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Viscosity solutions to PDE on Riemannian manifolds

October 19, 2007

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1. Eikonal equations

Let Ω be an open bounded subset of $\mathbb{R}^n$. Assume that $\partial\Omega$ is C^1. Then the function $v(x) = \mathrm{dist}(x, \partial\Omega)$ satisfies the equation $|\nabla v| = 1$ on a neighborhood of $\partial\Omega$ in $\mathbb{R}^n$, as well as the boundary condition $v = 0$ on $\partial\Omega$. However, there is no classical solution to the stationary eikonal problem

$$\begin{aligned} &|\nabla v(x)| = 1,\ x \in \Omega, \\ &v = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

There are lots of *almost everywhere solutions* which are everywhere differentiable, as shown by Deville and Matheron [12] (on $\mathbb{R}^n$) and Deville and Jaramillo [11] (on manifolds)

However, the only natural (at least from a geometric point of view) solution of this equation should be the Lipschitz function $\mathrm{dist}(\cdot, \partial\Omega)$, which is not everywhere differentiable.

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On the other hand, the function $u(t,x) = t - v(x) = t - \operatorname{dist}(x, \partial\Omega)$ satisfies the evolution eikonal equation

$$u_t - |\nabla u(x)| = 0 \tag{1.2}$$

on a neighborhood of $\{0\} \times \partial\Omega$ in $\mathbb{R} \times \mathbb{R}^n$, but again there is no classical global solution. This equation can be rewritten as

$$\frac{u_t}{|\nabla u|} = 1, \tag{1.3}$$

which means that the level sets $\Gamma_t := \{x : u(t,x) = 0\}$ evolve in time with normal velocity equal to 1, and $\Gamma_0 = \partial\Omega$. Again, there is no global solution to this equation and the only natural solutions should be those which satisfy the condition $u(t,\cdot)^{-1}(0) = \{x : \operatorname{dist}(x, \partial\Omega) = t\}$ (and hence cannot be everywhere differentiable).

Equation 1.3 is a very special example of a more general class of surface evolution equations.

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2. Classical evolution of hypersurfaces. Examples

In the last 30 years there has been a lot of interest in the evolution of hypersurfaces of $\mathbb{R}^n$ by functions of their curvatures. In this kind of problem one is asked to find a one parameter family of orientable, compact hypersurfaces Γ_t which are boundaries of open sets U_t and satisfy

$$\begin{aligned} &V = -G(\nu, D\nu) \text{ for } t > 0,\ x \in \Gamma_t, \quad \text{and} \\ &\Gamma_t\big|_{t=0} = \Gamma_0 \end{aligned} \tag{2.1}$$

for some initial set $\Gamma_0 = \partial U_0$, where V is the normal velocity of Γ_t, $\nu = \nu(t, \cdot)$ is a normal field to Γ_t at each x, and G is a given (nonlinear) function.

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Two of the most studied examples are the evolutions by

- mean curvature (when one takes $V = H := \operatorname{div} \nu$, the sum of all principal curvatures in the direction of ν); and by
- Gaussian curvature (when $V = K$ is the Gaussian curvature of Γ_t, that is the product of all principal curvatures in the direction of ν).

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In both cases (mean and Gaussian curvature), short time existence of classical solutions has been established.
For strictly convex initial data U_0, it has been shown that U_t shrinks to a point in finite time, and moreover, Γ_t looks more and more spherical at the end of the contraction.
This has been done by B. Andrews, M. Gage, M.A. Grayson, R. Hamilton, G. Huisken, K. Tso and others. See [2, 18, 23, 24, 26, 27, 39].

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In the case $n = 2$, it has moreover been proved (by Grayson) that (not necessarily convex) embedded plane curves Γ_t moving by their mean curvatures become convex in finite time, and afterwards they shrink to a point (with round limiting shape).
This is not true in the case of surfaces ($n \geq 3$).

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The classical motion of hypersurfaces by their mean curvatures has also been studied in the setting of Riemannian manifolds M of dimension n.
When $n = 2$, the mean curvature flow Γ_t is sometimes called *curve shortening*, because the flow lines in the the space of closed curves are tangent to the gradient for the length functional; this means that the curve is shrinking as fast as it can using only local information.

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Curve shortening processes can be used to find embedded geodesics on surfaces, especially spheres. For instance, Grayson [24] proved that if $\Gamma_0 : S^1 \longrightarrow M$ is a smooth curve embedded in M, then $\Gamma_t : S^1 \longrightarrow M$ exists for $t \in [0, t_\infty)$ satisfying

$$\frac{\partial C}{\partial t} = k\nu,$$

where k is the curvature of C and ν is its unit normal vector. If t_∞ is finite then $C(t)$ converges to a point. If t_∞ is infinite, then the curvature of C converges to 0 in the C^∞ norm (and one expects to find a geodesic in the limit).

Grayson used this result to give a rigorous proof of the Lusternik-Schnirelmann theorem: *a 2-sphere with a smooth Riemannian metric has at least three simple closed geodesics.*

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3. Development of singularities. Why not allow non-smooth initial data?

For dimension $n \geq 3$ it has been shown [22] that a hypersurface evolution Γ_t may develop singularities before it disappears.
Grayson's example consists of two disjoint spherical surfaces connected by a sufficiently thin neck. The inward curvature of the neck is so large that it will force the neck to pinch before the two spherical ends can shrink appreciably.

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Other examples: a thin torus shrinks to a ring. A fat torus becomes singular before it can shrink to a point.

Hence it is natural to try to develop weak notions of solutions to (2.1) which allow to deal with singularities of the evolutions, and even with nonsmooth initial data Γ_0.

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There are two mainstream approaches concerning weak solutions of (2.1): the first one uses geometric measure theory to construct (generally nonunique) varifold solutions [6, 29], while the second one adapts the theory of second order viscosity solutions developed in the 1980's [9] to show existence and uniqueness of level-set weak solutions to (2.1).

In this talk we will focus on this second approach, which was initiated in 1991 by L.C. Evans and Spruck [15] in the case of the mean curvature evolution, and independently by Y.G. Chen, Y. Giga and S. Goto [7] for more general equations (but not including the Gaussian curvature evolution).

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4. Level set method. Equations

A smooth function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ with $Du := D_x u \neq 0$ has the property that all its level sets evolve by (2.1) if and only if u is a solution of

$$u_t + F(Du, D^2u) = 0, \qquad (4.1)$$

where F is related to G in (2.1) through of the following formula:

$$F(p, A) = |p|\, G\left(\frac{p}{|p|}, \frac{1}{|p|}\left(I - \frac{p \otimes p}{|p|^2}\right) A\right). \qquad (4.2)$$

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Example 1. Mean curvature evolution. If u is a function on $[0,T]\times\mathbb{R}^n$ such that $Du(t,x)\neq 0$ for all t,x with $u(t,x)=c$, then each level set $\Gamma_t=\{u(t,\cdot)=c\}$ evolves according to its mean curvature if and only if u satisfies

$$\frac{u_t}{|Du|}=\operatorname{div}\left(\frac{Du}{|Du|}\right),$$

which in turn is equivalent to $u_t+F(Du,D^2u)=0$, where

$$F(\zeta,A)=-\operatorname{trace}\left(\left(I-\frac{\zeta\otimes\zeta}{|\zeta|^2}\right)A\right). \tag{4.3}$$

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Example 2. Gaussian curvature evolution. Now, if u is a function on $[0,T] \times \mathbb{R}^n$ such that $Du(t,x) \neq 0$ for all t,x with $u(t,x) = c$, then all level sets $\Gamma_t = \{u(t,\cdot) = c\}$ evolve according to their Gaussian curvature if and only if u satisfies

$$\frac{u_t}{|Du|} = \det\left(D^T\left(\frac{\nabla u}{|\nabla u|}\right)\right),$$

where D^T stands for the orthogonal projection onto $T\Gamma_t$ of the derivative in $\mathbb{R}^n$. This equation is equivalent to $u_t + H(Du, D^2u) = 0$, where

$$H(\zeta, A) = -|\zeta|\det\left(\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right)A + \frac{\zeta \otimes \zeta}{|\zeta|^2}\right).$$

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The level set method consists of the following steps

- For a given initial hypersurface Γ_0 which is the boundary of a bounded open set U_t we take an auxiliary function u_0 which is at least continuous and bounded and such that $\Gamma_0 = u_0^{-1}(0), U_0 = u_0^{-1}(0,\infty)$.
- We solve (in some weak sense) the initial value problem

$$u_t + F(Du, D^2u) = 0,\ \ u(0,x) = u_0(x)$$

- We then set $\Gamma_t = u(t,\cdot)^{-1}(0),\ \ \Gamma_t = u(t,\cdot)^{-1}(0,\infty)$, and hope that Γ_t is a generalized solution in some sense. In particular Γ_t should equal the classical motion whenever the latter exists, and should always be independent of the function u_0 chosen to represent Γ_0.

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Because equation (4.1) is very singular (meaning that F is undefined at $Du = 0$, admits no continuous extension to $\mathbb{R}^n \times \mathbb{R}^{n^2}$, and in general $F(p, A)$ is not bounded as $p \to 0$), classical PDE theory and Sobolev spaces are not useful to solve the problem.

It turns out that an adequate notion of solution of (4.1) is that of *viscosity solution*.

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5. Viscosity solutions to second order PDE

Second order subdifferentials, parabolic semi-jets.

Let $f : (0,T) \times \mathbb{R}^n \to (-\infty, +\infty]$ be a lower semicontinuous (LSC) function. We define the parabolic second order subjet of f at a point $z_0 = (t_0, x_0) \in (0,T) \times \mathbb{R}^n$ by

$$\mathcal{P}^{2,-} f(z_0) = \{(D_t\varphi(z_0), D_x\varphi(z_0), D_x^2\varphi(z_0)) \ : \\ \varphi \in C^2,\ f - \varphi \text{ attains a local minimum at } z_0\}.$$

Similarly, for an upper semicontinuous (USC) function $f : (0,T) \times \mathbb{R}^n \to [-\infty, +\infty)$, we define the parabolic second order superjet of f at (z_0) by

$$\mathcal{P}^{2,+} f(z_0) = \{(D_t\varphi(z_0), D_x\varphi(z_0), D_x^2\varphi(z_0)) \ : \\ \varphi \in C^2,\ f - \varphi \text{ attains a local maximum at } z_0\}.$$

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Standard notion of viscosity solution. Introduced by M. Crandall and P.-L. Lions in the 1980's.

Let $F : (0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ be a continuous function. Consider the equation

$$u_t + F(t, x, u, Du, D^2u) = 0, \qquad (5.1)$$

where u is a function of (t, x). We say that an upper semicontinuous function $u : (0,T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a viscosity subsolution of (5.1) provided that

$$a + F(t, x, u(t, x), \zeta, A) \le 0$$

for all $(t, x) \in (0,T) \times \mathbb{R}^n$ and $(a, \zeta, A) \in \mathcal{P}^{2,+}u(t, x)$.

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Similarly, a viscosity supersolution of (5.1) is a lower semicontinuous function $u : (0, T) \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$a + F(t, x, u(t, x), \zeta, A) \geq 0$$

for every $(t, x) \in (0, T) \times \mathbb{R}^n$ and $(a, \zeta, A) \in \mathcal{P}^{2,-}u(t, x)$. If u is both a viscosity subsolution and a viscosity supersolution of $u_t + F(t, x, u, Du, D^2u) = 0$, we say that u is a viscosity solution.

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Here is some motivation for this definition. Assume that F is elliptic (that is decreasing in the variable D^2u).

If u is a classical solution then we have $u_t(z) + F(Du(z), D^2u(z)) = 0$ for all $z = (t, x)$. Then, if φ is such that $u - \varphi$ attains a minimum at z, we have $\varphi_t(z) = u_t(z)$, $D\varphi(z) = Du(z)$, and $D^2u(z) \geq D^2\varphi(z)$. Since F is elliptic we get

$$\begin{aligned}&\varphi_t(z) + F(z, u(z), D\varphi(z), D^2\varphi(z)) \geq \\ &u_t(z) + F(z, u(z), Du(z), D^2u(z)) = 0,\end{aligned}$$

that is, u is a supersolution at z. A similar argument shows that u is a subsolution.

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Therefore every classical solution of $u_t + F(Du, D^2u) = 0$ is a viscosity solution.

This is no longer true if F is not elliptic, as shown by the example: $u(t,x) = t + x^2 - 2$, which is a classical solution of $u_t + u + u_{xx} - x^2 - 1 = 0$, but not a viscosity solution.

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Refined notion of viscosity solution. Introduced by H. Ishii and P. Souganidis in 1995 [32].

Due to the singularity of F in (4.1), the standard notion of viscosity solution does not make sense for functions φ at points $z_0 = (t_0, x_0)$ such that $D\varphi(z_0) = 0$ and $u - \varphi$ attains a maximum or a minimum at z_0.
In order to deal with this difficulty, Ishii and Souganidis suggested to make the class $\mathcal{A}(F)$ of test functions φ smaller, in a clever, rather technical way which we will not detail here, so that, at a point $z_0 = (t_0, x_0)$ where $D\varphi(z_0) = 0$, one can show that

$$\lim_{z \to z_0} F(D\varphi(z), D^2\varphi(z)) = 0,$$

and then to demand that a subsolution u of (4.1) should satisfy that if $u - \varphi$ has a maximum at z_0 and $D\varphi(z_0) = 0$ then

$$\varphi_t(z_0) \le 0.$$

A similar condition is required of u to be a supersolution.
The class $\mathcal{A}(F)$ of test functions is a proper subset of C^2, but remains dense in C^0.

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6. Assumptions on F

The function F is assumed to be of the form

$$F(\zeta, A) = |\zeta|\, G\left(\frac{\zeta}{|\zeta|}, \frac{1}{|\zeta|}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right) A\right), \qquad (6.1)$$

where G is any (nonlinear) function such that:

(A) F is continuous off $\{\zeta = 0\}$.

(B) F is *elliptic*, that is,

$$P \leq Q \implies F(\zeta, Q) \leq F(\zeta, P).$$

Besides, because $\left(I - \frac{\zeta\otimes\zeta}{|\zeta|^2}\right)(\zeta \otimes \zeta) = 0$, any function F of the form (6.1) also satisfies

(C) F is *geometric*, that is,

$$F(\lambda\zeta, \lambda A + \mu\zeta \otimes \zeta) = \lambda F(\zeta, A)$$

for every $\lambda > 0$, $\mu \in \mathbb{R}$.

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Examples of F's satisfying (A-C).

1. Mean curvature evolution equation for level sets.

2. Positive Gaussian curvature evolution equation for level sets. The Gaussian curvature evolution equation is given by $u_t + H(Du, D^2u) = 0$, where

$$H(\zeta, A) = -|\zeta| \det \left(\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2} \right) A + \frac{\zeta \otimes \zeta}{|\zeta|^2} \right).$$

However, the function H is not elliptic, so this problem cannot be treated, in its most general form, with the theory of viscosity solutions. Nevertheless, if our initial data $u(0,x) = g(x)$ satisfies that $D^2g(x) \geq 0$ (that is, if the initial hypersurface $\Gamma_0 = \{x \in M : g(x) = c\}$ has nonnegative Gaussian curvature) then it is reasonable, and consistent with the classical motion of convex surfaces by their Gaussian curvature, to assume that $D^2u(t,x) \geq 0$ for all (t,x) with $u(t,x) = c$ (that is, Γ_t will have nonnegative Gaussian curvature as long as it exists).

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In this case our equation becomes

$$u_t + F(Du, D^2u) = 0,$$

with

$$F(\zeta, A) = -|\zeta|\det{}_+\left(\frac{1}{|\zeta|}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right) A + \frac{\zeta \otimes \zeta}{|\zeta|^2}\right),$$

and where

$$\det{}_+(A) = \prod_{j=1}^{n} \max\{\lambda_j, 0\}$$

if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.
This F *is* elliptic and satisfies (A-C). This equation models the wearing process of a (nonconvex) stone at the seashore [31].

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7. Existence and uniqueness

Under assumptions (A-C) it is possible to show that there is a unique level set evolution of Γ_0, that is a unique family of sets Γ_t, D_t, of the form $\Gamma_t = \{x : u(t,x) = 0\}$, $D_t = \{x : u(t,x) > 0\}$, where u is a viscosity solution of

$$u_t + F(Du, D^2u) = 0$$

defined on $[0,\infty) \times \mathbb{R}^n$.

The proof is done it two steps: first one establishes a comparison principle (this is the hardest part of the theory, the proof involves deep results of real analysis and convex functions). Then one may construct a solution by combining the so-called Perron's method, comparison, and stability of the equation (that is, limits of subsolutions are subsolutions, etc), as follows.

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Comparison:

Let u, v be a viscosity subsolution and supersolution, respectively, of $u_t + F(Du, D^2u) = 0$, defined on $[0, T) \times \overline{\Omega}$. Assume that $u \leq v$ on the parabolic boundary $\{0\} \times \Omega \cup [0, T) \times \partial\Omega$. Then $u \leq v$ on $[0, T) \times \Omega$.

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Stability

Lemma 1. *Assume that u.s.c. (respectively l.s.c.) functions* u_k *are subsolutions (supersolutions, respectively). Assume also that* $\{u_k\}$ *converges locally uniformly to a function* u*. Then* u *is subsolution (supersolution, respectively).*

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Perron's method:
Assume comparison holds for the equation

$$\begin{cases} u_t + F(Du, D^2u) = 0 \\ u(0,x) = g(x). \end{cases} \tag{7.1}$$

Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of (7.1), respectively, satisfying $\underline{u}_*(0,x) = \overline{u}^*(0,x) = g(x)$. Then $w = \sup\{v : \underline{u} \leq v \leq \overline{u},\ v \text{ is a subsolution}\}$ is a solution of (7.1).

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Now we can prove existence of solutions to the surface evolution problem as follows. Assume that comparison holds for the equation

$$\begin{cases} u_t + F(Du, D^2u) = 0 \\ u(0,x) = g(x). \end{cases} \tag{7.2}$$

Take a continuous function g such that $g^{-1}(0) = \Gamma_0$. Since Γ_0 is compact we can assume that g equals some positive constant outside a closed ball B_0 containing Γ_0.

Let us first produce a solution of (7.2) for $g \in \mathcal{A}(F)$.

Let us define $\underline{u}(t,x) = -Kt + g(x)$, and $\overline{u}(t,x) = Kt + g(x)$, where $K := \sup_{x\in B} |F(Dg(x), D^2g(x))|$ (which is finite because $g \in \mathcal{A}(F)$ and B is compact). It is immediately seen that $\underline{u}$ is a subsolution and $\overline{u}$ is a supersolution of (7.2), and obviously $\underline{u}_*(0,x) = \overline{u}^*(0,x) = g(x)$. According to Perron's method and comparison, there exists a unique solution u of (7.2).

Now take g a continuous function. It is easy to check that the class $\mathcal{A}(F)$ of admissible test functions satisfies the hypotheses of the Stone-Weierstrass theorem, hence it is dense in the space of continuous functions. Therefore we can find a sequence g_k of functions in $\mathcal{A}(F)$ such that $g_k \to g$ uniformly on bounded sets. Let u_k be the unique solution of (7.2) with initial datum g_k. By comparison, for any ball $B \supset B_0$, (u_k) is a Cauchy sequence in $\mathcal{C}([0,\infty) \times B)$, hence it converges to some $u \in \mathcal{C}([0,\infty) \times \mathbb{R}^n)$ uniformly on bounded sets. By stability, it follows that u is a solution with initial datum $u(0,x) = g(x)$.

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8. Geometric consistency & Generalized evolution

Condition (C) (geometricity) and stability imply:

Theorem 2. *Let $\theta : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let u be a bounded continuous solution. Then $v = \theta \circ u$ is also a solution.*
Moreover, if θ is nondecreasing and u is a subsolution (resp. supersolution) then $v = \theta \circ u$ is a subsolution (resp. supersolution) as well.

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Let g be a continuous function on M with $\Gamma_0 = \{x \in M : g(x) = 0\}$, and assume that Γ_0 is compact. We may also assume that g is constant outside a bounded neighborhood of Γ_0, and in particular bounded. Let u be the unique solution of (4.1) with $u(0, \cdot) = g$. We define

$$\Gamma_t = \{x \in M : u(t, x) = 0\}.$$

Theorem 3. *Assume that comparison and existence hold for* (4.1). *Let* $\hat{g} : M \to \mathbb{R}$ *be a continuous function satisfying* $\Gamma_0 = \{x \in M : \hat{g}(x) = 0\}$ *and such that* $\hat{g}$ *is constant outside a bounded neighborhood of* Γ_0. *Let* $\hat{u}$ *be the unique continuous solution of* (4.1) *with initial condition* $\hat{g}$. *Then*

$$\Gamma_t = \{x \in M : \hat{u}(t, x) = 0\}.$$

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Corollary 4. *The definition of* $\Gamma_t = \{x \in M : u(t,x) = 0\}$ *does not depend upon the particular choice of the function* g *satisfying* $\Gamma_0 = \{x \in M : g(x) = 0\}$.

It can also be checked that the evolution $\Gamma_0 \mapsto \mathcal{K}(t)\Gamma_0 := \Gamma_t$ thus defined has the semigroup property

$$\mathcal{K}(t+s) = \mathcal{K}(t)\mathcal{K}(s).$$

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9. Consistency with the classical motion

Assume Γ_t is a classical evolution of Γ_0. More precisely, suppose $(\Gamma_t)_{t\in[0,T]}$ is a family of smooth, compact, orientable hypersurfaces in $\mathbb{R}^n$ evolving according to a classical geometric motion, locally depending only on its normal vector fields and second fundamental forms. In particular, we assume that Γ_t is the boundary of a bounded open set $U_t \subset \mathbb{R}^n$ and that there exists a family of diffeomorphisms of manifolds with boundary

$$\phi^t : \overline{U_0} \to \overline{U_t}, \qquad t \in [0,T],$$

such that:

(i) $\phi^0 =$Id, and,

(ii) for every $x \in \Gamma_0$ the following holds:

$$\frac{d}{dt}\phi^t(x) = G\left(\nu\left(t,\phi^t(x)\right), \nabla^\Gamma \nu\left(t,\phi^t(x)\right)\right), \tag{9.1}$$

where $\nu(t,\cdot)$ is a unit normal vector field to Γ_t, and the linear map

$$\nabla^\Gamma \nu(t,x) : (T\Gamma_t)_x \ni \xi \mapsto \nabla^T_\xi \nu(t,x) \in (T\Gamma_t)_x$$

and ∇^T stands for the orthogonal projection onto $(T\Gamma_t)_x$ of the derivative in $\mathbb{R}^n$.

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Classical motion by mean curvature corresponds to taking $G\left(\nu, \nabla^{\Gamma}\nu\right) = \operatorname{tr}\left(-\nabla^{\Gamma}\nu\right)\nu$, whereas classical motion by Gaussian curvature is defined by $G\left(\nu, \nabla^{\Gamma}\nu\right) = \det\left(-\nabla^{\Gamma}\nu\right)\nu$. The level set evolution equation induced by (9.1) is of the form (4.1) where F is related to G through formula (6.1). As before, we assume that F is continuous, elliptic, translation invariant and geometric.

Define $d : [0,T] \times M \to \mathbb{R}$, the signed distance function from Γ_t, as:

$$d\,(t,x) := \begin{cases} \operatorname{dist}\,(x, \Gamma_t) & \text{if } x \in U_t \\ -\operatorname{dist}\,(x, \Gamma_t) & \text{if } x \in M \setminus U_t. \end{cases}$$

Theorem 5. *Let u be the unique viscosity solution to the level set equation (4.1) with initial datum $u|_{t=0} = d|_{t=0}$. Then, for every $t \in [0,T]$, the zero level set of $u\,(t,\cdot)$ coincides with Γ_t:*

$$\Gamma_t = \{x \in M \;:\; u\,(t,x) = 0\}\,.$$

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10. Why not make it on manifolds?

In the case of the F corresponding to the mean curvature evolution, this theory has been extended to Riemannian manifolds by T. Ilmanen [28].

Recently, in a joint work with M. Jiménez-Sevilla and F. Macià [5], we have extended this theory to Riemannian manifolds of nonnegative curvature for all F's such that F is continuous off $\{Du = 0\}$, elliptic, geometric, and locally invariant by parallel translation. This includes the case of the (positive) Gaussian curvature evolution.

When M is not of nonnegative curvature, the same results hold if one additionally requires that F is uniformly continuous with respect to D^2u (but this excludes the motion by positive Gaussian curvature).

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Open problem. Is there a (well defined, unique, consistent) generalized evolution of level sets by their Gaussian curvatures in Riemannian manifolds of negative curvature?

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11. Counterexamples and conjectures in the setting of Riemannian manifolds

Several well known properties of the evolutions in $\mathbb{R}^n$ are no longer true in the case when the Riemannian manifold M has negative sectional curvature.

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Example 6. When (M, g) is the Euclidean space equipped with the canonical metric, Ambrosio and Soner have proved in [1] that the distance function $|d|$ is always a supersolution to the mean curvature equation for level sets. This is no longer the case of a general Riemannian manifold.
Conjecture: If M has nonnegative sectional curvature then $|d|$ is always a supersolution. If M has negative curvature then there always exists Γ_0 such that $|d|$ is not a supersolution.

Example 7. When $M = \mathbb{R}^n$, Evans and Spruck [15, Theorem 7.3] showed that if Γ_0, $\hat{\Gamma}_0$ are compact level sets and Γ_t, $\hat{\Gamma}_t$ are the corresponding generalized evolutions by mean curvature, then

$$\mathrm{dist}(\Gamma_0, \hat{\Gamma}_0) \le \mathrm{dist}(\Gamma_t, \hat{\Gamma}_t)$$

for all $t > 0$. This result fails for manifolds of negative curvature. For instance, let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 + z^2\}$ be a hyperboloid of revolution embedded in $\mathbb{R}^3$. Let

$$\Gamma_0 = \{(x, y, z) \in M : z = 0\},$$

and

$$\hat{\Gamma}_0 = \{(x, y, z) \in M : z = 1\}.$$

Then

$$\Gamma_t = \Gamma_0 \text{ for all } t > 0, \qquad \text{and} \qquad \mathrm{dist}(\Gamma_0, \hat{\Gamma}_0) > 0,$$

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but

$$\mathrm{dist}(\Gamma_t, \hat{\Gamma}_t) = \mathrm{dist}(\Gamma_0, \hat{\Gamma}_t) \to 0 \text{ as } t \to \infty.$$

Conjecture: Evans-Spruck's [15, Theorem 7.3] result holds true for all manifolds of nonnegative sectional curvature, but fails for all manifolds of negative curvature.

Example 8. In the case $M = \mathbb{R}^n$ it is well known that equation (4.1) preserves Lipschitz properties of the initial data. Namely, if g is L-Lipschitz and u is the unique solution of (4.1) then $u(t, \cdot)$ is L-Lipschitz too, for all $t > 0$; see [19, Chapter 3]. Since the proof of Theorem 7.3 in [15] remains valid for any manifold provided that one assumes the Lipschitz preserving property of (4.1), the preceding example also shows that (4.1) does not preserve Lipschitz constants when M is a hyperboloid of revolution.

Conjecture: The equation (4.1) has the Lipschitz preserving property if and only if M has nonnegative sectional curvature.

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12. General theory on manifolds

For general theory on viscosity solutions to PDE on manifolds, we refer to

D. Azagra, J. Ferrera, B. Sanz, *Viscosity solutions to second order partial differential equations I*, preprint, 2006 (math.AP/0612742v2),

for stationary equations, and to the Appendix in

D. Azagra, M. Jiménez-Sevilla, F. Macià, *Generalized motion of level sets by functions of their curvatures on Riemannian manifolds*, preprint, arXiv:0707.2012v2 [math.AP],

for evolution equations.

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