Topology at Infinity

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Abstract. These lectures will introduce modern tools of geometric topology to study noncompact spaces, especially manifolds and polyhedra. The main goal is to prove a Structure Theorem for Tame Ends of high dimensional manifolds: near infinity the manifold has a periodic shift map. By concentrating on this theorem the tools of controlled topology and approximate fibrations will be seen in action. The role of the end space as a homotopy model of the behavior of a space at infinity will be emphasised. The five lectures of 90 minutes each will be organized as described below.

Lecture 1. Introduction and overview.

(i) Statement of the Structure Theorem for Tame Ends (STTE)

(ii) Defintions and simple examples of terms in STTE:

- One end
- Tame = Reverse Tame plus Froward Tame. Examples: Jacob's ladder, Infinite tunnels, Stairway to heaven, Hocking and Young cover, Periodic dunce cap.
- MAF. Examples: Warsaw circle fiber as limit of bundles $T^2 \to S^1$, Inertial h-cobordisms.
- Finitely dominated. Easy reformulation in terms of deformation into compact subspace; construction of finitely dominated complexes that are not of finite type (assuming Wall's theory).

(iii) Rough outline of proof of STTE

- Include definition and examples of end spaces

(iv) History of the STTE and motivation. Why study tame ends?

Lecture 2. Tameness and bounded homotopy equivalences at infinity.

(i) Review of forward tameness and reverse tameness

(ii) More examples: Davis manifold, Whitehead continuum, mapping telescopes

(iii) Basic lemmas about tameness needed for STTE

(iv) Proof of Step 1 of the implication (1) $\Longrightarrow$ (2) in STTE

Lecture 3. Manifold approximate fibrations I.

(i) Survey of MAF theory, including the classification theorem

(ii) MAF sucking over $\mathbb{R}^1$, a torus trick, introduction to the approximation techniques of Chapman and Ferry

(iii) Proof of Steps 2 and 3 of the implication (1) $\Longrightarrow$ (2) in STTE

(iv) Easy proof of the implication (3) $\Longrightarrow$ (4) in STTE

Lecture 4. Manifold approximate fibrations II and infinite cyclic covers.

(i) Approximate isotopy covering theorem

(ii) Infinite cyclic coverings

(iii) Geometric wrapping up and the proof of the implication (2) $\Longrightarrow$ (3) in STTE

(iv) Tameness in bands (end duality of forward and reverse tameness)

(v) Proof of the implication (4) $\Longrightarrow$ (1) in STTE

Lecture 5. Conclusion, bits and pieces, and future directions. If time permits, a selection of these topics will touched on in the last lecture and some might be covered in earlier lectures.

(i) The fundamental group at infinity and stability

(ii) Siebenmann tameness and its equivalence with tameness in manifolds

(iii) Geometric relaxation

(iv) Siebenmann's and Farrell's theses; i.e., Siebenmann's end theorem and Farrell's fibering theorem

(v) Survey of controlled topology: pre- and early history

(vi) stratified spaces

(vii) Dihedral generalizations with Khan

(viii) Other developments: Guilbault's work; low dimensional manifolds; geometric group theory; trees and ultrametrics

(ix) Open problems

(x) Possible other topics: proper homotopy theory; number of ends of spaces; locally finite homology; homotopy inverse limits; chain complexes at infinity; the Hauptvermutung; the topological invariance of rational Pontryagin classes

Prerequisites. Basic courses in algebra and topology; some knowledge of ANRs, CW complexes, and manifolds would be helpful, but not necessary.

Main References

Ends of Complexes by B. Hughes and A. Ranicki (Cambridge, 1996); denoted HR in the notes.

The Topological Classification of Stratified Spaces by S. Weinberger (Chicago, 1994)

Conventions and notation.

Throughout these lectures it is assumed that ANR spaces are locally compact, separable and metric, and that CW complexes are locally finite.

A *map* is a continuous function.