A mathematical analysis of some hyperbolic - parabolic problems

Julien Jimenez

UCM - IMI

October, 20th 2009

Introduction

Aim of the talk

Mathematical analysis of the "model" problem

$$\begin{aligned} \partial_t u + \mathrm{div}\boldsymbol{F}_h(x,u) &= 0 && \text{in } (0,T)\times\Omega_h \\ \partial_t u + \mathrm{div}\boldsymbol{F}_p(x,u) &= \Delta\phi(u) && \text{in } (0,T)\times\Omega_p \end{aligned}$$

with

$$\Omega_p \cap \Omega_h = \Gamma_{hp} (\neq \emptyset)$$

Introduction

Aim of the talk

Mathematical analysis of the "model" problem

$$\begin{aligned} \partial_t u + \mathrm{div} \boldsymbol{F}_h(x,u) &= 0 && \text{in } (0,T)\times\Omega_h \\ \partial_t u + \mathrm{div} \boldsymbol{F}_p(x,u) &= \Delta\phi(u) && \text{in } (0,T)\times\Omega_p \end{aligned}$$

with

$$\Omega_p \cap \Omega_h = \Gamma_{hp} (\neq \emptyset)$$

Some applications

- Infiltration process
- Fluid dynamics

Outlines of the talk

1. Nonlinear hyperbolic problems
   - Notion of weak entropy solution
2. Nonlinear parabolic problems
   - The Schauder-Tychonoff fixed-point method
3. Case of a "nonlinear" flux : $\boldsymbol{F}_i(x,u) = b_i(x)\boldsymbol{f}_i(u)$ $(i = h, p)$
   - Definition of a weak entropy solution
   - Existence and uniqueness results
4. Case of a non "nonlinear" flux : $\boldsymbol{F}_i(x,u) = \boldsymbol{b}_i(x) f_i(u)$
   - Definition of a weak entropy solution
   - Existence and uniqueness results

Nonlinear hyperbolic problems

Problem (P_H)

Find a measurable and bounded function u such that

$$\partial_t u + \partial_x f(u) = 0,\ t > 0,\ x \in \mathbb{R},$$

with initial condition

$$u(x,0) = u_0(x),\ x \in \mathbb{R}.$$

Assumptions

- $u_0 \in \mathcal{C}_b^1(\mathbb{R})$
- $f \in \mathcal{C}^1(\mathbb{R})$

A simple example

- Consider Burgers equation : $f(u) = \frac{1}{2}u^2$

$$\partial_t u + u\partial_x u = 0$$

A simple example

- Consider Burgers equation : $f(u) = \frac{1}{2}u^2$

$$\partial_t u + u\partial_x u = 0$$

- Introduce the characteristic curves

$$\frac{dx}{dt} = f'(u(x(t), t)) = u(x(t), t)$$

- Compute

$$\begin{aligned} \frac{d}{dt}u(x(t), t) \quad &= \partial_t u(x,t) + \partial_x u(x,t)\frac{dx}{dt} \\ &= \partial_t u(x,t) + u(x,t)\partial_x u(x,t) \\ &= 0 \end{aligned}$$

A simple example

- Consider Burgers equation : $f(u) = \frac{1}{2}u^2$
$$\partial_t u + u \partial_x u = 0$$
- Introduce the characteristic curves
$$\frac{dx}{dt} = f'(u(x(t),t)) = u(x(t),t)$$
- Compute
$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \partial_t u(x,t) + \partial_x u(x,t)\frac{dx}{dt} \\ &= \partial_t u(x,t) + u(x,t)\partial_x u(x,t) \\ &= 0 \end{aligned}$$
- So u is constant along the characteristic curves and
$$\frac{dx}{dt} = u(x(0),0) = u_0(x_0) \quad \text{and} \quad x(t) = x_0 + t u_0(x_0)$$

A simple example

- Consider Burgers equation : $f(u) = \frac{1}{2}u^2$

$$\partial_t u + u\partial_x u = 0$$

- Introduce the characteristic curves

$$\frac{dx}{dt} = f'(u(x(t),t)) = u(x(t),t)$$

- Compute

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \partial_t u(x,t) + \partial_x u(x,t)\frac{dx}{dt} \\ &= \partial_t u(x,t) + u(x,t)\partial_x u(x,t) \\ &= 0 \end{aligned}$$

- So u is constant along the characteristic curves and

$$\frac{dx}{dt} = u(x(0),0) = u_0(x_0) \ \text{ and } \ x(t) = x_0 + tu_0(x_0)$$

- Choose u_0 such that $u_0(0) = 2$ and $u_0(1) = 1$
- So $u(2t,t) = u_0(0) = 2$ and $u(1+t,t) = u_0(1) = 1$

A simple example

- Consider Burgers equation : $f(u) = \frac{1}{2}u^2$

$$\partial_t u + u\partial_x u = 0$$

- Introduce the characteristic curves

$$\frac{dx}{dt} = f'(u(x(t),t)) = u(x(t),t)$$

- Compute

$$\begin{aligned} \frac{d}{dt}u(x(t),t) &= \partial_t u(x,t) + \partial_x u(x,t)\frac{dx}{dt} \\ &= \partial_t u(x,t) + u(x,t)\partial_x u(x,t) \\ &= 0 \end{aligned}$$

- So u is constant along the characteristic curves and

$$\frac{dx}{dt} = u(x(0),0) = u_0(x_0) \text{ and } x(t) = x_0 + tu_0(x_0)$$

- Choose u_0 such that $u_0(0) = 2$ and $u_0(1) = 1$
- So $u(2t,t) = u_0(0) = 2$ and $u(1+t,t) = u_0(1) = 1$
- For $t = 1$, $u(2,1) = 2 = 1$

Notion of weak solution

Weak solution

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution to (P_H) if $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t \varphi + f(u)\partial_x \varphi) dx dt + \int_0^\infty u_0 \varphi(0,x) dx = 0.$$

In particular,

$$\partial_t u + \partial_x f(u) = 0 \ \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+).$$

Notion of weak solution

Weak solution

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution to (P_H) if $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_0^\infty \int_{\mathbb{R}} (u\partial_t \varphi + f(u)\partial_x \varphi) dx dt + \int_0^\infty u_0 \varphi(0,x) dx = 0.$$

In particular,

$$\partial_t u + \partial_x f(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+).$$

Two important remarks

- Existence of a weak solution to (P_H)
- A weak solution is not unique

The "good" notion of solution

- Let $\eta \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ be a convex function and $F \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ s.t.

$$F'(u) = \eta'(u) f'(u)$$

- (η, F) is called an entropy pair

The "good" notion of solution

- Let $\eta \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ be a convex function and $F \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ s.t.

$$F'(u) = \eta'(u) f'(u)$$

- (η, F) is called an entropy pair

Definition 1

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak entropy solution to (P_H) if, for every entropy pairs (η, F),

$$\partial_t \eta(u) + \partial_x F(u) \leq 0 \;\text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+)$$

i.e $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\varphi \geq 0$,

$$\int_0^\infty \int_{\mathbb{R}} (\eta(u)\partial_t \varphi + \boldsymbol{F}(u)\partial_x \varphi) dx dt \geq 0$$

The "good" notion of solution

- For $k \in \mathbb{R}$, $\eta(u) = |u-k|$ and $F(u) = sgn(u-k)(f(u)-f(k))$

Definition 2

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak entropy solution to (P_H) if $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} (|u-k|\partial_t \varphi + sgn(u-k)(f(u)-f(k))\partial_x \varphi) dx dt \geq 0$$

The "good" notion of solution

- For $k \in \mathbb{R}$, $\eta(u) = |u-k|$ and $F(u) = sgn(u-k)(f(u)-f(k))$

Definition 2

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak entropy solution to (P_H) if $\forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} (|u-k|\partial_t \varphi + sgn(u-k)(f(u)-f(k))\partial_x \varphi) dxdt \geq 0$$

Important property (S.N. Kruzkhov)

The weak entropy solution u is the "limit" of $(u_\varepsilon)_{\varepsilon>0}$ where

$$\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon$$

Nonlinear parabolic problems : a fixed-point method

Problem

Find $u \in W(0,T)$ such that
$\forall v \in H_0^1(\Omega)$, for a.e. $t \in (0,T)$,

$$\langle \partial_t u, v \rangle + \int_\Omega (\nabla \phi(u) - \boldsymbol{b}(x) f(u)) \cdot \nabla v dx = 0,$$

$$u(0,.) = u_0 \text{ a.e. on } \Omega,$$

$$\text{for a.e. } (t,x) \in Q,\ u(t,x) \in [m,M].$$

Nonlinear parabolic problems : a fixed-point method

Problem

Find $u \in W(0,T)$ such that
$\forall v \in H_0^1(\Omega)$, for a.e. $t \in (0,T)$,

$$\langle \partial_t u, v \rangle + \int_\Omega (\nabla \phi(u) - \boldsymbol{b}(x) f(u)) \cdot \nabla v dx = 0,$$

$$u(0,.) = u_0 \text{ a.e. on } \Omega,$$

$$\text{for a.e. } (t,x) \in Q,\ u(t,x) \in [m, M].$$

Some assumptions and notations

- $W(0,T) = \{u \in L^2(0,T; H_0^1(\Omega)); \partial_t u \in L^2(0,T; H^{-1}(\Omega))\}$
- $\langle ., . \rangle :=$ the pairing between H^{-1} and H_0^1
- $\exists \alpha > 0, \forall \tau \in \mathbb{R}, \phi'(\tau) \geq \alpha$
- $u_0 \in L^\infty(\Omega)$, $m \leq u_0 \leq M$

A fixed-point Theorem

The Schauder-Tychonoff fixed point Theorem

Let X be a reflexive and separable Banach space. We suppose

- $K \subset X$, $K \neq \emptyset$, K is a closed, bounded and convex set
- The mapping $\mathcal{T} : K \mapsto K$ is "weakly-weakly" sequentially continuous, i.e. for any sequence $(x_n)_{n\in\mathbb{N}^*} \subset K$ that weakly converges towards x, the sequence $(\mathcal{T}(x_n))_{n\in\mathbb{N}^*}$ weakly converges towards $\mathcal{T}(x)$.

Then, $\mathcal{T}$ has at least one fixed-point in K.

A fixed-point Theorem

The Schauder-Tychonoff fixed point Theorem

Let X be a reflexive and separable Banach space. We suppose

- $K \subset X$, $K \neq \emptyset$, K is a closed, bounded and convex set
- The mapping $\mathcal{T} : K \mapsto K$ is "weakly-weakly" sequentially continuous, i.e. for any sequence $(x_n)_{n\in\mathbb{N}^*} \subset K$ that weakly converges towards x, the sequence $(\mathcal{T}(x_n))_{n\in\mathbb{N}^*}$ weakly converges towards $\mathcal{T}(x)$.

Then, $\mathcal{T}$ has at least one fixed-point in K.

Main idea

"associate" the nonlinear problem with a linear one via a mapping $\mathcal{T}$

- Troncation process. Consider the equivalent nonlinear problem
  Find $u \in W(0,T)$ such that

$$\begin{cases} \langle \partial_t u, v \rangle + \int_\Omega (\phi'(u^*)\nabla u - \boldsymbol{b}(x) f(u^*)) \cdot \nabla v dx = 0 \\ u(0,.) = u_0 \text{ a.e. on } Q \end{cases}$$

where, for a.e. $(t,x) \in Q$, $u^*(t,x) = \begin{cases} m & \text{if } u(t,x) < m \\ u(t,x) & \text{if } m \leq u(t,x) \leq M \\ M & \text{if } u(t,x) > M \end{cases}$

- Troncation process. Consider the equivalent nonlinear problem
  Find $u \in W(0,T)$ such that

$$\left\{ \begin{array}{l} \langle \partial_t u, v\rangle + \int_\Omega (\phi'(u^*)\nabla u - \boldsymbol{b}(x) f(u^*)) \cdot \nabla v dx = 0 \\ u(0,.) = u_0 \text{ a.e. on } Q \end{array} \right.$$

  where, for a.e. $(t,x) \in Q$, $u^*(t,x) = \left\{ \begin{array}{ll} m & \text{if } u(t,x) < m \\ u(t,x) & \text{if } m \leq u(t,x) \leq M \\ M & \text{if } u(t,x) > M \end{array} \right.$

- The linear problem : $w \in W(0,T)$ being fixed,
  $U_w \in W(0,T)$ is the unique solution of

$$\left\{ \begin{array}{l} \langle \partial_t U_w, v\rangle + \int_\Omega (\phi'(w^*)\nabla U_w - f(w^*)\boldsymbol{b}(x)) \cdot \nabla v dx = 0 \\ U_w(0,.) = u_0 \text{ a.e. on } \Omega \end{array} \right.$$

- Troncation process. Consider the equivalent nonlinear problem
  Find $u \in W(0,T)$ such that

$$\begin{cases} \langle \partial_t u, v\rangle + \int_\Omega (\phi'(u^*)\nabla u - \boldsymbol{b}(x) f(u^*)) \cdot \nabla v dx = 0 \\ u(0,.) = u_0 \text{ a.e. on } Q \end{cases}$$

  where, for a.e. $(t,x) \in Q$, $u^*(t,x) = \begin{cases} m & \text{if } u(t,x) < m \\ u(t,x) & \text{if } m \le u(t,x) \le M \\ M & \text{if } u(t,x) > M \end{cases}$

- The linear problem : $w \in W(0,T)$ being fixed,
  $U_w \in W(0,T)$ is the unique solution of

$$\begin{cases} \langle \partial_t U_w, v\rangle + \int_\Omega (\phi'(w^*)\nabla U_w - f(w^*)\boldsymbol{b}(x)) \cdot \nabla v dx = 0 \\ U_w(0,.) = u_0 \text{ a.e. on } \Omega \end{cases}$$

- We introduce the mapping

$$\begin{array}{cccc} \mathcal{T}: & W(0,T) & \to & W(0,T) \\ & w & \to & U_w \equiv \mathcal{T}(w) \end{array}$$

Parabolic problems : a fixed-point method

A priori estimates

- $\|U_w\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1$
- $\|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2$

Parabolic problems : a fixed-point method

A priori estimates

- $\|U_w\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1$
- $\|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2$

- We set

$$K = \{v \in W(0,T), \|v\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1, \|\partial_t v\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2; \; v(0,.) = v_0 \text{ a.e. on } \Omega\}$$

- K is convex, bounded, closed, $\mathcal{T}(K) \subset K$.

A priori estimates

- $\|U_w\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1$
- $\|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2$

- We set

$$K = \{v \in W(0,T), \|v\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1, \|\partial_t v\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2; \; v(0,.) = v_0 \text{ a.e. on } \Omega\}$$

- K is convex, bounded, closed, $\mathcal{T}(K) \subset K$.
- The "sequential" continuity : $w_n \rightharpoonup w$ in $W(0,T)$.

  We have to show that $\mathcal{T}(w_n) \equiv U_{w_n} \rightharpoonup \mathcal{T}(w)$ in $W(0,T)$.

- $W(0,T) \hookrightarrow\hookrightarrow L^2((0,T)\times\Omega) \Rightarrow w_n \to w$ in $L^2(Q)$ (up to a subsequence)
- $\|U_{w_n}\|_{W(0,T)} \le C \Rightarrow U_{w_n} \rightharpoonup U$ in $W(0,T)$ and $U_{w_n} \to U$ in $L^2(Q)$

- $W(0,T) \hookrightarrow\hookrightarrow L^2((0,T)\times\Omega) \Rightarrow w_n \to w$ in $L^2(Q)$ (up to a subsequence)
- $\|U_{w_n}\|_{W(0,T)} \leq C \Rightarrow U_{w_n} \rightharpoonup U$ in $W(0,T)$ and $U_{w_n} \to U$ in $L^2(Q)$
- We have :

$$\int_0^T \langle \partial_t U_{w_n}, v\rangle dt + \int_0^T \int_\Omega (\phi'(w_n^*)\nabla U_{w_n} - f(w_n^*)\boldsymbol{b}(x)) \cdot \nabla v dx dt = 0$$

- when n goes to $+\infty$:

$$\int_0^T \langle \partial_t U, v\rangle dt + \int_0^T \int_\Omega (\phi'(w^*)\nabla U - \boldsymbol{b}(x) f(w^*))\nabla v dx dt = 0$$

- $W(0,T) \hookrightarrow \mathcal{C}([0,T], L^2(\Omega)) \Rightarrow U(0,.) = u_0(.)$

- $W(0,T) \hookrightarrow\hookrightarrow L^2((0,T)\times\Omega) \Rightarrow w_n \to w$ in $L^2(Q)$ (up to a subsequence)
- $\|U_{w_n}\|_{W(0,T)} \leq C \Rightarrow U_{w_n} \rightharpoonup U$ in $W(0,T)$ and $U_{w_n} \to U$ in $L^2(Q)$
- We have :

$$\int_0^T \langle \partial_t U_{w_n}, v\rangle dt + \int_0^T \int_\Omega (\phi'(w_n^*)\nabla U_{w_n} - f(w_n^*)\boldsymbol{b}(x)) \cdot \nabla v dx dt = 0$$

- when n goes to $+\infty$:

$$\int_0^T \langle \partial_t U, v\rangle dt + \int_0^T \int_\Omega (\phi'(w^*)\nabla U - \boldsymbol{b}(x) f(w^*))\nabla v dx dt = 0$$

- $W(0,T) \hookrightarrow \mathcal{C}([0,T], L^2(\Omega)) \Rightarrow U(0,.) = u_0(.)$

Conclusion

- $U = \mathcal{T}(w)$
- the whole sequence $(\mathcal{T}(w_n))_n$ converges weakly towards $\mathcal{T}(w)$

The "nonlinear" coupled problem

Study of the problem

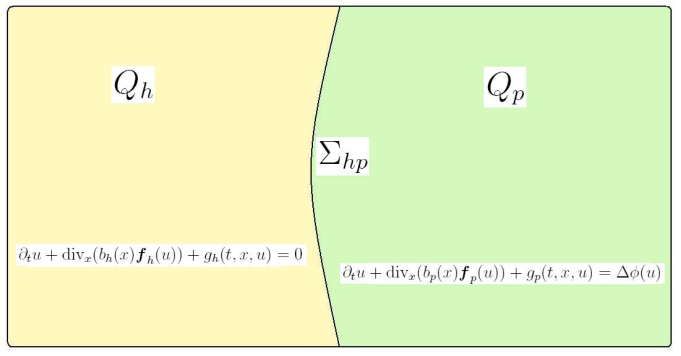
$$\begin{cases} \partial_t u + \mathrm{div}_x(b(x)\boldsymbol{f}(u)) + g(t,x,u) & = & \mathrm{div}_x(\mathbb{I}_{\Omega_p}(x)\nabla\phi(u)) & \text{in } (0,T)\times\Omega \\ u & = & 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,.) & = & u_0 & \text{on } \Omega \end{cases}$$

with

- $\Omega \subset \mathbb{R}^n$
- $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$, $\Omega_p \cap \Omega_h = \emptyset$
- $b(x)\boldsymbol{f}(u) = b_h(x)\boldsymbol{f}_h(u)\mathbb{I}_{\Omega_h} + b_p(x)\boldsymbol{f}_p(u)\mathbb{I}_{\Omega_p}$
- $g(t,x,u) = g_h(t,x,u)\mathbb{I}_{\Omega_h}(x) + g_p(t,x,u)\mathbb{I}_{\Omega_p}(x)$

Introduction

- $\Gamma_{hp} = \partial\Omega_h \cap \partial\Omega_p = \Gamma_h \cap \Gamma_p$
- $\Sigma_{hp} = (0, T) \times \Gamma_{hp}$



Assumptions and Notations

Main assumptions

- $u_0 \in L^\infty(\Omega)$
- Different nonlinearities on Ω_h and Ω_p
- ϕ is nondecreasing, ϕ^{-1} exists, $\phi(0) = 0$

Assumptions and Notations

Main assumptions

- $u_0 \in L^\infty(\Omega)$
- Different nonlinearities on Ω_h and Ω_p
- ϕ is nondecreasing, ϕ^{-1} exists, $\phi(0) = 0$

Notations

- $sgn_\eta :=$ Lipschitzian approximation of the function sgn
- $I_\eta, \boldsymbol{F}_\eta :=$ "regular" entropy pairs

$$I_\eta(a,b) = \int_b^a sgn_\eta(\phi(\tau) - \phi(b))d\tau$$

and

$$\boldsymbol{F}_{l,\eta}(a,b) = \int_{\phi(b)}^{\phi(a)} \boldsymbol{f}_l \circ \phi^{-1}(\tau) sgn'_\eta(\tau - \phi(b))d\tau.$$

$$\boldsymbol{F}_\eta = \boldsymbol{F}_{h,\eta}\mathbb{I}_{\Omega_h} + \boldsymbol{F}_{p,\eta}\mathbb{I}_{\Omega_p}$$

Notion of weak entropy solution

- $V = \{v \in H^1(\Omega_p), v = 0$ a.e. on $\Gamma_p \setminus \Gamma_{hp}\}$

Definition

- $u \in L^\infty(Q), \quad \phi(u) \in L^2(0,T;V)$
- $\forall \varphi \in \mathcal{D}(Q)$ with $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\begin{aligned}
&\int_Q I_\eta(u,k)\partial_t \varphi dxdt - \int_{Q_p} sgn_\eta(\phi(u)-\phi(k))\nabla\phi(u)\cdot\nabla\varphi dxdt \\
&+\int_Q b(x)\{sgn_\eta(\phi(u)-\phi(k))\boldsymbol{f}(u) - \boldsymbol{F}_\eta(u,k)\}\cdot\nabla\varphi dxdt \\
&-\int_Q \{sgn_\eta(\phi(u)-\phi(k))g(t,x,u) + \nabla b(x)\cdot\boldsymbol{F}_\eta(u,k)\}\varphi dxdt \\
&+\int_{\Sigma_{hp}} (b_h(\overline{\sigma})\boldsymbol{F}_{h,\eta}(u,k) - b_p(\overline{\sigma})\boldsymbol{F}_{p,\eta}(u,k))\varphi\cdot\boldsymbol{\nu}_h dtd\mathcal{H}^{n-1} \geq 0
\end{aligned} \tag{1}$$

Notion of weak entropy solution

Initial and boundary conditions

- $$\operatorname*{ess\,lim}_{t \to 0^+} \int_\Omega |u(t,x) - u_0(x)| dx = 0$$
- $\forall \zeta \in L^1(\Sigma_h \backslash \Sigma_{hp})$, $\zeta \geq 0$, $\forall k \in \mathbb{R}$,

$$\operatorname*{ess\,lim}_{s \to 0^-} \int_{\Sigma_{hp}} b(\overline{\sigma}) \mathcal{F}_h(u(\sigma + s\boldsymbol{\nu}_h), k) \cdot \boldsymbol{\nu}_h dt d\mathcal{H}^{n-1} \geq 0$$

where

$$\mathcal{F}_h(\tau, k) = \frac{1}{2}\{sgn(\tau)(\boldsymbol{f}_h(\tau) - \boldsymbol{f}_h(0)) \\ -sgn(k)(\boldsymbol{f}_h(k) - \boldsymbol{f}_h(0)) + sgn(\tau - k)(\boldsymbol{f}_h(\tau) - \boldsymbol{f}_h(k))\}$$

Some remarks

- If u satisfies the entropy inequality (1) then

$$\int_Q |u-k|\partial_t\varphi dxdt - \int_{Q_p} \nabla|\phi(u)-\phi(k)|\cdot\nabla\varphi dxdt$$
$$+\int_Q b(x)\mathbf{\Phi}(u,k)\cdot\nabla\varphi dxdt$$
$$-\int_Q sgn(u-k)(g(t,x,u)+\nabla b(x)\cdot\boldsymbol{f}(k))\varphi dxdt$$
$$+\int_{\Sigma_{hp}} sgn(\phi(u)-\phi(k))(b_h\boldsymbol{f}_h(k)-b_p f_p(k))\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \geq 0$$

where $\mathbf{\Phi}(u,k) = sgn(u-k)(\boldsymbol{f}(u)-\boldsymbol{f}(k))$ is the Kruzhkov flux

Some remarks

- If u is a weak entropy solution then, $\forall \varphi \in \mathcal{D}(Q)$,

$$\int_Q (u\partial_t \varphi + (b(x)\boldsymbol{f}(u) - \mathbb{I}_{\Omega_p} \nabla \phi(u)) \cdot \nabla \varphi - g(t,x,u)\varphi) dxdt = 0$$

Some remarks

- If u is a weak entropy solution then, $\forall \varphi \in \mathcal{D}(Q)$,

$$\int_Q (u\partial_t\varphi + (b(x)\boldsymbol{f}(u) - \mathbb{I}_{\Omega_p}\nabla\phi(u)) \cdot \nabla\varphi - g(t,x,u)\varphi)dxdt = 0$$

So u fulfills

$$\partial_t u + \mathrm{div}_x(b_h(x)\boldsymbol{f}_h(u)) + g_h(t,x,u) = 0 \text{ in } \mathcal{D}'(Q_h),$$

$$\partial_t u + \mathrm{div}_x(b_p(x)\boldsymbol{f}_p(u)) + g_p(t,x,u) = \Delta\phi(u) \text{ in } \mathcal{D}'(Q_p),$$

and the transmission condition (in a formal sense)

$$\big(b_p\boldsymbol{f}_p(u) - b_h\boldsymbol{f}_h(u)\big) \cdot \boldsymbol{\nu}_h = \nabla\phi(u) \cdot \boldsymbol{\nu}_h \text{ on } \Sigma_{hp}$$

The uniqueness property

Main assumption

$$\lambda \longmapsto \xi \cdot b(x)\boldsymbol{f}(\lambda) \text{ is not linear on any nondegenerate interval}$$

This nonlinear condition allows us

- to define "strong" trace on the hyperbolic zone for a weak entropy solution
- to obtain precompactness of sequence of solutions to approximate problems (existence)

The uniqueness property

Main assumption

$$\lambda \longmapsto \xi \cdot b(x)\boldsymbol{f}(\lambda) \text{ is not linear on any nondegenerate interval}$$

This nonlinear condition allows us

- to define "strong" trace on the hyperbolic zone for a weak entropy solution
- to obtain precompactness of sequence of solutions to approximate problems (existence)

Lemma (E. Yu. Panov)

Let u be a weak entropy solution. Then there exists a function $u^\tau \in L^\infty(\Sigma_h)$ such that, for every compact K of Σ_h and every regular Lipschitz deformation Ψ of Ω_h,

$$\operatorname{ess} \lim_{s \to 0^+} \int_K |u(\Psi(s,\sigma)) - u^\tau(\sigma)| dt d\mathcal{H}^{n-1} = 0$$

Uniqueness : hyperbolic zone

- Let u be a weak entropy solution. Then $\forall k \in \mathbb{R}$, $\forall \varphi \in \mathcal{D}((0,T) \times \mathbb{R}^n)$, $\varphi \geq 0$,

$$\begin{aligned} &\int_{Q_h} |u-k|\partial_t \varphi dxdt + \int_{Q_h} b_h(x)\mathbf{\Phi}_h(u,k)\cdot\nabla\varphi dxdt - \int_{Q_h} G_h(u,k)\varphi dxdt \\ \geq \quad & \int_{\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} + \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(0,k)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \\ & - \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\mathbf{\Phi}_h(u^\tau,0)\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \end{aligned}$$

where $G_h(u,k) = sgn(u-k)(g_h(t,x,u) + \nabla b_h(x)\cdot\boldsymbol{f}_h(k))$

Uniqueness : hyperbolic zone

- Let u be a weak entropy solution. Then $\forall k \in \mathbb{R}$, $\forall \varphi \in \mathcal{D}((0,T)\times\mathbb{R}^n)$, $\varphi \geq 0$,

$$\begin{aligned} &\int_{Q_h} |u-k|\partial_t\varphi dxdt + \int_{Q_h} b_h(x)\boldsymbol{\Phi}_h(u,k)\cdot\nabla\varphi dxdt - \int_{Q_h} G_h(u,k)\varphi dxdt \\ \geq\ & \int_{\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(u^\tau,k)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} + \int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(0,k)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} \\ & -\int_{\Sigma_h\setminus\Sigma_{hp}} b_h(\overline{\sigma})\boldsymbol{\Phi}_h(u^\tau,0)\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} \end{aligned}$$

where $G_h(u,k) = sgn(u-k)(g_h(t,x,u) + \nabla b_h(x)\cdot\boldsymbol{f}_h(k))$

- Method of doubling the variables $\Rightarrow$

$$\begin{aligned} -\int_{Q_h} |u-v|\gamma'(t)dxdt \leq & -\int_{Q_h} sgn(u-v)(g_h(t,x,u)-g_h(t,x,v))\gamma(t)dxdt \\ & -\int_{\Sigma_{hp}} sgn(u^\tau - v^\tau)(\boldsymbol{f}_h(u^\tau)-\boldsymbol{f}_h(v^\tau))\cdot\boldsymbol{\nu}_h b_h\gamma(t)d\mathcal{H}^{n-1}dt \end{aligned}$$

Uniqueness : parabolic zone

- Let u be a weak entropy solution. Then $\partial_t u \in L^2(0,T;V')$. Moreover, $\forall \varphi \in L^2(0,T;V)$,

$$\int_0^T \langle\langle \partial_t u, \varphi \rangle\rangle dt + \int_{Q_p} (\nabla\phi(u) - b_p \boldsymbol{f}_p(u)) \cdot \nabla\varphi dxdt + \int_{Q_p} g_p(t,x,u)\varphi dxdt - \int_{\Sigma_{hp}} b_h \boldsymbol{f}_h(u^\tau) \cdot \boldsymbol{\nu}_h \varphi dt d\mathcal{H}^{n-1} = 0$$

Uniqueness : parabolic zone

- Let u be a weak entropy solution. Then $\partial_t u \in L^2(0,T;V')$. Moreover, $\forall \varphi \in L^2(0,T;V)$,

$$\int_0^T \langle\langle \partial_t u, \varphi \rangle\rangle dt + \int_{Q_p} (\nabla\phi(u) - b_p \boldsymbol{f}_p(u)) \cdot \nabla\varphi dxdt$$
$$+ \int_{Q_p} g_p(t,x,u)\varphi dxdt - \int_{\Sigma_{hp}} b_h \boldsymbol{f}_h(u^\tau) \cdot \boldsymbol{\nu}_h \varphi dt d\mathcal{H}^{n-1} = 0$$

- Method of doubling the time variable $\Rightarrow$

$$-\int_{Q_p} |u-v|\gamma'(t)dxdt \leq M_{g_p} \int_{Q_p} |u-v|\gamma(t)dxdt$$
$$+\int_{\Sigma_{hp}} sgn(u^\phi - v^\phi) b_h (\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h \gamma(t) dt d\mathcal{H}^{n-1}$$

where $u^\phi = \phi^{-1}(\phi(u)_{|\Sigma_{hp}})$

Uniqueness : interface

Lemma : interface inequality

Let u be a weak entropy solution. Then a.e. in $(0,T)$, $\mathcal{H}^{n-1}$-a.e. on Γ_{hp}, for any $k \in I(u^\tau, u^\phi)$,

$$sgn(u^\tau - u^\phi) b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(k)) \cdot \boldsymbol{\nu}_h \geq 0$$

Uniqueness : last step

- We have

$$-\int_Q |u-v|\gamma'(t)dxdt \leq M_g \int_Q |u-v|\gamma(t)dxdt$$
$$+\int_{\Sigma_{hp}} b_h sgn(u^\phi - v^\phi)(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau) \cdot \boldsymbol{\nu}_h \gamma(t) d\mathcal{H}^{n-1}dt$$
$$-\int_{\Sigma_{hp}} b_h sgn(u^\tau - v^\tau)(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h \gamma(t) d\mathcal{H}^{n-1}dt$$

- We set

$$J = (sgn(u^\phi - v^\phi) - sgn(u^\tau - v^\tau))b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h$$

Uniqueness : last step

- We have

$$-\int_Q |u-v|\gamma'(t)dxdt \leq M_g \int_Q |u-v|\gamma(t)dxdt$$
$$+\int_{\Sigma_{hp}} b_h sgn(u^\phi - v^\phi)(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau) \cdot \boldsymbol{\nu}_h \gamma(t) d\mathcal{H}^{n-1}dt$$
$$-\int_{\Sigma_{hp}} b_h sgn(u^\tau - v^\tau)(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h \gamma(t) d\mathcal{H}^{n-1}dt$$

- We set

$$J = (sgn(u^\phi - v^\phi) - sgn(u^\tau - v^\tau))b_h(\boldsymbol{f}_h(u^\tau) - \boldsymbol{f}_h(v^\tau)) \cdot \boldsymbol{\nu}_h$$

- Interface inequality $\Rightarrow \ J \leq 0$
- Then

$$\int_\Omega |u(t,.) - v(t,.)|dx \leq e^{M_g t} \int_\Omega |u_0(.) - v_0(.)|dx$$

Existence : the viscous problem

- $\lambda_\mu(x) = \mathbb{I}_{\Omega_p}(x) + \mu \mathbb{I}_{\Omega_h}(x)$
- $\phi_\mu(u_\mu) = \phi(u_\mu) + \mu u_\mu$

Find a bounded and measurable function u_μ such that

$$\left\{ \begin{array}{rcll} \partial_t u_\mu + \mathrm{div}_x(b(x)\boldsymbol{f}(u_\mu)) + g(t,x,u_\mu) & = & \mathrm{div}_x(\lambda_\mu \nabla_x \phi_\mu(u_\mu)) & \text{in } Q \\ u_\mu & = & 0 & \text{on } \Sigma \\ u_\mu(0,.) & = & u_0 & \text{on } \Omega \end{array} \right.$$

Existence : main assumption

- We introduce a nondecreasing function M_1 such that

$$\left\{ \begin{array}{l} M_1(0) \geq M, \\ \forall t \in (0,T) \\ M_1'(t) + \nabla b(.) \cdot \boldsymbol{f}(M_1(t)) + g(t,.,M_1(t)) \geq 0 \text{ a.e. on } \Omega_L \cup \Omega_R \end{array} \right.$$

and a nonincreasing function M_2 such that

$$\left\{ \begin{array}{l} M_2(0) \leq m, \\ \forall t \in (0,T) \\ M_2'(t) + \nabla b(.) \cdot \boldsymbol{f}(M_2(t)) + g(t,.,M_2(t)) \leq 0 \text{ a.e. on } \Omega_L \cup \Omega_R \end{array} \right.$$

Existence : main assumption

- We introduce a nondecreasing function M_1 such that

$$\begin{cases} M_1(0) \geq M, \\ \forall t \in (0,T) \\ M_1'(t) + \nabla b(.) \cdot \boldsymbol{f}(M_1(t)) + g(t,.,M_1(t)) \geq 0 \text{ a.e. on } \Omega_L \cup \Omega_R \end{cases}$$

and a nonincreasing function M_2 such that

$$\begin{cases} M_2(0) \leq m, \\ \forall t \in (0,T) \\ M_2'(t) + \nabla b(.) \cdot \boldsymbol{f}(M_2(t)) + g(t,.,M_2(t)) \leq 0 \text{ a.e. on } \Omega_L \cup \Omega_R \end{cases}$$

Assumption (H)

For almost all $t \in (0,T)$, a.e. on Γ_{hp},

$$(b_p \boldsymbol{f}_p(M_1(t)) - b_h \boldsymbol{f}_h(M_1(t))) \cdot \boldsymbol{\nu}_h \geq 0$$
$$(b_p \boldsymbol{f}_p(M_2(t)) - b_h \boldsymbol{f}_h(M_2(t))) \cdot \boldsymbol{\nu}_h \leq 0$$

The viscous problem

- $W(0,T) = \{v \in L^2(0,T;H_0^1(\Omega)), \partial_t v \in L^2(0,T;H^{-1}(\Omega))\}$

Existence and uniqueness

Under (H), $\exists!\ u_\mu \in W(0,T) \cap L^\infty(Q)$ such that

- $\forall t \in [0,T], M_2(t) \leq u_\mu(t,.) \leq M_1(t)$ a.e. in Ω
- $u_\mu(0,.) = u_0$ a.e. in Ω
- For any $v \in H_0^1(\Omega)$, for almost all $t \in (0,T)$,

$$\langle \partial_t u_\mu, v \rangle + \int_\Omega ((\lambda_\mu(x)\nabla\phi_\mu(u_\mu) - b(x)\boldsymbol{f}(u_\mu)) \cdot \nabla v + g(t,x,u_\mu)v)dx = 0$$

The viscous problem

- $W(0,T) = \{v \in L^2(0,T;H_0^1(\Omega)), \partial_t v \in L^2(0,T;H^{-1}(\Omega))\}$

Existence and uniqueness

Under (H), $\exists! \; u_\mu \in W(0,T) \cap L^\infty(Q)$ such that

- $\forall t \in [0,T], M_2(t) \leq u_\mu(t,.) \leq M_1(t)$ a.e. in Ω
- $u_\mu(0,.) = u_0$ a.e. in Ω
- For any $v \in H_0^1(\Omega)$, for almost all $t \in (0,T)$,

$$\langle \partial_t u_\mu, v \rangle + \int_\Omega ((\lambda_\mu(x) \nabla \phi_\mu(u_\mu) - b(x)\boldsymbol{f}(u_\mu)) \cdot \nabla v + g(t,x,u_\mu)v)dx = 0$$

(Sketch of) Proof

- Assumption $(H) \Rightarrow L^\infty$-estimate
- Schauder-Tychonoff fixed-point Theorem $\Rightarrow$ Existence
- Holmgren-type duality method $\Rightarrow$ Uniqueness

Maximum principle

- $\mathcal{B}(a, b, c) = \max\{a, \min\{b, c\}\}$
- $u_\mu^\star = \mathcal{B}(M_2(t), u_\mu, M_1(t))$
- Study of the problem

$$\begin{cases} \text{Find } u_\mu \text{ in } W(0,T) \text{ such that a.e. on } (0,T) \text{ and for all } v \in H_0^1(\Omega), \\ \langle \partial_t u_\mu, v \rangle + \displaystyle\int_\Omega ((\lambda_\mu(x)\phi'_\mu(u_\mu^\star)\nabla u_\mu - b(x)\boldsymbol{f}(u_\mu^\star)) \cdot \nabla v + g(t,x,u_\mu^\star)v)dx = 0 \\ u_\mu(0,.) = u_0 \text{ a.e. on } \Omega \end{cases}$$

Maximum principle

- $\mathcal{B}(a,b,c) = \max\{a, \min\{b,c\}\}$
- $u_\mu^\star = \mathcal{B}(M_2(t), u_\mu, M_1(t))$
- Study of the problem

$$\begin{cases} \text{Find } u_\mu \text{ in } W(0,T) \text{ such that a.e. on } (0,T) \text{ and for all } v \in H_0^1(\Omega), \\ \langle \partial_t u_\mu, v \rangle + \int_\Omega ((\lambda_\mu(x)\phi'_\mu(u_\mu^\star)\nabla u_\mu - b(x)\boldsymbol{f}(u_\mu^\star)) \cdot \nabla v + g(t,x,u_\mu^\star)v)dx = 0 \\ u_\mu(0,.) = u_0 \text{ a.e. on } \Omega \end{cases}$$

- Test function $v_\eta = sgn_\eta(u_\mu - M_1(t))^+$
- For the convective term

$$-\int_{Q_s} b(x)\boldsymbol{f}(u_\mu^\star)) \cdot \nabla v_\eta dxdt = \sum_{i\in\{h,p\}} \int_{Q_{i,s}} \boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i(x) v_\eta dxdt$$
$$+ \int_{\Sigma_{hp}} (b_p \boldsymbol{f}_p(M_1(t)) - b_h \boldsymbol{f}_h(M_1(t))) \cdot \boldsymbol{\nu}_h v_\eta dt d\mathcal{H}^{n-1}$$

Maximum principle

- When η goes to 0

$$\int_\Omega (u_\mu(s,x) - M_1(s))^+ dx + \int_{Q_s} M_1'(t) sgn(u_\mu - M_1(t))^+ dxdt$$
$$+ \sum_{i\in\{h,p\}} \int_{Q_{i,s}} (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) sgn(u_\mu - M_1(t))^+ dxdt \leq 0$$

- By definition (of M_1),

$$M_1'(t) + (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) \geq 0$$

Maximum principle

- When η goes to 0

$$\int_\Omega (u_\mu(s,x) - M_1(s))^+ dx + \int_{Q_s} M_1'(t) sgn(u_\mu - M_1(t))^+ dxdt$$
$$+ \sum_{i\in\{h,p\}} \int_{Q_{i,s}} (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) sgn(u_\mu - M_1(t))^+ dxdt \leq 0$$

- By definition (of M_1),

$$M_1'(t) + (\boldsymbol{f}_i(M_1(t)) \cdot \nabla b_i + g_i(t,x,M_1(t))) \geq 0$$

- So

$$\int_\Omega (u_\mu(s,x) - M_1(s))^+ dx \leq 0$$

The viscous limit

A priori estimates

There exists a constant C independent of μ such that

$$\|(\lambda_\mu)^{1/2}\nabla\widehat{\phi}(u_\mu)\|^2_{L^2(Q)^n} + \|(\mu\lambda_\mu)^{1/2}\nabla u_\mu\|^2_{L^2(Q)^n} \leq C$$

$$\|\partial_t u_\mu\|_{L^2(0,T;H^{-1}(\Omega))} \leq C$$

where $\widehat{\phi}(x) = \int_0^x \sqrt{\phi'(\tau)}d\tau$

The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

$$\lambda \mapsto b_p(x)\boldsymbol{f}_p(\lambda) \cdot \xi \text{ and } \lambda \mapsto \phi(\lambda)\xi^2$$

are not linear simultaneously on any non-degenerate intervals

The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

$$\lambda \mapsto b_p(x)\boldsymbol{f}_p(\lambda) \cdot \xi \text{ and } \lambda \mapsto \phi(\lambda)\xi^2$$

are not linear simultaneously on any non-degenerate intervals

Consequences (E. Yu. Panov)

- The sequence $(u_\mu)_{\mu>0}$ is precompact in $L^1(Q_p)$

The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

$$\lambda \mapsto b_p(x)\boldsymbol{f}_p(\lambda) \cdot \xi \text{ and } \lambda \mapsto \phi(\lambda)\xi^2$$

are not linear simultaneously on any non-degenerate intervals

Consequences (E. Yu. Panov)

- The sequence $(u_\mu)_{\mu>0}$ is precompact in $L^1(Q_p)$
- Nonlinear flux on the hyperbolic zone $\Rightarrow$ $(u_\mu)_{\mu>0}$ is precompact in $L^1(Q_h)$
- $u_\mu \to u \in L^\infty(Q)$ in $L^1(Q)$

Study of the convective term

- Test function : $v_\mu^\eta = sgn_\eta(\phi(u_\mu) - \phi(k))\varphi_1\varphi_2$, $\varphi_1 \in \mathcal{C}_c^\infty([0,T))$, $\varphi_2 \in \mathcal{C}_c^\infty(\Omega)$.

$$\begin{aligned}
&\int_Q -b(x)\boldsymbol{f}(u_\mu)\cdot\nabla(sgn_\eta(\phi(u_\mu)-\phi(k))\varphi_1\varphi_2)dxdt \\
= \quad & -\sum_{i\in\{h,p\}}\int_{Q_i} b_i(x)\boldsymbol{f}_i(u_\mu)\cdot\nabla\phi(u_\mu)sgn'_\eta(\phi(u_\mu)-\phi(k))\varphi_1\varphi_2 dxdt \\
& -\sum_{i\in\{h,p\}}\int_{Q_i} b_i(x)sgn_\eta(\phi(u_\mu)-\phi(k))\varphi_1\boldsymbol{f}_i(u_\mu)\cdot\nabla\varphi_2 dxdt
\end{aligned}$$

Study of the convective term

- Test function : $v_\mu^\eta = sgn_\eta(\phi(u_\mu) - \phi(k))\varphi_1\varphi_2$, $\varphi_1 \in \mathcal{C}_c^\infty([0,T))$, $\varphi_2 \in \mathcal{C}_c^\infty(\Omega)$.

$$\begin{aligned}
&\int_Q -b(x)\boldsymbol{f}(u_\mu)\cdot\nabla(sgn_\eta(\phi(u_\mu)-\phi(k))\varphi_1\varphi_2)dxdt \\
= \quad & -\sum_{i\in\{h,p\}}\int_{Q_i} b_i(x)\boldsymbol{f}_i(u_\mu)\cdot\nabla\phi(u_\mu)sgn'_\eta(\phi(u_\mu)-\phi(k))\varphi_1\varphi_2dxdt \\
& -\sum_{i\in\{h,p\}}\int_{Q_i} b_i(x)sgn_\eta(\phi(u_\mu)-\phi(k))\varphi_1\boldsymbol{f}_i(u_\mu)\cdot\nabla\varphi_2dxdt
\end{aligned}$$

- $J_{\mu,\eta} = -\int_{Q_h} b_h(x)\boldsymbol{f}_h(u_\mu)\cdot\nabla\phi(u_\mu)sgn'_\eta(\phi(u_\mu)-\phi(k))\varphi_1\varphi_2dxdt$
- $J_{\mu,\eta} = -\int_{Q_h} b(x)\mathrm{div}\boldsymbol{F}_{h,\eta}(u_\mu,k)\varphi_1\varphi_2dxdt$

(with $\boldsymbol{F}_{h,\eta}(u_\mu,k) = \int_{\phi(k)}^{\phi(u_\mu)} \boldsymbol{f}_h\circ\phi^{-1}(\tau)sgn'_\eta(\tau-\phi(k))d\tau$)

Study of the convective term

$$J_{\mu,\eta} = \int_{Q_h} \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi_2 b_h) \varphi_1 dx dt$$
$$- \int_{\Sigma_{hp}} b_h \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot \boldsymbol{\nu}_h \varphi_1 \varphi_2 d\mathcal{H}^{n-1} dt$$

Study of the convective term

$$J_{\mu,\eta} = \int_{Q_h} \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi_2 b_h) \varphi_1 dxdt$$
$$- \int_{\Sigma_{hp}} b_h \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot \boldsymbol{\nu}_h \varphi_1 \varphi_2 d\mathcal{H}^{n-1} dt$$

- $\phi(u_\mu) \in L^2(0,T; H^1(\Omega)) \ \Rightarrow \ (\phi(u_\mu)_{|\Omega_h})_{|\Gamma_{hp}} = (\phi(u_\mu)_{|\Omega_p})_{|\Gamma_{hp}}$
- $(\boldsymbol{F}_{h,\eta}(u_\mu, k))_{\mu>0}$ converges strongly towards $\boldsymbol{F}_{h,\eta}(u, k)$ in $L^q(Q_p)^n$, $1 \leq q < \infty$

Study of the convective term

$$
\begin{aligned}
J_{\mu,\eta} = \; & \int_{Q_h} \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi_2 b_h)\varphi_1 dx dt \\
& - \int_{\Sigma_{hp}} b_h \boldsymbol{F}_{h,\eta}(u_\mu, k) \cdot \boldsymbol{\nu}_h \varphi_1 \varphi_2 d\mathcal{H}^{n-1} dt
\end{aligned}
$$

- $\phi(u_\mu) \in L^2(0,T;H^1(\Omega)) \;\Rightarrow\; (\phi(u_\mu)_{|\Omega_h})_{|\Gamma_{hp}} = (\phi(u_\mu)_{|\Omega_p})_{|\Gamma_{hp}}$
- $(\boldsymbol{F}_{h,\eta}(u_\mu, k))_{\mu>0}$ converges strongly towards $\boldsymbol{F}_{h,\eta}(u,k)$ in $L^q(Q_p)^n$, $1 \leq q < \infty$
- $(\boldsymbol{F}_{h,\eta}(u_\mu, k))_{\mu>0}$ is uniformly bounded in $L^2(0,T;V)^n \cap L^\infty(Q)^n$
- $(\boldsymbol{F}_{h,\eta}(u_\mu, k)\varphi_2)_{\mu>0}$ converges weakly, up to a subsequence, towards $\boldsymbol{F}_{h,\eta}(u,k)\varphi_2$ in $L^2(0,T;V)^n$
- $(\boldsymbol{F}_{h,\eta}(u_\mu, k)\varphi_2)_{\mu>0}$ converges weakly towards $\boldsymbol{F}_{h,\eta}(u,k)\varphi_2$ in $L^2(\Sigma_p)^n$

The non "nonlinear" coupling problem

Study of the problem

$$\begin{cases} \partial_t u + \mathrm{div}_x(\boldsymbol{b}(x)f(u)) + g(t,x,u) & = & \mathrm{div}_x(\mathbb{I}_{\Omega_p}(x)\nabla\phi(u)) & \text{in } Q, \\ u & = & 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,.) & = & u_0 & \text{on } \Omega, \end{cases}$$

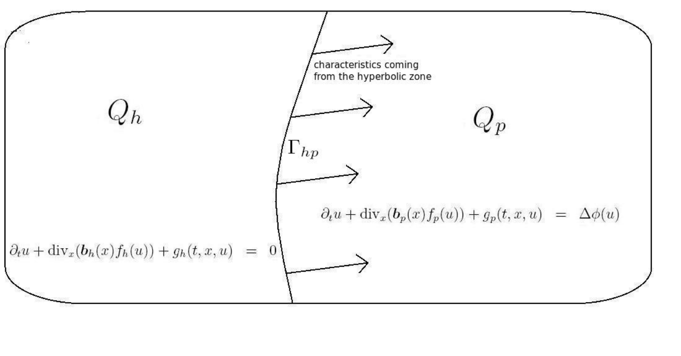
- $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$, $\Omega_p \cap \Omega_h = \emptyset$
- $\boldsymbol{b}(x)f(u) = \boldsymbol{b}_h(x)f_h(u)\mathbb{I}_{\Omega_h} + \boldsymbol{b}_p(x)f_p(u)\mathbb{I}_{\Omega_p}$
- $g(t,x,u) = g_h(t,x,u)\mathbb{I}_{\Omega_h}(x) + g_p(t,x,u)\mathbb{I}_{\Omega_p}(x)$

Main difference

The flux $\boldsymbol{b}(x)f(u)$ does not satisfy the nonlinear condition

The non "nonlinear" coupling problem

- $\Gamma_{hp} = \partial\Omega_h \cap \partial\Omega_p = \Gamma_h \cap \Gamma_p$
- $\Sigma_{hp} =]0, T[\times \Gamma_{hp}$



Assumptions

Main assumptions

- $u_0 \in L^\infty(\Omega)$
- Different nonlinearities on Ω_h and Ω_p
- $\boldsymbol{b}_i \in W^{1,\infty}(\Omega_i)^n$, $i = h, p$
- $\Gamma_{hp} \subset \{\bar{\sigma} \in \Gamma_h, \boldsymbol{b}_h(\overline{\sigma}) \cdot \boldsymbol{\nu}_h \geq 0\}$
- f_h is nondecreasing
- ϕ is nondecreasing, ϕ^{-1} exists, $\phi(0) = 0$

Notion of weak entropy solution

- $V = \{v \in H^1(\Omega_p), v = 0$ a.e. on $\Gamma_p \setminus \Gamma_{hp}\}$

Definition

- $u \in L^\infty(Q), \quad \phi(u) \in L^2(0,T;V)$.
- $\forall \varphi \in \mathcal{C}_c^\infty(Q)$, $\forall \varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\int_Q (|u-k|\partial_t \varphi + \boldsymbol{b}(x)\Phi(u,k)\cdot\nabla\varphi)dxdt - \int_{Q_p} \nabla|\phi(u)-\phi(k)|\cdot\nabla\varphi dxdt$$
$$-\int_Q sgn(u-k)(g(t,x,u) + \mathrm{div}\boldsymbol{b}(x)f(k))\varphi dxdt$$
$$+\int_{\Sigma_{hp}} \{\boldsymbol{b}_h f_h(k) - \boldsymbol{b}_p f_p(k)\}\cdot\boldsymbol{\nu}_h sgn(\phi(u)-\phi(k))\varphi dtd\mathcal{H}^{n-1} \geq 0.$$

$\Phi(u,k) = sgn(u-k)(f(u)-f(k))$

Notion of weak entropy solution

Initial and boundary conditions

- $$\operatorname{ess}\lim_{t\to 0^+}\int_\Omega |u(t,x)-u_0(x)| = 0,$$
- $\forall \varphi \in L^1(\Sigma_h \setminus \Sigma_{hp})$, $\varphi \geq 0$, $\forall k \in \mathbb{R}$,

$$\operatorname{ess}\lim_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} \mathcal{F}_h(u(\sigma+\tau\boldsymbol{\nu}_h),k)\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi dt d\mathcal{H}^{n-1} \geq 0,$$

where

$$\mathcal{F}_h(\tau,k) = \frac{1}{2}(|f_h(\tau)-f_h(0)| - |f_h(k)-f_h(0)| + |f_h(\tau)-f_h(k)|).$$

Uniqueness

Study in the hyperbolic zone

Let u be a weak entropy solution. $\forall\, k \in \mathbb{R}$, $\forall \varphi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,

$$\begin{aligned}
&\int_{Q_h} (|u-k|\partial_t\varphi + |f_h(u)-f_h(k)|\boldsymbol{b}_h\cdot\nabla\varphi - G_h(u,k)\varphi)dxdt \\
\geq\ & \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f_h(u(\sigma+\tau\boldsymbol{\nu}_h)) - f_h(k)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi(\sigma)dtd\mathcal{H}^{n-1} \\
& - \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} |f_h(u(\sigma+\tau\boldsymbol{\nu}_h)) - f_h(0)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \\
& + \int_{\Sigma_h\setminus\Sigma_{hp}} |f_h(k)-f_h(0)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1}
\end{aligned}$$

$$G_h(u,k) = sgn(u-k)(g(t,x,u) + \mathrm{div}\boldsymbol{b}_h(x)f_h(k))$$

Uniqueness

Study in the hyperbolic zone

Let u be a weak entropy solution. $\forall\, k \in \mathbb{R}$, $\forall \varphi \in \mathcal{C}_c^\infty(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,

$$\begin{aligned}
&\int_{Q_h} (|u-k|\partial_t\varphi + |f_h(u)-f_h(k)|\boldsymbol{b}_h\cdot\nabla\varphi - G_h(u,k)\varphi)dxdt \\
\geq\ & \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f_h(u(\sigma+\tau\boldsymbol{\nu}_h)) - f_h(k)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi(\sigma)dtd\mathcal{H}^{n-1} \\
&- \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\setminus\Sigma_{hp}} |f_h(u(\sigma+\tau\boldsymbol{\nu}_h)) - f_h(0)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1} \\
&+ \int_{\Sigma_h\setminus\Sigma_{hp}} |f_h(k)-f_h(0)|\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h\varphi dtd\mathcal{H}^{n-1}
\end{aligned}$$

$G_h(u,k) = sgn(u-k)(g(t,x,u) + \mathrm{div}\boldsymbol{b}_h(x)f_h(k))$

- Method of doubling variables $\Rightarrow$ Uniqueness (on the hyperbolic area)

Study in the parabolic zone

Let u be a weak entropy solution. Then $\partial_t u \in L^2(0,T;V')$, and for any $v \in L^2(0,T;V)$,

$$\int_0^T \langle\langle \partial_t u, v \rangle\rangle dt + \int_{Q_p} (\nabla \phi(u) - f_p(u)\boldsymbol{b}_p) \cdot \nabla v dx dt + \int_{Q_p} g_p(t,x,u) v dx dt$$
$$- \text{ess} \lim_{\tau \to 0^-} \int_{\Sigma_{hp}} f_h(u(\sigma + \tau \boldsymbol{\nu}_h)) \boldsymbol{b}_h \cdot \boldsymbol{\nu}_h v dt d\mathcal{H}^{n-1} = 0$$

Uniqueness

Study in the parabolic zone

Let u be a weak entropy solution. Then $\partial_t u \in L^2(0,T;V')$, and for any $v \in L^2(0,T;V)$,

$$\int_0^T \langle\langle \partial_t u, v\rangle\rangle dt + \int_{Q_p} (\nabla\phi(u) - f_p(u)\boldsymbol{b}_p)\cdot\nabla v dx dt + \int_{Q_p} g_p(t,x,u) v dx dt$$
$$-\text{ess}\lim_{\tau\to 0^-}\int_{\Sigma_{hp}} f_h(u(\sigma+\tau\boldsymbol{\nu}_h))\boldsymbol{b}_h\cdot\boldsymbol{\nu}_h v dt d\mathcal{H}^{n-1} = 0$$

- Method of doubling the time variable $\Rightarrow$ Uniqueness (on the parabolic zone)

Existence : the viscous problem

Find a bounded and measurable function u_μ such that

$$\left\{ \begin{array}{rcll} \partial_t u_\mu + \mathrm{div}(\boldsymbol{b}(x) f(u_\mu)) + g(t,x,u_\mu) & = & \mathrm{div}(\lambda_\mu \nabla \phi_\mu(u_\mu)) & \text{in } Q, \\ u_\mu & = & 0 & \text{on } \Sigma, \\ u_\mu(0,.) & = & u_0 & \text{on } \Omega. \end{array} \right.$$

Existence : the viscous problem

Find a bounded and measurable function u_μ such that

$$\begin{cases} \partial_t u_\mu + \mathrm{div}(\boldsymbol{b}(x) f(u_\mu)) + g(t,x,u_\mu) = \mathrm{div}(\lambda_\mu \nabla \phi_\mu(u_\mu)) & \text{in } Q, \\ u_\mu = 0 & \text{on } \Sigma, \\ u_\mu(0,.) = u_0 & \text{on } \Omega. \end{cases}$$

Assumption (H)

$$\sum_{l\in\{h,p\}} (g_i(.,.,m) + f_i(m)\mathrm{div}\boldsymbol{b}_i) \leq 0 \quad , \quad \sum_{l\in\{h,p\}} (g_i(.,.,M) + f_i(M)\mathrm{div}\boldsymbol{b}_i) \geq 0,$$

and a.e. on Γ_{hp},

$$(f_p(M)\boldsymbol{b}_p - f_h(M)\boldsymbol{b}_h)\cdot\boldsymbol{\nu}_h \geq 0,$$
$$(f_p(m)\boldsymbol{b}_p - f_h(m)\boldsymbol{b}_h)\cdot\boldsymbol{\nu}_h \leq 0.$$

The viscous problem

Existence and uniqueness

Under (H), $\exists!\ u_\mu \in W(0,T) \cap L^\infty(Q)$ such that

- $\forall t \in [0,T], m \le u_\mu(t,.) \le M$ a.e. in Ω
- $u_\mu(0,.) = u_0$ a.e. in Ω
- For any $v \in H_0^1(\Omega)$,

$$\langle \partial_t u_\mu, v\rangle + \int_\Omega ((\lambda_\mu(x)\nabla\phi_\mu(u_\mu) - \boldsymbol{b}(x)f(u_\mu)) \cdot \nabla v + g(t,x,u_\mu)v)dx = 0$$

The viscous limit

A priori estimates

There exists a constant C independent on μ such that

$$\|(\lambda_\mu)^{1/2}\nabla\widehat{\phi}(u_\mu)\|^2_{L^2(Q)^n} + \|(\mu\lambda_\mu)^{1/2}\nabla u_\mu\|^2_{L^2(Q)^n} \leq C,$$

$$\|\partial_t u_\mu\|_{L^2(0,T;H^{-1}(\Omega))} \leq C,$$

where $\widehat{\phi}(x) = \displaystyle\int_0^x \sqrt{\phi'(\tau)}d\tau$

The viscous limit

- ϕ^{-1} is Hölder continuous with an exponent $\tau \in (0,1)$

Proposition

There exists a function u in $L^\infty(Q)$ with $\phi(u)$ in $L^2(0,T;V)$ and such that up to a subsequence when μ goes to 0^+,

$$u_\mu \rightharpoonup u \text{ in } L^\infty(Q) \text{ weak} - \star$$
$$u_\mu \to u \text{ in } L^q(Q_p) \text{ strongly for any finite } q \text{ and a.e. on } Q_p$$

Besides we also have

$$\nabla\phi(u_\mu) \rightharpoonup \nabla\phi(u) \text{ weakly in } L^2(Q_p)^n$$
$$\mu\nabla\phi(u_\mu) \to 0,\ \lambda_\mu\mu\nabla u_\mu \to 0 \text{ strongly in } L^2(Q_h)^n$$

The viscous limit : notion of process

Theorem (Eymard, Gallouet, Herbin)

Let $(u_m)_{m>0}$ be a sequence of measurable functions on $\mathcal{O}$ with

$$\exists\ M > 0, \forall m > 0, \|u_m\|_{L^\infty(\mathcal{O})} \leq M$$

Then $\exists\ (u_{\varphi(m)})_{m>0}$ and $\exists\ \pi \in L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous and bounded functions h on $\mathcal{O} \times (-M, M)$,

$$\forall \xi \in L^1(\mathcal{O}),\ \lim_{m\to+\infty} \int_{\mathcal{O}} h(x, u_{\varphi(m)})\xi dx = \int_{]0,1[\times\mathcal{O}} h(x, \pi(\alpha, x)) d\alpha \xi dx$$

The viscous limit : notion of process

Theorem (Eymard, Gallouet, Herbin)

Let $(u_m)_{m>0}$ be a sequence of measurable functions on $\mathcal{O}$ with

$$\exists\ M > 0, \forall m > 0, \|u_m\|_{L^\infty(\mathcal{O})} \leq M$$

Then $\exists\ (u_{\varphi(m)})_{m>0}$ and $\exists\ \pi \in L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous and bounded functions h on $\mathcal{O} \times (-M, M)$,

$$\forall \xi \in L^1(\mathcal{O}),\ \lim_{m\to+\infty} \int_{\mathcal{O}} h(x, u_{\varphi(m)})\xi dx = \int_{]0,1[\times\mathcal{O}} h(x, \pi(\alpha, x)) d\alpha \xi dx$$

Consequence

$(u_{\mu|\Omega_h})_{\mu>0}$ " $\rightarrow$ " $\pi \in L^\infty((0,1) \times Q_h)$

The viscous limit on the hyperbolic zone

The process π fulfills

- $\forall \varphi \in \mathcal{C}_c^\infty(Q_h)$, $\varphi \geq 0$,

$$\int_0^1 \int_{Q_h} (|\pi - k|\partial_t \varphi + \boldsymbol{b}(x)\Phi(\pi, k) \cdot \nabla\varphi) d\alpha dx dt$$
$$- \int_0^1 \int_{Q_h} sgn(\pi - k)(g(t, x, \pi) + \mathrm{div}\boldsymbol{b}(x) f(k))\varphi d\alpha dx dt \geq 0$$

- $\operatorname{ess}\lim_{\tau \to 0^-} \int_0^1 \int_{\Sigma_h \setminus \Sigma_{hp}} \mathcal{F}_h(\pi(\alpha, \sigma + \tau\boldsymbol{\nu}_h), k)\boldsymbol{b}_h \cdot \boldsymbol{\nu}_h \varphi d\alpha d\sigma \geq 0$
- $\operatorname{ess}\lim_{t \to 0^+} \int_0^1 \int_\Omega |\pi(\alpha, t, x) - u_0(x)| d\alpha dx = 0$

The viscous limit on the hyperbolic zone

The process π fulfills

- $\forall \varphi \in \mathcal{C}_c^\infty(Q_h)$, $\varphi \geq 0$,

$$\int_0^1 \int_{Q_h} (|\pi - k|\partial_t \varphi + \boldsymbol{b}(x)\Phi(\pi, k) \cdot \nabla \varphi) d\alpha dx dt$$
$$- \int_0^1 \int_{Q_h} sgn(\pi - k)(g(t, x, \pi) + \mathrm{div}\boldsymbol{b}(x) f(k))\varphi d\alpha dx dt \geq 0$$

- $\displaystyle \text{ess}\lim_{\tau \to 0^-} \int_0^1 \int_{\Sigma_h \setminus \Sigma_{hp}} \mathcal{F}_h(\pi(\alpha, \sigma + \tau \boldsymbol{\nu}_h), k)\boldsymbol{b}_h \cdot \boldsymbol{\nu}_h \varphi d\alpha d\sigma \geq 0$
- $\displaystyle \text{ess}\lim_{t \to 0^+} \int_0^1 \int_\Omega |\pi(\alpha, t, x) - u_0(x)| d\alpha dx = 0$

Consequence

For a.e. α in $(0, 1)$,

$$u(.,.) = \pi(\alpha, ., .) \text{ a.e. on } Q_h$$

(with $u_\mu \rightharpoonup u$ in $L^\infty(Q)$ weak $- \star$)

About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

$$\forall\ \delta > 0, \quad H_\delta(\tau,k) = \left((\mathrm{dist}(\tau, I[0,k]))^2 + \delta^2\right)^{\frac{1}{2}} - \delta$$
$$Q_{h,\delta}(\tau,k) = \int_k^\tau \partial_1 H_\delta(\lambda,k) f_h'(\lambda) d\lambda$$

About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

$$\forall\ \delta > 0, \quad H_\delta(\tau, k) = \left((\mathrm{dist}(\tau, I[0,k]))^2 + \delta^2\right)^{\frac{1}{2}} - \delta$$
$$Q_{h,\delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k) f_h'(\lambda) d\lambda$$

- We choose $v = \partial_1 H_\delta(u_\mu, k)\varphi$ in the variational equality fulfilled by u_μ

$$\int_{(0,1)\times Q_h} (H_\delta(\pi, k)\partial_t \varphi + Q_{h,\delta}(\pi, k)\boldsymbol{b}_h \cdot \nabla\varphi d\alpha dx dt \geq 0$$

About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

$$\forall\ \delta > 0, \quad H_\delta(\tau, k) = \left((\text{dist}(\tau, I[0,k]))^2 + \delta^2\right)^{\frac{1}{2}} - \delta$$
$$Q_{h,\delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k) f_h'(\lambda) d\lambda$$

- We choose $v = \partial_1 H_\delta(u_\mu, k)\varphi$ in the variational equality fulfilled by u_μ

$$\int_{(0,1)\times Q_h} (H_\delta(\pi, k)\partial_t \varphi + Q_{h,\delta}(\pi, k)\boldsymbol{b}_h \cdot \nabla\varphi d\alpha dx dt \geq 0$$

- So

$$\text{ess}\lim_{\tau \to 0^-} \int_{(0,1)\times \Sigma_h \setminus \Sigma_{hp}} Q_{h,\delta}(\pi(\alpha, \sigma + \tau\boldsymbol{\nu}), k)\boldsymbol{b}_h(\overline{\sigma}) \cdot \boldsymbol{\nu}_h \varphi d\alpha dt d\mathcal{H}^{n-1} \geq 0$$

About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

$$\forall\ \delta > 0, \quad H_\delta(\tau,k) = \left((\mathrm{dist}(\tau, I[0,k]))^2 + \delta^2\right)^{\frac{1}{2}} - \delta$$
$$Q_{h,\delta}(\tau,k) = \int_k^\tau \partial_1 H_\delta(\lambda,k) f_h'(\lambda) d\lambda$$

- We choose $v = \partial_1 H_\delta(u_\mu, k)\varphi$ in the variational equality fulfilled by u_μ

$$\int_{(0,1)\times Q_h} (H_\delta(\pi,k)\partial_t\varphi + Q_{h,\delta}(\pi,k)\boldsymbol{b}_h \cdot \nabla\varphi d\alpha dx dt \geq 0$$

- So

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{(0,1)\times\Sigma_h\setminus\Sigma_{hp}} Q_{h,\delta}(\pi(\alpha,\sigma+\tau\boldsymbol{\nu}),k)\boldsymbol{b}_h(\overline{\sigma}) \cdot \boldsymbol{\nu}_h \varphi d\alpha dt d\mathcal{H}^{n-1} \geq 0$$

- When δ goes to 0^+

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_0^1 \int_{\Sigma_h\setminus\Sigma_{hp}} \mathcal{F}_h(\pi(\alpha,\sigma+\tau\boldsymbol{\nu}_h),k)\boldsymbol{b}_h \cdot \boldsymbol{\nu}_h \varphi d\alpha d\sigma \geq 0$$