ',num2str(k),' es: ',num2str(cputime-ti)])
end
resultado=intervalos;
end
```

Finally, we will use the following code to perform the estimates that we have been obtaining throughout all this last section:

```
function resultado=multiples_6(n)
format long;
L0=-2.2654;
resultado=zeros(2,n);
p =[0,1,0,L0,0,1,0];
q=p;
N=pol_norm(p);
for k=1:n
resultado(1,k)=normp(q,12*k/(6*k+1))/N∧k;
resultado(2,k)=exp((log(resultado(1,k)))/(6*k));
q=conv(p,q);
end
disp(resultado(1,n))
disp(resultado(2,n))
```

Remark that in all the previous samples, we have been working for finding the estimates for the homogeneous polynomials of degree 6.

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