Simple Mathematical Models related to fluids and heat conduction: the Boussinesq problem.

J.M. Rakotoson

July 24, 2008

The purpose of this course is to discuss the resolution of an evolution parabolic system. This simplified system presents the same difficulties as the full models of Navier-Stokes coupled with the heat equation (with or without the Boussinesq approximation). We shall first recall some basic tools in Functional Analysis and the mathematical background for the Navier-Stokes coupled with the heat equation. Nevertheless, to understand the course, the students should at least know the notion of distribution or generalized derivatives (see [25, 2])

Contents

1 The models

1.1 The simplified model.

The system of equations that we shall study is the following:

Let Ω a smooth open set of the plane R^2, (here, we mean by smooth at least a C^2 domain), $T > 0$, $Q_T = \Omega\times]0, T[$, we are looking for a couple of functions (v, θ) satisfying (BS):

- $$v_t - \Delta v = \rho(\theta) \text{ in } Q_T,$$

- $$\rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2 \text{ in } Q_T.$$

We shall complete this system by the following boundary conditions and the initial values.

$v(x,t) = 0 = \dfrac{\partial\theta}{\partial t}(x,t),\ x \in \partial\Omega, t \in]0,T[.$

$v(x,0) = v(0), \theta(x,0) = \theta(0), x \in \Omega.$

Before giving the conditions on ρ, the initial data and starting the resolution of such system, let us give the full model from which such a consideration comes from.

The results concerning those equations were obtained with J.I. Díaz and P. Schmidt and were published in [6, 7].

1.2 The full model (by P. Schmidt).

This section was written by Paul Schmidt one of the co-authors in [6, 7, 8].

The following equations have been derived from the first principles governing the flow of a viscous, heat -conducting fluid:

A.) Balance momentum

$$\rho V_t + \rho(V.\nabla)V - \nabla.S(V,p) = -\rho\nabla\phi$$

B.) Balance of mass

$$\rho_t + \nabla\cdot(\rho V) = 0$$

C.) Balance of internal energy

$$\rho c\theta_t + \rho c(V.\nabla)\theta - \nabla.(\kappa\nabla\theta) = S(V,p) : \nabla V$$

The unknown are the velocity V(vector field), p (pressure), ρ (density), and θ (the temperature); $S(V,p)$ is the stress tensor, ϕ is the gravitational potential; c and κ denote the heat capacity and thermal conductivity respectively. In 3D (resp 2D) we have only five (four) equations for six (resp. three) unknowns, the system $(A),(B),(C)$ must be supplemented by a constitutive relation between the thermodynamic quantities p,ρ,θ called : equation of state.

The simplest would be $\rho =$ constant. Under this assumption, $(A),(B)$ are reduced to the classical Navier-Stokes system and is decoupled from the last equation (B); clearly, this is useless if we want to model the buoyancy-driven flow. To model buoyancy, we should assume that : $\rho > 0$, is a decreasing function of θ eventually bounded; but then, (B) becomes an evolution equation for θ!

For this reason , we shall first introduce the simplification that : ρ is constant but only is the balance of mass: $\nabla.\ V = 0$ (divergence free) (the fluid is said to be incompressible) and this implies

$$S(V,p) = \mu(\nabla V + \nabla V^{tr}) - pId \text{ and } S(V,p) : \nabla V = \frac{\mu}{2}|\nabla V + \nabla V^{tr}|^2,$$

where μ is the viscosity, the symbol "*tr*" means the trace an "*Id*" is the identity matrix. As a consequence, we get:

D.)

$$\rho V_t + \rho(V.\nabla)V - \nabla.(\mu(\nabla V + \nabla V^{tr})) + \nabla p = \rho g$$

with $g = -\nabla\phi$.
E.)

$$div(V) = \nabla.V = 0$$

F.)

$$\rho c\theta_t + \rho c(V.\nabla)\theta - \nabla.(\kappa\nabla\theta) = \frac{\mu}{2}|\nabla V + \nabla V^{tr}|^2$$

In principle, ρ, μ, c, κ should all be positive functions of θ.

1.3 Classical Boussinesq approximation:

Assume that μ, c, κ are constant, and that $\rho = \rho_0$ is constant everywhere except on the right hand side of $D.$), where we set $\rho = \rho_0(1 - \alpha\theta)$ with $\alpha > 0$. We assume that this system may not have global-in-time solutions.
We set μ, c, κ, ρ_0 equal to one, therefore our final system that we should consider at the end of this session will be :
G.)

$$V_t + (V.\nabla)V - \Delta V + \nabla p = \rho(\theta)g$$

H.)

$$\nabla.V = 0$$

I.)

$$\rho(\theta)\theta_t + \rho(\theta)(V.\nabla)\theta - \Delta\theta = |\nabla V + \nabla V^{tr}|^2$$

The pressure p plays the role of Lagrange multiplier associated to the divergence constraint on v. For this reason, in the simplified model we took away the ∇p and the condition $div(V) = 0$.

2 Preliminary material: functional spaces

In this section, we shall announce some useful tools on functional. The reader might found those details and complements in [28, 11, 18, 25, 2].

2.1 "Non evolution" case: Some basic and usual spaces

For an open set Ω (bounded all the time in this course) and $1 \leqslant p \leqslant \infty$, we set
$L^p(\Omega) = \left\{ v : \Omega \to R, \text{ measurable }, \int_\Omega |v|^p(x)dx < \infty \text{ if } p \text{ is finite.} \right\}$
$L^\infty(\Omega) = \left\{ v : \Omega \to R, \text{ measurable }, \operatorname{ess\,sup}_\Omega |v(x)| < \infty \right\}$.
Those spaces are endowed with their usual norms, that is :

$$|v|_p^p = \int_\Omega |v|^p(x)dx, \text{ if } p < \infty,$$

$$|v|_\infty = \text{ess sup }_\Omega |v(x)| \text{ otherwise.}$$

The associate and dual space of $L^p(\Omega)$ for $1 < p < \infty$ is $L^{p'}(\Omega)$ with p' being the conjuguate of p, i.e. $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

We have the

Lemma 1.

1. *Holder's inequality:*
$$\int_\Omega |vw|(x)dx \leqslant |v|_p |w|_{p'},$$
with $v \in L^p(\Omega)$ *and* $w \in L^{p'}(\Omega)$ *(the inequality is true for* $p = 1,\ p' = +\infty$*).*

2. *Interpolation inequality: Let* $1 \leqslant p < r < q \leqslant +\infty$ *and* $\theta \in [0,1] : \dfrac{1}{r} = \dfrac{\theta}{p} + \dfrac{1-\theta}{q}$*. Then for all* $u \in L^p(\Omega) \cap L^q(\Omega)$*, we have:*
$$|u|_r \leqslant |u|_p^\theta |u|_q^{1-\theta}.$$

Comments: One proves 2.) using 1.).

Definition 1 (of $\mathbf{H^s(\Omega)},\ \mathbf{s \in [0, +\infty[}$).

1. $s = 0 : H^0(\Omega) = L^2(\Omega).$

2. $s \in \mathbb{N},\ s \neq 0$
$$H^s(\Omega) = \Big\{ v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega),\ |\alpha| \leqslant s \Big\}.$$
Here
$$D^\alpha = \frac{\partial^{\alpha_1 + \ldots + \alpha_N}}{\partial x_1^{a_1} \ldots \partial x_N^{\alpha_N}},\ (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N;\ |\alpha| = \alpha_1, \ldots + \alpha_N.$$

3. $s \in [0, +\infty[,\ s \notin \mathbb{N}$. *We shall start with*

   (a) *If* $s = \sigma \in]0,1[$.
$$H^\sigma(\Omega) = \Big\{ v \in L^2(\Omega) : v_\sigma(x,y) = \frac{|v(x) - v(y)|}{|x-y|^{\sigma + \frac{N}{2}}} \in L^2(\Omega \times \Omega) \Big\}.$$

   (b) *If* $s > 1,\ s \notin \mathbb{N}$*, we set*
$$[s] = \text{ the biggest integer less than } s \text{ and let } \sigma = s - [s] \in]0,1[\ ,$$
then
$$H^s(\Omega) = \Big\{ v \in H^{[s]}(\Omega) : D^\alpha v \in H^\sigma(\Omega) \text{ for } |\alpha| = [s] \Big\}.$$

Properties 1. $H^s(\Omega),\ s \in [0, +\infty[$ *are Hilbert spaces under the following norms:*

1. *For* $s \in \mathbb{N}, : v \in H^s(\Omega)$
$$|v|^2_{H^s(\Omega)} = \sum_{|\alpha| \leqslant s} |D^\alpha v|^2_{L^2(\Omega)}.$$

2. *For* $s = \sigma \in]0,1[$
$$|v|^2_{H^s(\Omega)} = |v|^2_{L^2(\Omega)} + |v_\sigma|^2_{L^2(\Omega \times \Omega)}.$$

3. *For* $s > 1,\ s \notin \mathbb{N}$, *and* $\sigma = s - [s]$

$$|v|^2_{H^s(\Omega)} = |v|^2_{H^{[s]}(\Omega)} + |v_\sigma|^2_{L^2(\Omega\times\Omega)}.$$

Before giving more properties, there is another definition of $H^s(\Omega),\ s \in [0,+\infty[$ using the Fourier transform.
We recall first the

Lemma 2. *[***Extension properties***]*
Assume that Ω *is an open bounded set of class* $C^\ell,\ \ell \geqslant 1$. *then, there exists a continuous linear operator* P_ℓ *from* $H^m(\Omega),\ m \in \mathbb{N},\ m \leqslant \ell$ *onto* $H^m(\mathbb{R}^N)$ *such that:*

1. $P_\ell u = u$ *for all* $u \in H^m(\Omega)$.
2. $\exists\ c(\Omega,\ell) > 0 : |P_\ell u|_{H^m(\mathbb{R}^N)} \leqslant c(\Omega,\ell)|u|_{H^m(\Omega)}$.

Definition 2 (**of** $\mathbf{H^s(\mathbb{R}^N)},\ \mathbf{s \in [0,+\infty[}$)**.**
Let $v \in L^2(\mathbb{R}^N)$, *we denote by* $\widehat{v}$ *the Fourier transform of* v *(* $\widehat{v} = \mathcal{F}(v)$*). We recall the Plancherel Formula*

$$|\widehat{v}|_{L^2(\mathbb{R}^N)} = |v|_{L^2(\mathbb{R}^N)}.$$

$$H^s(\mathbb{R}^N) = \Big\{ v : \mathbb{R}^N \to \mathbb{R} : \int_{\mathbb{R}^N} (1+|\xi|^2)^s |\widehat{v}(\xi)|^2 d\xi < +\infty \Big\}.$$

The norm of $H^s(\mathbb{R}^N)$ *given previously is equivalent to*

$$|v|^2_{H^s(\mathbb{R}^N)} \approx \int_{\mathbb{R}^N)} (1+|\xi|^2)^s |\widehat{v}(\xi)|^2 d\xi.$$

Definition 3 (**of** $\mathbf{H^s(\Omega)},\ \mathbf{s \in [0,+\infty[}$)**.**
Assume that Ω *is an open bounded set of* $\mathbb{R}^N$ *of class* $C^\ell, \ell \geqslant 1$. *Then, for* $s \leqslant \ell$

$$H^s(\Omega) = \Big\{ v \in H^{[s]}(\Omega) : P_\ell v \in H^s(\mathbb{R}^N) \Big\}.$$

$$|v|_{H^s(\Omega)} = |P_\ell v|_{H^s(\mathbb{R}^N)}.$$

Definition 4 (**of** $\mathbf{H^{-s}(\Omega)},\ \mathbf{s \in [0,+\infty[}$)**.**
We set

$$C_c^\infty(\Omega) = \Big\{ v : \Omega \to \mathbb{R} \textit{ indefinitely differentiable with compact support} \Big\}.$$

For $s \in [0,+\infty[$, *one define*

$$H_0^s(\Omega) = \overline{C_c^\infty(\Omega)}^{H^s(\Omega)},$$

and

$$H^{-s}(\Omega) \textit{denotes the dual space of } H_0^s(\Omega).$$

Lemma 3 (**Compact embeddings for** $\mathbf{H^s(\Omega)}$ **and interpolation inequalities between** $\mathbf{H^s(\Omega)}$)**.**
Let Ω *be a bounded set of class* $C^\ell,\ \ell \geqslant 1$.
Then, the injection $H^{s_1}(\Omega) \hookrightarrow H^{s_2}(\Omega)$ *is compact if* $0 \leqslant s_2 < s_1 \leqslant \ell$.
Moreover if $s = \theta s_1 + (1-\theta)s_2 \in]s_2, s_1[,\ \theta \in]0,1[$,
then, there exists a constant $c = c(\Omega) > 0$*:*

$$|u|_{H^s(\Omega)} \leqslant c|u|^\theta_{H^{s_1}(\Omega)} |u|^{1-\theta}_{H^{s_2}(\Omega)} \textit{ for } u \in H^{s_1}(\Omega).$$

Comments:

- Continuous injection shall be denoted by $\subset_{>}$.
- Use the second definition of $H^s(\Omega)$: $|u|_{H^{s_i}(\Omega)} = |P_\ell u|_{H^{s_i}(\mathbb{R}^N)}$ and apply the Hölder inequality, noticing that $1 = \theta + (1-\theta)$. In fact the regularity of Ω can be weaken here, this why another proof can be done, using the first definition or an equivalent one.

Lemma 4 (Sobolev embeddings).
One has:

1. $H^s(\mathbb{R}^N) \subset_{>} L^q(\mathbb{R}^N)$ *if* $s < \dfrac{N}{2}$ *and* $\dfrac{1}{q} = \dfrac{1}{2} - \dfrac{s}{N}$.
2. $|D^\alpha u|_{L^\infty(\mathbb{R}^N)} \leqslant c(s,\alpha)|u|_{H^s(\mathbb{R}^N)}$ *if* $s > \dfrac{N}{2}$, $|\alpha| < s - \dfrac{N}{2}$
3. *If* Ω *is a bounded subset of* $\mathbb{R}^N$ *of class* C^ℓ, $\ell \geqslant 2$,
$$H^s(\Omega) \subset_{>} L^q(\Omega) \text{ if } s < \frac{N}{2},\ \frac{1}{q} = \frac{1}{2} - \frac{s}{2}.$$
4. *If* Ω *is a bounded open set of class* C^ℓ, $\ell > [s]$, *then*
$$H^s(\Omega) \subset_{>} C(\overline{\Omega}) \text{ if } s > \frac{N}{2}.$$
The injection is compact.
More generally,
$$|D^\alpha u|_{L^\infty(\Omega} \leqslant c(\Omega, s, \alpha)|u|_{H^s(\Omega)} \text{ with } |\alpha| < s - \frac{N}{2}.$$

Lemma 5 (Poincaré-Sobolev inequality).
Let Ω *be an open bounded set of* $\mathbb{R}^N$.
There exists $c(\Omega) = c > 0$ *such that*
$$|u|_{L^2(\Omega)} \leqslant c(\Omega)|\nabla u|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega).$$
In particular
$$|u|_{H^1(\Omega)} \approx |\nabla u|_{L^2(\Omega)} \text{ on } H_0^1(\Omega).$$

Two last interpolation inequalities that we shall use are the Gagliardo-Niremberg interpolation inequalities and the Agmon's inequality.

Lemma 6 (Gagliardo-Niremberg).
Let Ω *be an open set of class* C^3 *in* $\mathbb{R}^2$.
Then, there exists $c > 0$
$$|u|_{L^4(\Omega)} \leqslant c|u|_{L^2(\Omega)}^{\frac{1}{2}}|u|_{H^1(\Omega)}^{\frac{1}{2}} \quad \forall u \in H^1(\Omega).$$

Lemma 7 (Agmon's Lemma).
Let Ω *be a bounded open set of class* C^4 *in* $\mathbb{R}^N$.
Then, there exists $c(\Omega) = c > 0$*:*
If $N = 2$ *then* $|u|_{L^\infty(\Omega)} \leqslant c|u|_{L^2(\Omega)}^{\frac{1}{2}}|u|_{H^2(\Omega)}^{\frac{1}{2}}$.
If $N = 3$ *then* $|u|_{L^\infty(\Omega)} \leqslant c|u|_{H^1(\Omega)}^{\frac{1}{2}}|u|_{H^2(\Omega)}^{\frac{1}{2}}$.

2.2 Functional spaces for evolution time dependance

For simplicity, we shall restrict to the case of Hilbert spaces which are separable.

If V is a real Hilbert space with a norm denoted by $||\cdot|| = |\cdot|_V$ and the scalar product is $((\cdot,\cdot)) = (\cdot,\cdot)_V$.

Then we have

Definition 5.
Let $1 \leqslant p \leqslant +\infty,\ T \in]0,\infty[$.

$$L^p(0,T;V) = \Big\{ v : [0,T] \to V \textit{ Bochner measurable, } \textit{such that } \begin{cases} \displaystyle\int_0^T ||v(t)||^p dt < +\infty & p < +\infty, \\ \operatorname*{ess\,sup}_{t\in[0,T]} ||v(t)|| < +\infty & \textit{otherwise.} \end{cases} \Big\}$$

Endowed with the natural norm,

$$|v|^p_{L^p(0,T;V)} = \int_0^T ||v(t)||^p dt \textit{ for } p < +\infty,$$

and

$$|v|_{L^\infty(0,T;V)} = \operatorname*{ess\,sup}_{t\in[0,T]} ||v(t)|| \textit{otherwise.}$$

These spaces are complete.

In particular

Lemma 8 (Hilbert space properties).
$L^2(0,T;V)$ *is an Hilbert space, whose dual is* $L^2(0,T;V')$ *if* V' *denotes the dual of* V.

Definition 6.

$$C([0,T];V) = \Big\{ v : [0,T] \to V \textit{ continuous} \Big\}$$

is a Banach space endowed with

$$|v|_{C([0,T];V)} = \operatorname*{Max}_{[0,T]} ||v(t)||.$$

Definition 7.
$\mathcal{D}(0,T) = C_c^\infty(]0,T[)$ *and* $\mathcal{D}(0,T;V) = C_c^\infty(]0,T[;V)$ *whose dual is* $\mathcal{D}'(0,T;V)$.

Definition 8.
If $v \in L^1(0,T;V)$ *we set* $v_t = v' = \dfrac{dv}{dt} \in \mathcal{D}'(0,T;V)$ *defined by*

$$\frac{dv}{dt}(\varphi) = -\int_0^T v(t)\varphi'(t)dt, \quad \forall \varphi \in \mathcal{D}(0,T).$$

Lemma 9 (Embedding).
Let $(V, ||\cdot||)$ *and* $(H, |\cdot|)$ *be two Hilbert spaces with the following conditions*

1. *V is campactly embedded in H,*

2. V is dense in H and we have

$$V \subset_{>} H = H' \subset_{>} V'.$$

We define

$$W(0,T) = \Big\{ v \in L^2(0,T;V') : v' \in L^2(0,T;V') \Big\}.$$

Then,

1. $W(0,T \subset_{>} C([0,T];H)$,

2. $\dfrac{d}{dt}|v(t)|^2 = 2 < v(t), v'(t) >_{V,V'}$.

The main compactness lemma for time evolution equation is

Theorem 1 (Compactness for evolution equation).
Let $X \underset{c}{\subset_{>}} Y \subset_{>} Z$ be three Banach spaces (or Hilbert in our cases).
Let $1 \leqslant p \leqslant +\infty,\ 1 \leqslant q \leqslant +\infty$ and set

$$W_{p,q}(0,T) = \Big\{ v \in L^p(0,T;X) : v' \in L^q(0,T;Z) \Big\}$$

1. If $p < +\infty$ then $W_{p,1}(0,T) \underset{c}{\subset_{>}} L^p(0,T;Y)$,

2. If $p = +\infty$ then $W_{\infty,q} \underset{c}{\subset_{>}} C([0,T],Y)$ provided that $q > 1$.

As a corollary of the two last theorems and H^s-spaces:

Corollary 1 (Compactness for $\mathbf{L^2(0,T;H^m(\Omega))}$).
Let $m \geqslant 1$, and Ω a bounded C^m open set of $\mathbb{R}^N$.
Then $W(0,T;H^m(\Omega)) = \Big\{ v \in L^2(0,T;H^m(\Omega))$ such that $v' \in L^2(0,T;(H^m(\Omega))') \Big\}$ is compactly embedded in $C([0,T],H^s(\Omega))$ for all $s < m$.

This corollary was widely used in [26].

3 The simplified model and main results

Let $V = H_0^1(\Omega),\ H = H^1(\Omega)$, with $\Omega \subset \mathbb{R}^N$, be a smooth bounded (say of class C^4) set with $N = 2$. We suppose that the function ρ is

$$\rho(\sigma) = (1-\sigma)_+ \tag{1}$$

We denote by Φ a primitive of ρ such that

$$\Phi(s) = \int_0^s \rho(\sigma)d\sigma = -\frac{1}{2}(1-s)_+^2 + \frac{1}{2}.$$

We shall discuss about the local existence of a couple (v,θ), where v is the simplified velocity and θ is the simplified temperature satisfying

$$(BS) \begin{cases} v_t - \Delta v = (1-\theta)_+ & \text{in } Q_T = \Omega \times]0,T[, \\ (1-\theta)_+ \theta_t - \Delta\theta = |\nabla v|^2 & \text{in } Q_T, \\ v(x,t) = 0 = \dfrac{\partial \theta}{\partial n}(x,t) & \text{on } \Sigma_T = \partial\Omega \times]0,T[. \end{cases}$$

Here $\vec{n}(x) = n(x)$ denotes the outernormal at a point $x \in \partial\Omega$.

The initial data are $v(x,0) = v_0(x)$ and $\theta(x,0) = \theta_0(x),\ x \in \Omega \subset \mathbb{R}^2$.
The main difficulty in this model is the θ-equation, since it degenerates because of the term $(1-\theta)_+$ which makes the equation to be "parabolic" on $\{\theta < 1\}$ and "elliptic" on $\{\theta = 1\}$. We shall show that the solution might not exist (but a simple **formal** analysis can show that assertion).
We shall introduce the following definition of truncated problem :

Definition 9.
Let T be in $]0,\infty[$. A couple (θ, v) such that $\theta \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$ with $\Phi(\theta) \in L^2(Q_T)$ and $v \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$ is called a "(weak in θ and strong in v) solution" for the following truncated system (TBS) associated to the equations given in section 1.1, if there exist a real α and a function $g_v \in [|\nabla v|^2 \chi_{\{\theta<\alpha\}}, |\nabla v|^2]$ a.e. in Q_T such that

$$\frac{d}{dt}\int_\Omega v\varphi dx + \int_\Omega \nabla v \cdot \nabla \varphi dx = \int_\Omega (1-\theta)_+ \varphi dx, \ \text{in } \mathcal{D}'(0,T),\ \forall \varphi \in H_0^1(\Omega),$$

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega (1-\theta)_+^2 \psi dx + \int_\Omega \nabla\theta \cdot \nabla\psi dx = \int_\Omega g_v \psi dx \ \text{in } \mathcal{D}'(0,T),\ \forall \psi \in H^1(\Omega),$$

A weak solution (θ, v) is called an "exact (weak in θ and strong in v) solution" on Q_T if it satisfies the following condition :

$$|\nabla v|^2 = g_v,\ \text{a.e. in } Q_T.$$

A weak solution (θ, v) is called an "almost exact (weak in θ and strong in v) solution" on Q_T if :

$$g_v = |\nabla v|^2 \chi_{\{\theta<\alpha\}} \ \text{a.e. in } Q_T.$$

An exact solution (resp. almost exact solution with $\theta \in L^2(0,T; H^2(\Omega))$ is called a (strong in θ and strong in v) exact solution (resp. strong-strong and almost exact solution).

If there is a time $T_0 < T$ for which one those definitions are fulfilled, we will say that it is a local exact (respectively almost exact) solution.

Remark 1. We note that

$$|\nabla v|^2 \chi_{\{\theta<\alpha\}} = |\nabla v|^2 - |\nabla v|^2 \chi_{\{\theta \geqslant \alpha\}}$$

which proves the relationship with the dissipative (in θ) term $|\nabla v|^2 \chi_{\{\theta\geqslant\alpha\}}$ (for a prescribed v). Moreover, if (θ, v) is a (weak in θ and strong in v) solution and $\alpha \geqslant \theta_\infty$, with $\theta_\infty = \operatorname*{ess\,sup}_{Q_T} \theta$. Then,

$$|\nabla v|^2 \chi_{\{\theta<\theta_\infty\}} \leqslant |\nabla v|^2 \chi_{\{\theta<\alpha\}}$$

and equality holds if (θ, v) is an almost exact solution.

Now, we give some sufficient conditions to obtain an almost solution and an exact solution:

Proposition 1.
Let θ be a function such that (θ, v) is a weak solution for the truncated system with $\theta \in L^{\frac{3}{2}}_{loc}(0,T;W^{2,\frac{3}{2}}_{loc}(\Omega))$, $\Phi(\theta) \in L^1_{loc}(0,T;W^{1,1}_{loc}(\Omega))$ and $\theta_\infty = \operatorname{ess\,sup}_{Q_T} \theta \leqslant \alpha$. Assume that $g_v \in L^{\frac{3}{2}}(Q_T)$. Then

$$g_v = |\nabla v|^2 \chi_{\{\theta<\theta_\infty\}}.$$

Furthermore, if $\theta \in C(\overline{Q}_T)$ and $\theta_0 < \alpha - \delta$ for some $\delta > 0$ then the couple (v, θ) is a local exact solution.

Proof of Proposition 1. Let us observe that θ satisfy

$$\frac{\partial \Phi(\theta)}{\partial t} - \Delta\theta = g_v \ in\ \Omega.$$

If $\theta \in L^{\frac{3}{2}}_{loc}(0,T;W^{2,\frac{3}{2}}_{loc}(\Omega))$, and $g_v \in L^{\frac{3}{2}}(Q_T)$ then $\dfrac{\partial \Phi(\theta)}{\partial t} \in L^{\frac{3}{2}}_{loc}(Q_T)$. Thus by a Stampacchia result (see e.g. [16]) we have $\Delta\theta = \dfrac{\partial \Phi(\theta)}{\partial t} = 0$ a.e. on the set

$$E = \Big\{(t,x) \in Q_T : \theta(t,x) = \theta_\infty\Big\}.$$

This means $g_v(t,x) = 0$ a.e. on E, since $g_v = |\nabla v|^2$ on $\{\theta < \theta_\infty\}$, then we have the result. If $\theta \in C(\overline{Q}_T)$ then the choice of $\delta > 0$ so that $\theta_0 + \delta < \alpha$ and the continuity of θ imply that there exists a time $T_0 > 0$, such that $\theta(t,x) < \alpha - \frac{\delta}{2}$ for all $(t,x) \in Q_{T_0}$. Therefore, one has

$$|\nabla v|^2 \chi_{\{\theta<\alpha\}} = |\nabla v|^2, \text{ in } Q_{T_0}.$$

This shows that the couple is a local exact solution. ■

Theorem 2.
Let $0 \leqslant \theta_0 \leqslant 1$, $(\theta_0, v_0) \in H^1(\Omega) \times H^1_0(\Omega)$. Then for all $T > 0$, there exist a function $\theta \in L^2(0,T;H^1(\Omega))$, $0 \leqslant \theta \leqslant 1$ with $\theta \in C([0,T];L^2(\Omega))$, $v \in C([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega))$ satisfying $\forall \varphi \in H^1_0(\Omega), \forall \psi \in H^1(\Omega)$ that

$$\frac{d}{dt}\int_\Omega v(t,x)\varphi(x)dx + \int_\Omega \nabla\varphi(x)\cdot\nabla v(t,x)dx = \int_\Omega \varphi(x)(1-\theta(t,x))dx$$

and

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega (1-\theta)^2\psi(x)dx + \int_\Omega \nabla\psi(x)\cdot\nabla\theta(t,x)(t,x)dx = \int_\Omega \psi(x)g_v(t,x)dx, \ in\ \mathcal{D}'(0,T)$$

with $g_v \in [|\nabla v|^2\chi_{\{\theta<1\}}, |\nabla v|^2]$, $v(0) = v_0$, $\theta(0) = \theta_0$ *in* Ω.

Proof.
We consider $\varepsilon \in]0,1[$ and the following "nondegenerate" parabolic system $(BS)_\varepsilon$

Find a "regular" couple $(v^\varepsilon, \theta^\varepsilon)$ satisfying

$$\begin{cases} v^\varepsilon_t - \Delta v^\varepsilon = (1-\theta^\varepsilon_+)_+ & in\ Q_T \\ \theta^\varepsilon_t - \dfrac{\Delta\theta^\varepsilon}{(1-\theta^\varepsilon_+)_+ + \varepsilon} = \dfrac{S_\varepsilon(\theta^\varepsilon)}{(1-\theta^\varepsilon_+)_+ + \varepsilon}|\nabla v^\varepsilon| & in\ Q_T \\ v^\varepsilon = \dfrac{\partial\theta^\varepsilon}{\partial n} = 0 & on\ \Sigma_T \\ v^\varepsilon(x,0) = v_0(x),\ \theta^\varepsilon(x,0) = \theta_0(x), & x \in \Omega. \end{cases}$$

Here S_ε is a continuous function from $\mathbb{R}$ into $[0,1]$. For convenience, we shall set

$$\rho_\varepsilon(\theta^\varepsilon) = (1-\theta^\varepsilon_+)_+ + \varepsilon, \quad H_\varepsilon(\theta_\varepsilon) = \frac{S_\varepsilon(\theta^\varepsilon)}{\rho_\varepsilon(\theta^\varepsilon)}, \quad \rho_0(\theta^\varepsilon) = (1-\theta^\varepsilon_+)_+.$$

One has the

Lemma 10.
There exist a couple $(v^\varepsilon, \theta^\varepsilon)$ satisfying the system $(BS)_\varepsilon$ with the following regularity:

1. $v^\varepsilon \in C([0,T; H^s(\Omega) \cap H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega))\ \forall s \in [0,2[$.
2. $\theta^\varepsilon \in C([0,T]; H^s(\Omega)) \cap L^2(0,T; H^2(\Omega))\ \forall s < 2$.
3. v^ε_t *and* θ^ε_t *are in* $L^2(Q_T)$.

Proof
The proof Lemma 10 using the Galerkin approximation:
We shall use the following eigenfunctions which are elements of $C^\infty(\Omega) \cap H^2(\Omega)$

$$-\Delta\varphi_j = \lambda^D_j \varphi_j \text{ in } \Omega, \quad \varphi_j = 0 \text{ on } \partial\Omega,\ j = 1, 2, \ldots.$$

$$-\Delta\psi_j + \psi_j = \lambda^N_j \psi_j \text{ in } \Omega, \quad \frac{\partial\psi_j}{\partial n} = 0 \text{ on } \partial\Omega \quad j = 1, 2, \ldots.$$

(we note that ψ_1 is the constant function 1). For $T > 0$, we set $Q_T =]0,T[\times\Omega$. We set $V_m = \text{span}\{\varphi_j, j \leqslant m\}$, $H_m = \text{span}\{\psi_j,\ j \leqslant m\}$ for $m \geqslant 1$.

We recall that $\bigcup_{m\geqslant 1} V_m$ (resp. $\bigcup_{m\geqslant 1} H_m$) (see e.g. [25], [18]) is dense in V (resp. in H).
We will use the following orthogonal projections $P_m : L^2(\Omega) \to V_m, Q_m : L^2(\Omega) \to H_m$.

From the Cauchy-Peano's theorem, there exist for all $m \geqslant 1$ $\theta_m \in C^1([0,T_m); H_m)$ and $v_m \in C^1([0,T_m); V_m)$ for some $0 < T_m \leqslant T$, satisfying : $\forall\varphi \in V_m$, $\forall\psi \in H_m$, for all $t \in [0,T_m)$, $\theta_m(0) = Q_m\theta_0$, $v_m(0) = P_m v_0$

$$\frac{d}{dt}\int_\Omega v_m(t)\varphi dx + \int_\Omega \nabla v_m(t)\cdot\nabla\varphi dx = \int_\Omega \rho_0(\theta_m(t))\varphi dx, \tag{2}$$

$$\frac{d}{dt}\int_\Omega \theta_m(t)\psi + \int_\Omega \nabla\theta_m(t)\cdot\nabla\left(\frac{\psi}{\rho_\varepsilon(\theta_m(t))}\right)dx = \int_\Omega \frac{\psi}{\rho_\varepsilon(\theta_m(t))}|\nabla v_m(t)|^2 S_\varepsilon(\theta_m(t))dx. \tag{3}$$

To show that $T_m = T$, we need some estimates on v_m and θ_m. Those estimates will be uniform in m. ∎

Lemma 11. *For all $t \in [0,T_m)$*

(a) $\displaystyle\frac{d}{dt}\int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx \leqslant |\Omega|,$ *in* $\mathcal{D}'(0,T_m)$,

(b) $\displaystyle\frac{d}{dt}\int_\Omega |\nabla\theta_m(t)|^2 + \int_\Omega \frac{|\Delta\theta_m(t)|^2}{\rho_\varepsilon(\theta_m(t))} \leqslant \frac{1}{\varepsilon^2}|\Omega|,$ *in* $\mathcal{D}'(0,T_m)$.

Proof. To prove (a) we use the fact that $v_m \in C^1([0,T_m);V_m)$, for each $t \in (0,T_m)$. Then we have :

$$-\Delta v_m(t) \in H_0^1(\Omega) \text{ and, } \frac{d}{dt}\int_\Omega v_m(t)\varphi dx = \int_\Omega \frac{\partial v_m}{\partial t}(t)\varphi(x)dx \;\forall \varphi \in H_0^1(\Omega),$$

and therefore, we can $\varphi = -\Delta v_m(t)$. An integration by part yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx = -\int_\Omega \rho_0(\theta_m(t))\Delta v_m(t)dx.$$

Since $0 \leqslant \rho_0(\theta_m) \leqslant 1$, then by the Young's inequality we deduce

$$\frac{d}{dt}\int_\Omega |\nabla v_m(t)|^2 dx + \int_\Omega |\Delta v_m(t)|^2 dx \leqslant |\Omega|.$$

(b) A similar argument holds for θ_m. Choosing $\psi = -\Delta\theta_m(t)$ and noticing that $\dfrac{\partial \psi}{\partial n} = 0$ on $\partial\Omega$, an integration by parts gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta_m(t)|^2 + \int_\Omega \frac{|\Delta\theta_m(t)|^2}{\rho_\varepsilon(\theta_m)}dx \leqslant \int_\Omega \frac{|\Delta\theta_m(t)|}{\rho_\varepsilon(\theta_m(t))}dx.$$

But $\varepsilon \leqslant \rho_\varepsilon(\theta_m(t))$, thus the Young's inequality yields

$$\frac{d}{dt}\int_\Omega |\nabla\theta_m(t)|^2 + \int_\Omega \frac{|\Delta\theta_m(t)|^2}{\rho_\varepsilon(\theta_m)} \leqslant \frac{1}{\varepsilon^2}|\Omega|.$$

■

Lemma 1 shows that $T_m = T$. Moreover, one has an uniform boundedness for v_m as $m \to +\infty$. Indeed, since $v_m(t) \in H_0^1(\Omega)$, the Sobolev-Poincaré inequality with estimate (a) implies that v_m remains in a bounded set of $L^2(0,T;H^2(\Omega))$ and in $L^\infty(0,T;H_0^1(\Omega))$. While for θ_m, we need to control for instance $\int_\Omega \theta_m(t,x)^2 dx$. To do this, we shall denote by c or c_i where i is an integer greater than one, various constants independent of m and ε. If we want to emphasize the dependence of a constant with respect to ε, we shall note c_ε.

Lemma 12. *For all* $t \in [0,T]$

$$\int_\Omega |\theta_m(t,x)|^2\, dx \leqslant c_\varepsilon.$$

Proof. We take $\psi = \theta_m(t)$ in relation (3). An integration by part and relation (3) yield

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_m^2(t,x)dx \leqslant c_\varepsilon \int_\Omega |\theta_m(t,x)|dx + \int_\Omega \frac{\Delta\theta_m(t,x)}{\rho_\varepsilon(\theta_m(t))}\theta_m(t,x)dx. \tag{4}$$

The statement (b) of Lemma 1 implies that

$$\int_0^T \int_\Omega \frac{|\Delta\theta_m|^2(t,x)}{\rho_\varepsilon(\theta_m(t))}dxdt \leqslant c_\varepsilon(T,\theta_0). \tag{5}$$

Relation (4) gives the following Gronwall inequality,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \theta_m^2(t,x)dx \leqslant c_\varepsilon + \frac{c}{\varepsilon}\int_\Omega \theta_m^2(t,x)dx + \int_\Omega \frac{|\Delta\theta_m|^2(t,x)}{\rho_\varepsilon(\theta_m(t))}dx. \tag{6}$$

From relations (5) and (6), we conclude the Lemma 2. ■

The Lemma 1 and Lemma 2 show that θ_m remains in a bounded set of $L^2(0,T;H^2(\Omega))$ and in $L^\infty(0,T;H^1(\Omega))$ as $m \to +\infty$. While for the time derivative, those uniform estimates combined with the equations satisfied by v_m and θ_m imply :

Lemma 13. *We have:*

i) $\left|\frac{\partial v_m}{\partial t}\right|_{L^2(Q_T)} \leqslant |\Delta v_m|_{L^2(Q_T)} + |\rho_0(\theta_m)|_{L^2(Q_T)} \leqslant c,$

ii) $\left|\frac{\partial \theta_m}{\partial t}\right|_{L^2(Q_T)} \leqslant \left|\frac{\Delta\theta_m}{\rho_\varepsilon(\theta_m)}\right|_{L^2(Q_T)} + \frac{(T|\Omega|)^{\frac{1}{2}}}{\varepsilon^2} \leqslant c_\varepsilon.$

In particular, v_m and θ_m remains in a bounded set of $W(0,T;H^2(\Omega))$ as m varies.

Proof. The time derivatives satisfy the following equations for each time t:

$$\frac{\partial v_m}{\partial t}(t) = \Delta v_m(t) + P_m(\rho_0(\theta_m(t))), \tag{7}$$

$$\frac{\partial \theta_m}{\partial t}(t) = Q_m\left(\frac{\Delta\theta_m}{\rho_\varepsilon(\theta_m)}(t)\right) + Q_m\left(\frac{S_\varepsilon(\theta_m)|\nabla v_m|^2(t)}{\rho_\varepsilon(\theta_m)}\right) \tag{8}$$

Since the projection is a contraction, relations (7), (8) with Lemma 2 imply Lemma 3. ■

Proof of Lemma10 (continuation).
By Aubin-Lions-Simon's classical compactness results (see Theorem 1 and its corollary), we have as $m \to +\infty$

$$\text{a couple } (\theta^\varepsilon, v^\varepsilon) \text{ such that } \begin{cases} v_m \to v^\varepsilon \begin{cases} \text{strongly in } C([0,T], H^s(\Omega)\cap H_0^1(\Omega)) \text{ for all } s<2, \\ \text{weakly in } L^2(0,T;H^2(\Omega)), \end{cases} \\ \textit{and} \\ \theta_m \to \theta^\varepsilon \begin{cases} \text{strongly in } C([0,T], H^s(\Omega)) \text{ for all } s<2, \\ \text{weakly in } L^2(0,T;H^2(\Omega)). \end{cases} \end{cases}$$

Moreover, we have the following weak convergences in $L^2(Q_T)$:

$$\frac{\partial v_m}{\partial t} \rightharpoonup \frac{\partial v^\varepsilon}{\partial t},$$

$$\frac{\partial \theta_m}{\partial t} \rightharpoonup \frac{\partial \theta^\varepsilon}{\partial t}.$$

Due to the above convergences, one can see easily that the couple $(\theta^\varepsilon, v^\varepsilon)$ is a solution of :

$$\frac{\partial v^\varepsilon}{\partial t} = \Delta v^\varepsilon + \rho_0(\theta^\varepsilon) \tag{9}$$

$$\frac{\partial \theta^\varepsilon}{\partial t} = \frac{\Delta \theta^\varepsilon}{\rho_\varepsilon(\theta^\varepsilon)} + \frac{S_\varepsilon(\theta^\varepsilon)}{\rho_\varepsilon(\theta^\varepsilon)}|\nabla v^\varepsilon|^2, \tag{10}$$

with the initial data $v^\varepsilon(0) = v_0$ and $\theta^\varepsilon(0) = \theta_0$. Moreover, on the boundary $\partial\Omega$, we have that the normal trace of $\theta^\varepsilon(t)$, $t \in [0,T]$: $\dfrac{\partial \theta^\varepsilon(t)}{\partial n}$ and the trace of $v^\varepsilon(t)$ are zero. This system is equivalent to the following one in $\mathcal{D}'(0,T)$: for all $\varphi \in H_0^1(\Omega)$, for all $\psi \in H^1(\Omega)$

$$\frac{d}{dt}\int_\Omega v_\varepsilon \varphi + \int_\Omega \nabla v^\varepsilon \cdot \nabla \varphi dx = \int_\Omega \varphi \rho_0(\theta^\varepsilon) dx, \tag{11}$$

$$\frac{d}{dt}\int_\Omega \Phi_\varepsilon(\theta^\varepsilon)\psi dx + \int_\Omega \nabla \theta^\varepsilon \cdot \nabla \psi dx = \int_\Omega S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2 dx. \tag{12}$$

Here, $\Phi_\varepsilon(s) = \displaystyle\int_0^s \rho_\varepsilon(y)dy$. For the function θ^ε, we need to show first the :

Lemma 14.
Consider from now

$$S_\varepsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \leqslant 1-\varepsilon, \\ 0 & \text{if } \sigma \geqslant 1, \\ \dfrac{1}{\varepsilon}(1-\sigma) & \text{if } 1-\varepsilon < \sigma < 1. \end{cases}$$

If $0 \leqslant \theta_0 \leqslant 1$ *a.e. in* Ω *then* $0 \leqslant \theta^\varepsilon \leqslant 1$ *a.e. in* Q_T.

Proof. We multiply the equation by $\rho_\varepsilon(\theta^\varepsilon)\theta^\varepsilon_-$. Relation (10) gives

$$\int_\Omega \frac{\partial \theta^\varepsilon}{\partial t}\rho_\varepsilon(\theta^\varepsilon)\theta^\varepsilon_- dx + \int_\Omega \nabla\theta^\varepsilon \cdot \nabla\theta^\varepsilon_- dx = \int_\Omega S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2(\theta^\varepsilon)_- dx.$$

Since the right hand side is non negative and $(1-s_+)s_- = s_-$, then one has :

$$-\frac{1+\varepsilon}{2}\frac{d}{dt}\int_\Omega (\theta^\varepsilon_-)^2 dx - \int_\Omega |\nabla \theta^\varepsilon_-|^2 dx \geqslant 0,$$

thus one has :

$$\int_\Omega ((\theta^\varepsilon)_-(t,x))^2 dx \leqslant \int_\Omega ((\theta_0^\varepsilon)_-)^2(x) dx = 0,$$

and so a.e. in Q_T : $\theta^\varepsilon \geqslant 0$. Multiplying the equation by $\rho_\varepsilon(\theta^\varepsilon)(\theta^\varepsilon - 1)_+$ equation (10)

$$\int_\Omega (\theta^\varepsilon - 1)_+ \rho_\varepsilon(\theta^\varepsilon)\frac{\partial \theta^\varepsilon}{\partial t} dx + \int_\Omega |\nabla(\theta^\varepsilon-1)_+|^2 dx = \int_\Omega (\theta^\varepsilon-1)_+ S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2 dx = 0.$$

That is

$$\frac{d}{dt}\int_\Omega \int_0^{\theta^\varepsilon} \big[\rho_0(\sigma)+\varepsilon\big](\sigma-1)_+ d\sigma + \int_\Omega |\nabla(\theta^\varepsilon-1)_+|^2 dx = 0.$$

Then for all t :

$$\int_\Omega \int_0^{\theta^\varepsilon(t,x)} \big[\rho_0(\sigma)+\varepsilon\big](\sigma-1)_+ d\sigma dx \leqslant \int_\Omega \int_0^{\theta_0(x)} \big[\rho_0(\sigma)+\varepsilon\big](\sigma-1)_+ d\sigma dx = 0$$

since for $\sigma \geqslant 0$, $\rho_0(\sigma) = (1-\sigma)_+$, then

$$((1-\sigma)_+ + \varepsilon)(\sigma-1)_+ = \varepsilon(\sigma-1)_+.$$

Therefore, we have

$$\int_\Omega \int_0^{\theta^\varepsilon(t,x)} \big[\rho_0(\sigma)+\varepsilon\big](\sigma-1)_+ d\sigma dx = \varepsilon \int_\Omega \int_0^{\theta^\varepsilon} (\sigma-1)_+^2 d\sigma dx = 0$$

we deduce $\theta^\varepsilon \leqslant 1,$ a.e. in Q_T. ■

To get some uniform a priori estimates in ε on v^ε, we recall firstly that Lemma 1, with the previous convergence (or using directly the above equation (9)) imply :

Corollary 2. *We have:*

(a) $\displaystyle \frac{d}{dt}\int_\Omega |\nabla v^\varepsilon(t)|^2 dx + \int_\Omega |\Delta v^\varepsilon(t)|^2 dx \leqslant |\Omega|,$ *in* $\mathcal{D}'(0,T)$.

(b) $\displaystyle \left|\frac{\partial v^\varepsilon}{\partial t}\right|_{L^2(Q_T)} \leqslant c.$

In particular v^ε *is in a bounded set of* $W(0,T;H^2(\Omega))$ *as* ε *varies.*

Thus, we can conclude as before, by Corollary 1 of Theorem 1, that $v^\varepsilon \to v$ strongly in $C([0,T],H^s(\Omega)\cap H_0^1(\Omega))$ for all $s<2$ and weakly in $L^2(0,T;H^2(\Omega))$. Moreover, we have the following weak convergence in $L^2(Q_T)$:

$$\frac{\partial v^\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}.$$

Lemma 15.
θ^ε *remains in a bounded set of* $L^2(0,T;H^1(\Omega))$ *as* $\varepsilon \to 0$.

Proof. We multiply the equation (10) by $\theta^\varepsilon \rho_\varepsilon(\theta^\varepsilon)$ to get:

$$\int_\Omega \theta^\varepsilon \rho_\varepsilon(\theta^\varepsilon)\frac{\partial \theta^\varepsilon}{\partial t} dt + \int_\Omega |\nabla \theta^\varepsilon|^2 dx = \int_\Omega \theta^\varepsilon S_\varepsilon(\theta^\varepsilon)|\nabla v^\varepsilon|^2 dx, \tag{13}$$

$$\int_0^T dt \int_\Omega |\nabla\theta^\varepsilon|^2 dx \leqslant -\int_0^T dt\int_\Omega \theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t} + \int_0^T\int_\Omega |\nabla v^\varepsilon|^2 dxdt, \tag{14}$$

and

$$\int_0^T dt \int_\Omega \rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t} dx = \int_0^T \frac{d}{dt}\left[\int_\Omega dx \int_0^{\theta^\varepsilon} \sigma\rho_\varepsilon(\sigma) d\sigma\right] dt. \tag{15}$$

Since

$$\rho_\varepsilon(\sigma) = (1-\sigma_+)_+ + \varepsilon \leqslant 2,$$

we have, for all t

$$\int_\Omega dx \int_0^{\theta^\varepsilon(x,t)} \sigma\rho_\varepsilon(\sigma)d\sigma \leqslant \int_\Omega (\theta^\varepsilon)^2(x,t)dx.$$

$$\left|\int_0^T dt\int_\Omega \theta^\varepsilon\rho_\varepsilon(\theta^\varepsilon)\frac{\partial\theta^\varepsilon}{\partial t}dx\right| \leqslant \left[\int_\Omega (\theta^\varepsilon)^2(T,x)dx + \int_\Omega \theta_0^2(x)dx\right] \leqslant c_1. \tag{16}$$

Thus relation (14) with corollary 2 give

$$\int_0^T\int_\Omega |\nabla\theta^\varepsilon|^2 dxdt \leqslant c_1 + 4T|\Omega| + \int_\Omega |\nabla v_0|^2 dx = c_2.$$

■

End of the proof of Theorem 2.
Let

$$\Phi_\varepsilon(\theta^\varepsilon) = \int_0^{\theta^\varepsilon}(1-\sigma)d\sigma + \varepsilon\theta^\varepsilon = -\frac{1}{2}(1-\theta^\varepsilon)^2 + \frac{1}{2} + \varepsilon\theta^\varepsilon.$$

then from equation (10), we have :

$$\left|\frac{\partial \Phi_\varepsilon(\theta^\varepsilon)}{\partial t}\right|_{H^{-1}(\Omega)} \leqslant |\nabla\theta^\varepsilon|_{L^2(\Omega)} + \left||\nabla v^\varepsilon|^2\right|_{L^2(\Omega)}.$$

Using Gagliardo-Nirenberg's inequality, one has

$$\left||\nabla v^\varepsilon|^2\right|_{L^2(\Omega)} = |\nabla v^\varepsilon|^2_{L^4(\Omega)} \leqslant c|\nabla v^\varepsilon|_{L^2(\Omega)}|v^\varepsilon|_{H^2(\Omega)} \leqslant c|v^\varepsilon|_{H^2(\Omega)}. \tag{17}$$

Since v^ε remains in a bounded set of $L^\infty(0,T;H^1_0(\Omega))$. Thus using the equation satisfied by θ^ε (see relation (10)), we have

$$\int_0^T \left|\frac{\partial \Phi_\varepsilon(\theta^\varepsilon)}{\partial t}\right|^2_{H^{-1}(\Omega)} dt \leqslant c\left[|\nabla\theta^\varepsilon|^2_{L^2(Q_T)} + |v^\varepsilon|^2_{L^2(0,T;H^2(\Omega))}\right] \leqslant c_3. \tag{18}$$

Thus, $\Phi_\varepsilon(\theta^\varepsilon)_t$ remains in a bounded set of $L^2(0,T;H^{-1}(\Omega))$. Since, we have

$$|\nabla\Phi_\varepsilon(\theta^\varepsilon)|^2_{L^2(Q_T)} = \int_{Q_T}(\rho_\varepsilon(\theta^\varepsilon))^2|\nabla\theta^\varepsilon|^2 dx \leqslant 4\int_{Q_T}|\nabla\theta^\varepsilon|^2 dxdt \leqslant c_8,$$

This shows that $w^\varepsilon \doteq \Phi_\varepsilon(\theta^\varepsilon)$ belongs to a bounded set of $W(0,T;H^1(\Omega))$, the Aubin-Lions-Simon's compactness result (see Theorem 1 and its corollary) implies the existence of a function w satisfying $\Phi_\varepsilon(\theta^\varepsilon)$ converges to w strongly in $C([0,T];L^2(\Omega))$ and a.e. in Q_T. Therefore, $\displaystyle\int_0^{\theta^\varepsilon(t,x)}\rho(\sigma)d\sigma$ converges to w strongly in $C([0,T];L^2(\Omega))$ and a.e. in Q_T and

$$0 \leqslant w \leqslant \int_0^1 \rho(\sigma)d\sigma = \frac{1}{2}, \qquad w(0,x) = \int_0^{\theta_0(x)}\rho(\sigma)d\sigma.$$

Since the restriction of $\Phi_0(\sigma)$ to $[0,1]$, that is the map $\Phi_0 : [0,1] \to \mathbb{R}^+$ given by $\Phi_0(\sigma) = \displaystyle\int_0^\sigma \rho(s)ds = -\frac{(1-\sigma)^2}{2} + \frac{1}{2}$, is invertible from $[0,1]$ to its range, its inverse Φ_0^{-1} is continuous, we deduce that :

$$\Phi_0^{-1}\left(\int_0^{\theta^\varepsilon(t,x)}\rho(\sigma)d\sigma\right) = \theta^\varepsilon(t,x) \to \Phi_0^{-1}(w)(t,x) \text{ a.e. on } Q_T.$$

Then, we can define $\theta \doteq \Phi_0^{-1}(w) = 1 - \sqrt{1-2w}$. Thus $\theta \in L^2(0,T;H^1(\Omega))$ and $0 \leqslant \theta \leqslant 1$ a.e. in Q_T. Hence, we have the following convergences: $\theta^\varepsilon \rightharpoonup \theta$ weakly in $L^2(0,T;H^1(\Omega))$, $\Phi_\varepsilon(\theta^\varepsilon) \to \Phi(\theta)$ strongly in $C([0,T];L^2(\Omega))$ and a.e. in Q_T. Therefore $\rho_0(\theta^\varepsilon) \to \rho_0(\theta) = 1-\theta$ in any $L^p(Q_T)$, $p < +\infty$ and $S_\varepsilon(\theta^\varepsilon) \to 1$ on $\{\theta < 1\}$.
To show that $\displaystyle\lim_{t\to t_0}\int_\Omega |\theta(t,x)-\theta(t_0,x)|^p dx = 0$, it is sufficent to show the case $p=1$. We may assume that $t_0 = 0$. We know that

$$\lim_{t\to 0}\int_\Omega |w(t,x)-w(0,x)|dx = 0,$$

thus

$$\lim_{t\to 0}\int_\Omega |\Phi_0^{-1}(w(t,x)) - \Phi_0^{-1}(w(0,x))|dx = 0,$$

(arguing by contradiction and using the continuity of Φ_0^{-1}), that is

$$0 = \lim_{t\to 0}\int_\Omega |\theta(t,x) - \Phi_0^{-1}(w(0,x))|dx \quad \text{and} \quad \Phi_0^{-1}(w(0,x)) = \theta_0(x).$$

Passing to the limit in equation (11) and (12), we deduce that (v,θ) is a solution of

$$\frac{d}{dt}\int_\Omega v\varphi dx + \int_\Omega \nabla v\cdot\nabla\varphi = \int_\Omega \varphi(1-\theta)dx,$$

$$\frac{d}{dt}\int_\Omega \Phi(\theta)\psi + \int_\Omega \nabla\theta\cdot\nabla\psi dx = \int_\Omega \psi g_v dx,$$

with $g_v \in \left[|\nabla v|^2\chi_{\{\theta<1\}}, |\nabla v|^2\right]$ which proves the required question.
We first note that if $\theta < \gamma < 1$ implies that $\rho_\varepsilon(\theta) \geqslant 1-\gamma > 0$ .We recall that $\rho_\varepsilon(\theta) \leqslant 2$. From relation (10) one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta^\varepsilon|^2 dx + \int_\Omega \frac{|\Delta\theta^\varepsilon|^2}{\rho_\varepsilon(\theta^\varepsilon)}dx \leqslant \int_\Omega \frac{|\nabla v^\varepsilon|^2|\Delta\theta^\varepsilon|}{\rho_\varepsilon(\theta^\varepsilon)}dx. \tag{19}$$

From which we deduce from the above equation and Young inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\theta^\varepsilon|^2 dx + \frac{1}{4}\int_\Omega |\Delta\theta^\varepsilon|^2 dx \leqslant \frac{1}{2(1-\gamma)}\int_\Omega |\nabla v^\varepsilon|^4 dx. \tag{20}$$

Since v^ε belongs to a bounded set of $L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$, we know that if N=2 , $|\nabla v^\varepsilon|$ belongs to a bounded set of $L^4(Q_T)$. This shows that

$$\int_0^T\int_\Omega |\Delta\theta^\varepsilon|^2 dxdt + \sup_t \int_\Omega |\nabla\theta^\varepsilon(t,x)|^2 dx \leqslant c. \tag{21}$$

Therefore, θ_t^ε remains in a bounded set of $L^2(0,T;L^2(\Omega)$. We conclude that $(v^\varepsilon,\theta^\varepsilon)$ is in a bounded set of $W(0,T;H^2(\Omega))$, then using compactness result (see Corollary 1 of theorem 1) : $(v^\varepsilon,\theta^\varepsilon)$ converges to (v,θ) strongly in $C([0,T];H^s(\Omega))^2$ for all $s<2$ and weakly in $L^2(0,T;H^2(\Omega))^2$. This allows to pass easily to the limit in the equation. If $\theta_0 < 1-\delta$ with some $\delta > 0$, then this weak solution is a local exact solution since one has $\theta \in C([0,T];H^s(\Omega)) \subset C(\overline{Q}_T)$ for $s>1$ .Thus, we may apply the first proposition to arrive to the following additionnal conclusion.

Corollary 3. . *Let N=2, $\theta_0 \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $0 \leqslant \underset{\Omega}{\text{Min}}\,\theta_0 \leqslant \underset{\Omega}{\text{Max}}\,\theta_0 = a_0 < 1-\delta$, for some $\delta > 0$ and $v_0 \in H_0^1(\Omega)$.Then there is a couple (θ,v) in $[L^2(0,T;H^2(\Omega)) \cap C([0,T];H^s(\Omega))]^2$ for all $s<2$, with $\frac{\partial\theta}{\partial t}$ and $\frac{\partial v}{\partial t}$ in $L^2(Q_T)$ satisfying :*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = 1-\theta, \text{ in } Q_T \\ (1-\theta\,)\ \dfrac{\partial\theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<1-\delta\}} \text{ in } Q_T, \\ \dfrac{\partial\theta}{\partial n} = v = 0 \text{ on } (0,T)\times\partial\Omega, \\ \theta(0) = \theta_0, : v(0) = v_0 \text{ in } \Omega. \end{cases}$$

■

4 Some Extensions and Qualitative Properties

The above method can be applied if we replace v_t in the v-equation by $(1-\theta)v_t$, we have

Corollary 4. . *Let N=2, $\theta_0 \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $0 \leqslant \underset{\overline{\Omega}}{\mathrm{Min}}\, \theta_0 \leqslant \underset{\overline{\Omega}}{\mathrm{Max}}\, \theta_0 = a_0 < 1-\delta$, for some $\delta > 0$ and $v_0 \in H_0^1(\Omega)$. Then there is a couple (θ, v) in $[L^2(0,T;H^2(\Omega)) \cap C([0,T];H^s(\Omega))]^2$ for all $s < 2$, with $\dfrac{\partial \theta}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ in $L^2(Q_T)$ satisfying :*

$$\begin{cases} (1-\theta)\dfrac{\partial v}{\partial t} - \Delta v = 1-\theta, & \text{in } Q_T; \\ (1-\theta\,)\,\dfrac{\partial \theta}{\partial t} - \Delta\theta = |\nabla v|^2 \chi_{\{\theta < 1-\delta\}}, & \text{in } Q_T; \\ \dfrac{\partial \theta}{\partial n} = v = 0, & \text{on } (0,T)\times\partial\Omega; \\ \theta(0) = \theta_0,\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

Moreover, $0 \leqslant \underset{\overline{\Omega}}{\mathrm{Min}}\, \theta(t) \leqslant \underset{\overline{\Omega}}{\mathrm{Max}}\, \theta(t) \leqslant 1-\delta$ for all $t \geqslant 0$. This solution is a local strong and exact solution, that is a solution of (BS) with the density $(1-\theta)$ with the time derivative in v.

We have a non existence result of exact local-in-time if the initial data exceed the value one. More precisely

Theorem 3 (A non existence result).
Let $\theta_0 \in L^2(\Omega)$, $\theta_0(x) \geqslant 1$, for a.e. $x \in \Omega$, $v_0 \neq 0$ and $\rho(\sigma) = (1-\sigma)_+$, $\sigma \in \mathbb{R}$. Then there exists no local exact solution satisfying (TBS).

■

5 Recalling Navier-Stokes equation framework

The results of this section can be found in [11, 28] where additional details . Let Ω be an open smooth set of $\mathbb{R}^N$, $N=2, f : \Omega\times]0,T[\to \mathbb{R}^2$ and $u_0 : \Omega \to \mathbb{R}^2$ being given. The Navier Stokes equations consist in finding a vector field $u = (u_1, u_2) : \Omega\times]0,T[\to \mathbb{R}^2$ and $p : \Omega\times]0,T[\to \mathbb{R}$ a scalar function such that:

$$(N.S.) \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + \displaystyle\sum_{i=1}^{2} u_i \cdot \dfrac{\partial u}{\partial x_i} + \nabla p = f & \text{in } \Omega\times]0,T[, \\ \operatorname{div} u = \displaystyle\sum_{i=1}^{N} \dfrac{\partial u_i}{\partial x_i} = 0 & \text{in } \Omega\times]0,T[, \\ u = 0 & \text{in } \partial\Omega\times]0,T[, \\ u(0,x) = u_0(x) & \text{(initial data)}, x \in \Omega. \end{cases}$$

The spaces H and V defined by

$$H = \text{closure in } L^2(\Omega)^2 \text{ of } \mathcal{V} = \Big\{ v \in C_c^\infty(\Omega)^2, \operatorname{div} v = 0 \Big\}$$

$$V = \text{closure in } H_0^1(\Omega)^2 \text{ of } \mathcal{V}$$

For the variational formulation, we need the following bilinear form on V and scalar products:

$$\begin{cases} b(u,v,w) = \displaystyle\sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx & \begin{cases} u = (u_1, u_2) \\ v = (v_1, v_2) \\ w = (w_1, w_2) \end{cases}, \\ ((u,v)) = \displaystyle\int_\Omega \nabla u \cdot \nabla v dx, \\ (u,v) = \displaystyle\int_\Omega u \cdot v dx, \\ < T, w >= \text{the duality product between } T \in V' \text{ and } w \in V. \end{cases}$$

V' is the dual of V. The bilinear form and the scalar product $((\cdot,\cdot))$ define operators by setting, for $u \in V$,

$$\begin{cases} < B(u,u), w >= b(u,v,w) & \forall w \in V, \\ < Au, w >= ((u,w)) & \forall w \in V; \end{cases}$$

Thus for $u \in Ł^2(0,T;V), T < +\infty$, we have $B(u,u) \in L^1(0,T;V')$ and $Au \in L^2(0,T;V')$, where A is linear. Thus, the variational formulation of $(N.S.)$ for $f \in L^2(0,T;H),\ u_0 \in V$ gives

$$\begin{cases} \dfrac{d}{dt}(u,v) + \nu((u,v)) + b(u,u,v) = (f,v) & \forall v \in V, \\ u(0) = u_0. \end{cases}$$

Which is equivalent to

$$\begin{cases} \dfrac{du}{dt} + \nu Au + B(u,u) = f & \text{(equality in } V'), \\ u(0) = u_0. \end{cases}$$

The main properties of B are given in [28, 29]. In particular, $b(u,u,v) = -b(u,v,u)$, $b(u,u,u) = 0$.
These formulas can be obtained by integration by part with the condition $\operatorname{div}(u) = 0$.
To solve the above problem, we follow the same scheme as for the "simple" model:

1st step: Use Galerkin method.
Introducing $\varphi_j \in V$ such that, $j \geqslant 1$

$$((\varphi_j, v)) = \lambda_j(\varphi_j, v),\ \forall v \in V,$$

where $\lambda_j > 0$ is the j^{th} eigenvalue of the operator satisfying

$$< A\varphi, v >= ((\varphi, v)),\ \forall \varphi \in V,\ \forall v \in V.$$

We have $\varphi_j \in H^{\ell+2}(\Omega)$ if $\Omega \in C^\ell,\ \ell \geqslant 1$.

Set $V_m = \text{span}\Big\{\varphi_j, j \leqslant m\Big\}$.
Then, by Cauchy- Peano's theorem, one has

$$u_m \in C^1([0,T_m), V_m),\ T_m \in]0,T].$$

$$\frac{d}{dt}(u_m(t),\varphi_j)+\nu((u_m(t),\varphi_j))+b(u_m(t),u_m(t),\varphi_j)=(f(t),\varphi_j),\quad j=1,\dots,m,$$

$$u_m(0)=P_m u_0 \quad (P_m : H\to V_m \text{ orthogonal projection}).$$

2^{nd} step: Uniform proiri estimates in m

Proposition 2. *One has*

$$\frac{1}{2}\frac{d}{dt}|u_m(t)|^2+\nu||u_m(t)||^2=(f(t),u_m(t)).$$

Then

$$|u_m(t)|^2+\nu\int_0^T ||u_m(t)||^2 \leqslant c(T,u_0,|f|_{L^2(0,T;H)}).$$

In particular,

u_m belongs to a bounded set of $L^\infty(0,T,H)\cap L^2(0,T;V)$ as $m\to+\infty$.

Proof:
We take $u_m(t)$ as a test function and use the fact that $b(u_m(t),u_m(t),u_m(t))=0$. For convenience we recall that one has for the operator B the

Definition 10. $(u,v,\psi)\in V^3,\ u=(u_1,u_2),\ v=(v_1,v_2),\ \psi=(\psi_1,\psi_2),$

$$< B(u,v),\psi > \doteqdot b(u,v,\psi)=\sum_{i,j\leqslant 2}\int_\Omega u_j\frac{\partial v_i}{\partial x_j}\psi_i dx.$$

Remark 1 Since $V\subset L^4(\Omega)^2$ this inclusion is continuous, we then have from Holder inequality

$$\int_\Omega u_j\frac{\partial v_i}{\partial x_j}\psi_i dx \leqslant ||u||\,||v||\,||\psi||$$

Lemma 16.
Au_m belongs to a bounded set of $L^2(0,T;V')$ as $m\to+\infty$,
and $B(u_m,u_m)$ belongs to a bounded set of $L^2(0,T;V')$ as $m\to+\infty$.

Proof:
$\forall v\in V: ||v||=1$, one has

$$|< Au_m(t),v >|\leqslant ||u_m(t)||\cdot||v||=||u_m(t)||,$$

this shows that $|A_m(t)|_{V'}\leqslant ||u_m(t)||$. We conclude with Proposition 2.

Corollary 5. *One has:* $|B(u_m(t),u_m(t))|_{V'}\leqslant c|u_m(t)|_H||u_m(t)||$

$$\begin{aligned}
b(u_m(t),u_m(t),\psi) &= -b(u_m(t),u_m(t),\psi)\\
&\leqslant c|u_m(t)|_{L^4(\Omega)}||\psi||\,|u_m(t)|_{L^4(\Omega)}\\
&\leqslant c|u_m(t)|^{\frac12}_{L^2(\Omega)}||u_m(t)||^{\frac12}||\psi||\,|u_m(t)|^{\frac12}_{L^2(\Omega)}||u_m(t)||^{\frac12}\\
&\leqslant c|u_m(t)|_H||u_m(t)||\,||\psi||.
\end{aligned}$$

Corollary 6 (of Lemma16).
$\frac{du_m}{dt}$ *belongs to a bounded set of* $L^2(0,T;V')$ *as* $m \to +\infty$.

Proof:

$$\frac{du_m}{dt}(t) + \nu A u_m(t) + P_m B(u_m(t), u_m(t)) = P_m f.$$

Using Lemma 16 and knowing that

$$|P_m B(u_m(t), u_m(t))|_{V'} \leqslant |B(u_m(t), u_m(t))|_{V'}$$

we get the result. ∎

Conclusion By compactness result (see theorem 2) there exist $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ a subsequence still denoted u_m so that:

$$\begin{aligned} u_m &\to u \text{ in } L^2(0,T;H)\text{-strong} \\ u_m &\to u \text{ in } C([0,T],V')\text{-strong} \\ u_m &\rightharpoonup u \text{ in } L^2(0,T;V)\text{-weak.} \end{aligned}$$

Notice that $\forall \psi \in V \cap C_c^\infty(\Omega)$

$$\int_{Q_T} u_{mj}(t) \frac{\partial u_{mi}}{\partial x_j}(t)\psi dxdt = \int_{Q_T} (u_{mj} - u_j) \frac{\partial u_{mi}}{\partial x_j}(t)\psi dxdt + \int_{Q_T} u_j \frac{\partial u_{mi}}{\partial x_j}(t)\psi dxdt = I_{1m} + I_{2m}.$$

$$|I_{1m} \leqslant |u_{mj} - u_j|_{L^2(0,T;H)} |u_m|_{L^2(0,T;H)} |\psi|_\infty \xrightarrow[m \to +\infty]{} 0,$$

$$I_{2m} \to \int_{Q_T} u \frac{\partial u_i}{\partial x_j} \psi dxdt.$$

∎

Therefore

$$b(u_m(t), u_m(t), \psi) \to b(u(t), u(t), \psi) \text{ in } \mathcal{D}'(0,T).$$

One can show that $u(0) = u_0$ by usual argument.

6 Navier-Stokes equations coupled with the heat equation

We adopt the notations of the preceeding section on Navier-Stokes. The aim of this section is to study the full system G.), H.), I.) given in the first section, when $\rho(\theta) = (1-\theta)_+ \vec{g}$, where $\vec{g}$ is the gravitational field. Roughly speaking, using the same method as for the simplified model, we can show that :

- if $\theta_0 \in C(\overline{\Omega})$, $\sup_{\overline{\Omega}} \theta_0(x) < 1$, then we have a local strong solution (we assume here that $N = 2$).

- if $\theta_0 \in L^2(\Omega)$ with $\theta_0(x) \geqslant 0$ then, we have no solution for the θ-equation satisfying $|\nabla u| \neq 0$ (that is $u_0 \not\equiv 0$).

More precisely, we can show the following

Theorem 4.
Let $T \in]0, +\infty[, u_0 \in V$ *and* $\theta_0 \in H^1(\Omega) \cap C(\overline{\Omega})$, Ω *is a smooth bounded open set of* $\mathbb{R}^2$.

If $\sup_{\overline{\Omega}} \theta_0(x) < 1$, *then there exist a time* $T_1 \in]0, T]$ *(that can be chosen maximal), a vector valued function* $u \in C([0, T_1], V) \cap L^2(0, T_1; V \cap H^2(\Omega))$, $\theta \in C([0, T_1], V) \cap L^2(0, T_1; H^2(\Omega)), \theta < 1$ *in* Q_{T_1} *satisfying*

1. $\dfrac{d}{dt}(u(t), \varphi) + \nu((u(t), u(t), \varphi)) + b(u(t), u(t), \varphi) = ((1-\theta)\vec{g}, \varphi)$ *in* $\mathcal{D}'(0, T_1)$, $\forall \varphi \in V$,
2. $(1-\theta)\theta_t - \Delta\theta + u \cdot \nabla\theta = |\nabla u|^2$ *in* $Q_{T_1} = \Omega \times]0, T_1[$,
3. $\dfrac{\partial \theta}{\partial n} = 0$ *on* $\partial\Omega \times]0, T_1[$,
4. $u(0) = u_0, \theta(0) = \theta_0$.

Acknowledgements. This course was taught during a visit of the author to UCM in 2008 (Programa de Visitantes Distinguidos, Grupo Santander). He wants to thank the university for its hospitality. A special thanks to Professor Ivorra Benjamin for reading and correcting some misprints in this paper.

The following references have been used in my papers with J.I Diaz and P. Schmidt, and also for the course. I add them for information for the reader.

References

[1] G.K. Batchelor, "An Introduction to Fluid Dynamics," Cambridge University Press (1967).

[2] H. Brezis *Analyse fonctionnelle Théorie et Applications.* Masson, 1983.

[3] N. Bruyère, *Existence et unicité de la solution faible-renormalisée pour un système non linéaire de Boussinesq,* C. R. Acad. Sci. Paris Sér. I Math., to appear.

[4] J.I. Díaz and G. Galiano, *Existence and uniqueness of solutions of the Boussinesq system with nonlinear thermal diffusion,* Topol. Methods Nonlinear Anal., 11 (1998), 59–82.

[5] J.I. Díaz, M. Lazzo, and P.G. Schmidt, *Large solutions for a system of elliptic equations arising from fluid dynamics,* SIAM J. Math. Anal., 37 (2005), 490–513.

[6] J.I. Díaz, J.M. Rakotoson, and P.G. Schmidt, *A parabolic system involving a quadratic gradient term related to the Boussinesq approximation,* Rev. R. Acad. Cienc. Ser. A Mat., 101 (2007), 113–117.

[7] J.I. Díaz, J.M. Rakotoson, and P.G. Schmidt, *Mathematical issues concerning the Boussinesq approximation for thermally coupled viscous flows,* Proc. Appl. Math. Mech., to appear.

[8] J.I. Díaz, J.M. Rakotoson, and P.G. Schmidt, *Weak solutions of a modified Navier-Stokes-Boussinesq model for buoyancy-driven flow with viscous heating,* in preparation.

[9] J.I. Díaz and I.I. Vrabie, *Compactness of the Green operator of nonlinear diffusion equations: Application to Boussinesq type systems in fluid dynamics,* Topol. Methods Nonlinear Anal., 4 (1994), 399–416 (volume dedicated to Jean Leray).

[10] E. Feireisl and J. Málek, *On the Navier-Stokes equations with temperature-dependent transport coefficients,* Differ. Equ. Nonlinear Mech., Art. ID 90616 (2006), 14 pp. (electronic).

[11] P. Constantin, and C. Foias, *Navier-Stokes Equations* Chicago Lectures in Mathematica (1988).

[12] T. Hishida, *Existence and regularizing properties of solutions for the nonstationary convection problem,* Funkcial. Ekvac., 34 (1991), 449–474.

[13] Y. Kagei, *On weak solutions of nonstationary Boussinesq equations,* Differential Integral Equations, 6 (1993), 587–611.

[14] Y. Kagei, *Attractors for two-dimensional equations of thermal convection in the presence of the dissipation function,* Hiroshima Math. Journal, 25 (1995), 251–311.

[15] Y. Kagei, M. Růžička, M., and G. Thäter, *Natural convection with dissipative heating,* Comm. Math. Phys., 214 (2000), 287–313.

[16] Kinderlehrer and Stampacchia "Variational inequality " Cambridge University press (1980).

[17] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. 23, Amer. Math. Soc., Providence, R.I. (1968).

[18] J.L. Lions " Quelques méthodes de résolutions de problémes aux limites" Dunod 1968

[19] P.L. Lions, "Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models," Oxford University Press (1996).

[20] J.M. Mihaljan, *A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid,* Astrophys. J., 136 (1962), 1126–1133.

[21] H. Morimoto, *Nonstationary Boussinesq equations,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., 39 (1992), 61–75.

[22] J. Naumann, *On the existence of weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids,* Math. Meth. Appl. Sci., 29 (2006), 1883–1906.

[23] J. Nečas and T. Roubíček, *Buoyancy-driven viscous flow with L^1-data,* Nonlinear Anal., 46 (2001), 737–755.

[24] K.R. Rajagopal, M. Růžička, and A.R. Srinivasa, *On the Oberbeck-Boussinesq approximation,* Math. Models Methods Appl. Sci., 6 (1996), 1157–1167.

[25] J.E Rakotoson and J.M. Rakotoson "Analyse fonctionnelle appliquée aux équations aux dérivées partielles" Presses Universitaires de France (1999).

[26] J.E Rakotosonv J.M. Rakotoson, C. Verbeke **Higher order equations related to thin film : blow-up and global existence, the influence of initial data.** *version revise Juin 2007 pour Journal of Differential Equation* .

[27] J.M. Rakotoson and R. Temam, *An optimal compactness theorem and application to elliptic-parabolic systems,* Appl. Math. Lett., 14 (2001), 303-306.

[28] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis* North Holland (1984).

[29] R. Temam, *Navier-Stokes Equations, Theory and Nonlinear Fucntional Analysis* (1983).