A mathematical analysis of some hyperbolic - parabolic problems

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Aim of the talk

Mathematical analysis of the "model" problem

\[ \partial_t u + \text{div} \ F_h(x, u) = 0 \quad \text{in} \ (0, T) \times \Omega_h \]

\[ \partial_t u + \text{div} \ F_p(x, u) = \Delta \phi(u) \quad \text{in} \ (0, T) \times \Omega_p \]

with

\[ \Omega_p \cap \Omega_h = \Gamma_{hp} (\neq \emptyset) \]
Introduction

Aim of the talk

Mathematical analysis of the "model" problem

\[ \partial_t u + \text{div} \mathbf{F}_h(x, u) = 0 \quad \text{in} \ (0, T) \times \Omega_h \]

\[ \partial_t u + \text{div} \mathbf{F}_p(x, u) = \Delta \phi(u) \quad \text{in} \ (0, T) \times \Omega_p \]

with

\[ \Omega_p \cap \Omega_h = \Gamma_{hp} (\neq \emptyset) \]

Some applications

- Infiltration process
- Fluid dynamics
Outlines of the talk

1. Nonlinear hyperbolic problems
   - Notion of weak entropy solution

2. Nonlinear parabolic problems
   - The Schauder-Tychonoff fixed-point method

3. Case of a "nonlinear" flux: \( F_i(x, u) = b_i(x)f_i(u) \) \((i = h, p)\)
   - Definition of a weak entropy solution
   - Existence and uniqueness results

4. Case of a non "nonlinear" flux: \( F_i(x, u) = b_i(x)f_i(u) \)
   - Definition of a weak entropy solution
   - Existence and uniqueness results
Problem \( (P_H) \)

Find a measurable and bounded function \( u \) such that

\[
\partial_t u + \partial_x f(u) = 0, \ t > 0, \ x \in \mathbb{R},
\]

with initial condition

\[
u(x, 0) = u_0(x), \ x \in \mathbb{R}.
\]

Assumptions

- \( u_0 \in C^1_b(\mathbb{R}) \)
- \( f \in C^1(\mathbb{R}) \)
A simple example

- Consider Burgers equation: 
  \[ f(u) = \frac{1}{2}u^2 \]
  \[ \partial_t u + u\partial_x u = 0 \]
A simple example

- Consider Burgers equation: \( f(u) = \frac{1}{2}u^2 \)

\[
\partial_t u + u\partial_x u = 0
\]

- Introduce the characteristic curves

\[
\frac{dx}{dt} = f'(u(x(t), t)) = u(x(t), t)
\]

- Compute

\[
\frac{d}{dt} u(x(t), t) = \partial_t u(x, t) + \partial_x u(x, t) \frac{dx}{dt} = \partial_t u(x, t) + u(x, t) \partial_x u(x, t)
\]

\[
= 0
\]
A simple example

- Consider Burgers equation: \( f(u) = \frac{1}{2} u^2 \)
  \[ \partial_t u + u \partial_x u = 0 \]

- Introduce the characteristic curves
  \[ \frac{dx}{dt} = f'(u(x(t), t)) = u(x(t), t) \]

- Compute
  \[ \frac{d}{dt} u(x(t), t) = \partial_t u(x, t) + \partial_x u(x, t) \frac{dx}{dt} \]
  \[ = \partial_t u(x, t) + u(x, t) \partial_x u(x, t) \]
  \[ = 0 \]

- So \( u \) is constant along the characteristic curves and
  \[ \frac{dx}{dt} = u(x(0), 0) = u_0(x_0) \quad \text{and} \quad x(t) = x_0 + tu_0(x_0) \]
A simple example

- Consider Burgers equation: \( f(u) = \frac{1}{2}u^2 \)
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- Introduce the characteristic curves
  \[ \frac{dx}{dt} = f'(u(x(t), t)) = u(x(t), t) \]

- Compute
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  \frac{d}{dt} u(x(t), t) = \partial_t u(x, t) + \partial_x u(x, t) \frac{dx}{dt}
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  = 0
  \]

- So \( u \) is constant along the characteristic curves and
  \[
  \frac{dx}{dt} = u(x(0), 0) = u_0(x_0) \quad \text{and} \quad x(t) = x_0 + tu_0(x_0)
  \]

- Choose \( u_0 \) such that \( u_0(0) = 2 \) and \( u_0(1) = 1 \)

- So \( u(2t, t) = u_0(0) = 2 \) and \( u(1 + t, t) = u_0(1) = 1 \)
Consider Burgers equation:

\[ f(u) = \frac{1}{2} u^2 \]

\[ \partial_t u + u \partial_x u = 0 \]

Introduce the characteristic curves

\[ \frac{dx}{dt} = f'(u(x(t), t)) = u(x(t), t) \]

Compute

\[ \frac{d}{dt} u(x(t), t) = \partial_t u(x, t) + \partial_x u(x, t) \frac{dx}{dt} \]
\[ = \partial_t u(x, t) + u(x, t) \partial_x u(x, t) \]
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So \( u \) is constant along the characteristic curves and

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So \( u(2t, t) = u_0(0) = 2 \) and \( u(1 + t, t) = u_0(1) = 1 \)

For \( t = 1 \), \( u(2, 1) = 2 = 1 \)
Notion of weak solution

Weak solution

A function \( u \in L^\infty(\mathbb{R} \times \mathbb{R}_+) \) is a weak solution to \((P_H)\) if \( \forall \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+) \),

\[
\int_0^\infty \int_\mathbb{R} (u \partial_t \varphi + f(u) \partial_x \varphi) dx \, dt + \int_0^\infty u_0 \varphi(0, x) dx = 0.
\]

In particular,

\[
\partial_t u + \partial_x f(u) = 0 \quad \text{in} \ D'(\mathbb{R} \times \mathbb{R}_+).\]
Notion of weak solution

**Weak solution**

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak solution to $(PH)$ if

\[ \forall \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}_+) , \]

\[ \int_0^\infty \int_\mathbb{R} \left( u \partial_t \varphi + f(u) \partial_x \varphi \right) dx dt + \int_0^\infty u_0 \varphi(0, x) dx = 0. \]

In particular,

\[ \partial_t u + \partial_x f(u) = 0 \text{ in } D'(\mathbb{R} \times \mathbb{R}_+). \]

**Two important remarks**

- Existence of a weak solution to $(PH)$
- A weak solution is not unique
Let \( \eta \in C^1(\mathbb{R}, \mathbb{R}) \) be a convex function and \( F \in C^1(\mathbb{R}, \mathbb{R}) \) s.t.

\[
F'(u) = \eta'(u)f'(u)
\]

\((\eta, F')\) is called an entropy pair.
The "good" notion of solution

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  \[ F'(u) = \eta'(u)f'(u) \]

- $(\eta, F)$ is called an entropy pair

**Definition 1**

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak entropy solution to $(P_H)$ if, for every entropy pairs $(\eta, F)$,

\[ \partial_t \eta(u) + \partial_x F(u) \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R} \times \mathbb{R}_+) \]

i.e \( \forall \varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}_+), \varphi \geq 0, \)

\[ \int_0^\infty \int_\mathbb{R} \left( \eta(u) \partial_t \varphi + F(u) \partial_x \varphi \right) dx dt \geq 0 \]
The "good" notion of solution

- For $k \in \mathbb{R}$, $\eta(u) = |u - k|$ and $F(u) = \text{sgn}(u - k)(f(u) - f(k))$

**Definition 2**

A function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is a weak entropy solution to $(P_H)$ if

$\forall \varphi \in C^\infty_c(\mathbb{R} \times \mathbb{R}_+), \varphi \geq 0, \forall k \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}} (|u - k| \partial_t \varphi + \text{sgn}(u - k)(f(u) - f(k)) \partial_x \varphi)dxdt \geq 0$$
The ”good” notion of solution

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**Definition 2**

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\forall \varphi \in C_\infty(\mathbb{R} \times \mathbb{R}_+), \varphi \geq 0, \forall k \in \mathbb{R},
\]

\[
\int_0^\infty \int_{\mathbb{R}} (|u - k| \partial_t \varphi + \text{sgn}(u - k)(f(u) - f(k)) \partial_x \varphi) \, dx \, dt \geq 0
\]

**Important property (S.N. Kruzkhov)**

The weak entropy solution \( u \) is the ”limit” of \((u_\varepsilon)_{\varepsilon>0}\) where

\[
\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \varepsilon \partial_{xx} u_\varepsilon
\]
Nonlinear parabolic problems: a fixed-point method

Problem

Find \( u \in W(0, T) \) such that
\[ \forall v \in H^1_0(\Omega), \text{ for a.e. } t \in (0, T), \]
\[
\langle \partial_t u, v \rangle + \int_{\Omega} (\nabla \phi(u) - b(x)f(u)) \cdot \nabla v \, dx = 0,
\]
\[ u(0, .) = u_0 \text{ a.e. on } \Omega, \]
for a.e. \((t, x) \in Q, \ u(t, x) \in [m, M].\]
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u(0, .) = u_0 \text{ a.e. on } \Omega,
\]

for a.e. \( (t, x) \in Q, \ u(t, x) \in [m, M]. \)

Some assumptions and notations

- \( W(0, T) = \{ u \in L^2(0, T; H^1_0(\Omega)) ; \partial_t u \in L^2(0, T; H^{-1}(\Omega)) \} \)
- \( \langle ., . \rangle := \text{the pairing between } H^{-1} \text{ and } H^1_0 \)
- \( \exists \alpha > 0, \forall \tau \in \mathbb{R}, \phi'(\tau) \geq \alpha \)
- \( u_0 \in L^\infty(\Omega), \ m \leq u_0 \leq M \)
A fixed-point Theorem

The Schauder-Tychonoff fixed point Theorem

Let $X$ be a reflexive and separable Banach space. We suppose

- $K \subset X$, $K \neq \emptyset$, $K$ is a closed, bounded and convex set
- The mapping $T : K \mapsto K$ is “weakly-weakly” sequentially continuous, i.e. for any sequence $(x_n)_{n \in \mathbb{N}^*} \subset K$ that weakly converges towards $x$, the sequence $(T(x_n))_{n \in \mathbb{N}^*}$ weakly converges towards $T(x)$.

Then, $T$ has at least one fixed-point in $K$. 
A fixed-point Theorem

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Then, $T$ has at least one fixed-point in $K$.

Main idea

"associate" the nonlinear problem with a linear one via a mapping $T$
Troncation process. Consider the equivalent nonlinear problem
Find \( u \in W(0,T) \) such that

\[
\begin{cases}
\langle \partial_t u, v \rangle + \int_{\Omega} \left( \phi'(u^*) \nabla u - b(x)f(u^*) \right) \cdot \nabla v \, dx = 0 \\
u(0,.) = u_0 \text{ a.e. on } Q
\end{cases}
\]

where, for a.e. \((t,x) \in Q\), \( u^*(t,x) = \begin{cases} m & \text{if } u(t,x) < m \\ u(t,x) & \text{if } m \leq u(t,x) \leq M \\ M & \text{if } u(t,x) > M \end{cases} \)
Troncation process. Consider the equivalent nonlinear problem
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\end{cases} \)

The linear problem : \( w \in W(0, T) \) being fixed, \( U_w \in W(0, T) \) is the unique solution of

\[
\begin{cases}
\langle \partial_t U_w, v \rangle + \int_{\Omega} (\phi'(w^*) \nabla U_w - f(w^*) b(x)) \cdot \nabla v dx = 0 \\
U_w(0, .) = u_0 \ \text{a.e. on } \Omega
\end{cases}
\]
Troncation process. Consider the equivalent nonlinear problem

Find \( u \in W(0, T) \) such that

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\begin{cases}
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The linear problem: \( w \in W(0, T) \) being fixed, \( U_w \in W(0, T) \) is the unique solution of

\[
\begin{cases}
\langle \partial_t U_w, v \rangle + \int_{\Omega} (\phi'(w^*) \nabla U_w - f(w^*)b(x)) \cdot \nabla v dx = 0 \\
U_w(0, .) = u_0 \text{ a.e. on } \Omega
\end{cases}
\]

We introduce the mapping

\[
\mathcal{T} : W(0, T) \rightarrow W(0, T) \\
w \rightarrow U_w \equiv \mathcal{T}(w)
\]
**A priori estimates**

- \( \|U_w\|_{L^2(0,T;H^1_0(\Omega))} \leq C_1 \)
- \( \|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2 \)
A priori estimates

- $\|U_w\|_{L^2(0,T;H^1_0(\Omega))} \leq C_1$
- $\|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2$

We set

$$K = \{ v \in W(0,T), \| v \|_{L^2(0,T;H^1_0(\Omega))} \leq C_1, \| \partial_t v \|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2;\ v(0,.) = v_0 \text{ a.e. on } \Omega \}$$

$K$ is convex, bounded, closed, $T(K) \subset K.$
Parabolic problems: a fixed-point method

A priori estimates

- \( \|U_w\|_{L^2(0,T;H^1_0(\Omega))} \leq C_1 \)
- \( \|\partial_t U_w\|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2 \)

We set

\[
K = \{ v \in W(0,T), \| v \|_{L^2(0,T;H^1_0(\Omega))} \leq C_1, \| \partial_t v \|_{L^2(0,T;H^{-1}(\Omega))} \leq C_2; v(0,.) = v_0 \text{ a.e. on } \Omega \}
\]

- \( K \) is convex, bounded, closed, \( T(K) \subset K \).
- The ”sequential” continuity: \( w_n \rightharpoonup w \) in \( W(0,T) \).

We have to show that \( T(w_n) \equiv U_{w_n} \rightharpoonup T(w) \) in \( W(0,T) \).
\[ W(0, T) \hookrightarrow L^2((0, T) \times \Omega) \Rightarrow w_n \to w \text{ in } L^2(Q) \text{ (up to a subsequence)} \]
\[ \|U_{w_n}\|_{W(0,T)} \leq C \Rightarrow U_{w_n} \rightharpoonup U \text{ in } W(0,T) \text{ and } U_{w_n} \to U \text{ in } L^2(Q) \]
\begin{itemize}
  \item $W(0, T) \hookrightarrow L^2((0, T) \times \Omega) \Rightarrow w_n \to w$ in $L^2(Q)$ (up to a subsequence)
  \item $\|U_{w_n}\|_{W(0,T)} \leq C \Rightarrow U_{w_n} \to U$ in $W(0,T)$ and $U_{w_n} \to U$ in $L^2(Q)$
  \item We have :
    \[
    \int_0^T \langle \partial_t U_{w_n}, v \rangle dt + \int_0^T \int_{\Omega} (\phi'(w_n^*) \nabla U_{w_n} - f(w_n^*)b(x)) \cdot \nabla v dx dt = 0
    \]
  \item when $n$ goes to $+\infty$ :
    \[
    \int_0^T \langle \partial_t U, v \rangle dt + \int_0^T \int_{\Omega} (\phi'(w^*) \nabla U - b(x)f(w^*)) \nabla v dx dt = 0
    \]
  \item $W(0, T) \hookrightarrow C([0, T], L^2(\Omega)) \Rightarrow U(0, .) = u_0(.)$
\end{itemize}
\( W(0, T) \hookrightarrow L^2((0, T) \times \Omega) \Rightarrow w_n \rightarrow w \) in \( L^2(Q) \) (up to a subsequence)

\[ \|U_{w_n}\|_{W(0,T)} \leq C \Rightarrow U_{w_n} \rightarrow U \text{ in } W(0,T) \text{ and } U_{w_n} \rightarrow U \text{ in } L^2(Q) \]

We have:

\[
\int_0^T \langle \partial_t U_{w_n}, v \rangle dt + \int_0^T \int_\Omega (\phi'(w_n^*) \nabla U_{w_n} - f(w_n^*)b(x)) \cdot \nabla v dx dt = 0
\]

when \( n \) goes to \(+\infty\) :

\[
\int_0^T \langle \partial_t U, v \rangle dt + \int_0^T \int_\Omega (\phi'(w^*) \nabla U - b(x)f(w^*)) \nabla v dx dt = 0
\]

\( W(0, T) \hookrightarrow C([0, T], L^2(\Omega)) \Rightarrow U(0, .) = u_0(.) \)

**Conclusion**

\( U = \mathcal{T}(w) \)

the whole sequence \((\mathcal{T}(w_n))_n\) converges weakly towards \(\mathcal{T}(w)\)
The "nonlinear" coupled problem

Study of the problem

\[
\begin{align*}
\partial_t u + \text{div}_x (b(x)f(u)) + g(t, x, u) &= \text{div}_x (\mathbb{I}_{\Omega_p}(x) \nabla \phi(u)) \quad \text{in } (0, T) \times \Omega \\
u &= 0 \quad \text{on } (0, T) \times \partial \Omega \\
u(0, .) &= u_0 \quad \text{on } \Omega
\end{align*}
\]

with

- \( \Omega \subset \mathbb{R}^n \)
- \( \overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p, \ \Omega_p \cap \Omega_h = \emptyset \)
- \( b(x)f(u) = b_h(x)f_h(u)\mathbb{I}_{\Omega_h}(x) + b_p(x)f_p(u)\mathbb{I}_{\Omega_p}(x) \)
- \( g(t, x, u) = g_h(t, x, u)\mathbb{I}_{\Omega_h}(x) + g_p(t, x, u)\mathbb{I}_{\Omega_p}(x) \)
- \( \Gamma_{hp} = \partial \Omega_h \cap \partial \Omega_p = \Gamma_h \cap \Gamma_p \)
- \( \Sigma_{hp} = (0, T) \times \Gamma_{hp} \)
Main assumptions

- $u_0 \in L^\infty(\Omega)$
- Different nonlinearities on $\Omega_h$ and $\Omega_p$
- $\phi$ is nondecreasing, $\phi^{-1}$ exists, $\phi(0) = 0$
Assumptions and Notations

**Main assumptions**

- \( u_0 \in L^\infty(\Omega) \)
- Different nonlinearities on \( \Omega_h \) and \( \Omega_p \)
- \( \phi \) is nondecreasing, \( \phi^{-1} \) exists, \( \phi(0) = 0 \)

**Notations**

- \( sgn_\eta := \text{Lipschitzian approximation of the function } sgn \)
- \( I_\eta, F_\eta := \text{"regular" entropy pairs} \)
- \[
I_\eta(a, b) = \int_b^a sgn_\eta(\phi(\tau) - \phi(b))d\tau
\]
- \[
F_{l,\eta}(a, b) = \int_{\phi(b)}^{\phi(a)} f_l \circ \phi^{-1}(\tau)sgn'_\eta(\tau - \phi(b))d\tau.
\]
- \[
F_\eta = F_{h,\eta} + F_{p,\eta}
\]
Notion of weak entropy solution

- \( V = \{ v \in H^1(\Omega_p), v = 0 \text{ a.e. on } \Gamma_p \setminus \Gamma_{hp} \} \)

**Definition**

- \( u \in L^\infty(Q), \quad \phi(u) \in L^2(0,T;V) \)
- \( \forall \varphi \in D(Q) \text{ with } \varphi \geq 0, \forall k \in \mathbb{R}, \)

\[
\int_Q I_\eta(u,k) \partial_t \varphi dx dt - \int_{Q_p} sgn_\eta(\phi(u) - \phi(k)) \nabla \phi(u) \cdot \nabla \varphi dx dt \\
+ \int_Q b(x) \{ sgn_\eta(\phi(u) - \phi(k)) f(u) - F_\eta(u,k) \} \cdot \nabla \varphi dx dt \\
- \int_Q \{ sgn_\eta(\phi(u) - \phi(k)) g(t,x,u) + \nabla b(x) \cdot F_\eta(u,k) \} \varphi dx dt \\
+ \int_{\Sigma_{hp}} (b_h(\sigma) F_{h,\eta}(u,k) - b_p(\sigma) F_{p,\eta}(u,k)) \varphi \cdot \nu_h dtd\mathcal{H}^{n-1} \geq 0
\]
Notion of weak entropy solution

Initial and boundary conditions

\[ \text{ess lim }_{t \to 0^+} \int_{\Omega} |u(t, x) - u_0(x)| \, dx = 0 \]

\[ \forall \zeta \in L^1(\Sigma_h \setminus \Sigma_{hp}), \; \zeta \geq 0, \; \forall k \in \mathbb{R}, \]
\[ \text{ess lim }_{s \to 0^-} \int_{\Sigma_{hp}} b(\sigma) F_h(u(\sigma + s \nu_h), k) \cdot \nu_h \, dt \, d\mathcal{H}^{n-1} \geq 0 \]

where

\[ F_h(\tau, k) = \frac{1}{2} \{ sgn(\tau)(f_h(\tau) - f_h(0)) \]
\[ - sgn(k)(f_h(k) - f_h(0)) + sgn(\tau - k)(f_h(\tau) - f_h(k)) \}\]
Some remarks

If $u$ satisfies the entropy inequality (1) then

$$\int_Q |u - k| \partial_t \varphi dx dt - \int_{Q_p} \nabla |\phi(u) - \phi(k)| \cdot \nabla \varphi dx dt$$

$$+ \int_Q b(x) \Phi(u, k) \cdot \nabla \varphi dx dt$$

$$- \int_Q \text{sgn}(u - k)(g(t, x, u) + \nabla b(x) \cdot f(k)) \varphi dx dt$$

$$+ \int_{\Sigma_{hp}} \text{sgn}(\phi(u) - \phi(k))(b_h f_h(k) - b_p f_p(k)) \cdot \nu_h \varphi dt \mathcal{H}^{n-1} \geq 0$$

where $\Phi(u, k) = \text{sgn}(u - k)(f(u) - f(k))$ is the Kruzhkov flux
Some remarks

If $u$ is a weak entropy solution then, $\forall \varphi \in \mathcal{D}(Q)$,

$$
\int_Q \left( u \partial_t \varphi + (b(x)f(u) - \mathbb{I}_{\Omega_p} \nabla \phi(u)) \cdot \nabla \varphi - g(t, x, u) \varphi \right) dx dt = 0
$$
Some remarks

- If $u$ is a weak entropy solution then, $\forall \varphi \in \mathcal{D}(Q)$,

$$
\int_Q (u \partial_t \varphi + (b(x)f(u) - \mathbb{1}_{\Omega_p} \nabla \phi(u)) \cdot \nabla \varphi - g(t,x,u) \varphi) \, dx \, dt = 0
$$

So $u$ fulfills

$$
\partial_t u + \text{div}_x (b_h(x)f_h(u)) + g_h(t,x,u) = 0 \text{ in } \mathcal{D}'(Q_h),
$$

$$
\partial_t u + \text{div}_x (b_p(x)f_p(u)) + g_p(t,x,u) = \Delta \phi(u) \text{ in } \mathcal{D}'(Q_p),
$$

and the transmission condition (in a formal sense)

$$
(b_p f_p(u) - b_h f_h(u)) \cdot \nu_h = \nabla \phi(u) \cdot \nu_h \text{ on } \Sigma_{hp}
$$
The uniqueness property

Main assumption

\[ \lambda \mapsto \xi \cdot b(x) f(\lambda) \] is not linear on any nondegenerate interval

This nonlinear condition allows us

- to define ”strong” trace on the hyperbolic zone for a weak entropy solution
- to obtain precompactness of sequence of solutions to approximate problems (existence)

Lemma (E. Yu. Panov)

Let \( u \) be a weak entropy solution. Then there exists a function \( u_{\tau} \in L^\infty(\Sigma_h) \) such that, for every compact \( K \) of \( \Sigma_h \) and every regular Lipschitz deformation \( \Psi \) of \( \Omega_h \),

\[
\text{ess} \lim_{\varepsilon \to 0^+} \int_K |u(\Psi(s, \sigma)) - u_{\tau}(\sigma)| \, dt \, dH^{n-1} = 0
\]
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Lemma (E. Yu. Panov)

Let \( u \) be a weak entropy solution. Then there exists a function \( u^\tau \in L^\infty(\Sigma_h) \) such that, for every compact \( K \) of \( \Sigma_h \) and every regular Lipschitz deformation \( \Psi \) of \( \Omega_h \),

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\]
Let $u$ be a weak entropy solution. Then $\forall k \in \mathbb{R}, \forall \varphi \in \mathcal{D}((0, T) \times \mathbb{R}^n)$, $\varphi \geq 0,$

$$\int_{Q_h} |u - k| \partial_t \varphi dxdt + \int_{Q_h} b_h(x) \Phi_h(u, k) \cdot \nabla \varphi dxdt - \int_{Q_h} G_h(u, k) \varphi dxdt \geq \int_{\Sigma_{hp}} b_h(\sigma) \Phi_h(u^\tau, k) \cdot \nu_h \varphi dtd\mathcal{H}^{n-1} + \int_{\Sigma_h \setminus \Sigma_{hp}} b_h(\sigma) \Phi_h(0, k) \cdot \nu_h \varphi dtd\mathcal{H}^{n-1}$$

$$- \int_{\Sigma_h \setminus \Sigma_{hp}} b_h(\sigma) \Phi_h(u^\tau, 0) \cdot \nu_h \varphi dtd\mathcal{H}^{n-1}$$

where $G_h(u, k) = \text{sgn}(u - k)(g_h(t, x, u) + \nabla b_h(x) \cdot f_h(k))$
**Uniqueness : hyperbolic zone**

- Let \( u \) be a weak entropy solution. Then \( \forall k \in \mathbb{R}, \forall \varphi \in D((0, T) \times \mathbb{R}^n), \varphi \geq 0, \)

\[
\int_{Q_h} |u - k| \partial_t \varphi dx dt + \int_{Q_h} b_h(x) \Phi_h(u, k) \cdot \nabla \varphi dx dt - \int_{Q_h} G_h(u, k) \varphi dx dt \\
\geq \int_{\Sigma_{hp}} b_h(\sigma) \Phi_h(u^\tau, k) \cdot \nu_h \varphi dt d\mathcal{H}^{n-1} + \int_{\Sigma_h \setminus \Sigma_{hp}} b_h(\sigma) \Phi_h(0, k) \cdot \nu_h \varphi dt d\mathcal{H}^{n-1} \\
- \int_{\Sigma_h \setminus \Sigma_{hp}} b_h(\sigma) \Phi_h(u^\tau, 0) \cdot \nu_h \varphi dt d\mathcal{H}^{n-1}
\]

where \( G_h(u, k) = \text{sgn}(u - k)(g_h(t, x, u) + \nabla b_h(x) \cdot f_h(k)) \)

- **Method of doubling the variables** \( \Rightarrow \)

\[
- \int_{Q_h} |u - v| \gamma'(t) dx dt \leq - \int_{Q_h} \text{sgn}(u - v)(g_h(t, x, u) - g_h(t, x, v)) \gamma(t) dx dt \\
- \int_{\Sigma_{hp}} \text{sgn}(u^\tau - v^\tau)(f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h b_h \gamma(t) d\mathcal{H}^{n-1} dt
\]
Let $u$ be a weak entropy solution. Then $\partial_t u \in L^2(0, T; V')$. Moreover, for all $\varphi \in L^2(0, T; V)$,

$$
\int_0^T \langle \langle \partial_t u, \varphi \rangle \rangle dt + \int_{Q_p} (\nabla \phi(u) - b_p f_p(u)) \cdot \nabla \varphi dx dt \\
+ \int_{Q_p} g_p(t, x, u) \varphi dx dt - \int_{\Sigma_{hp}} b_h f_h(u^\tau) \cdot \nu_h \varphi dt d\mathcal{H}^{n-1} = 0
$$
Uniqueness: parabolic zone

- Let $u$ be a weak entropy solution. Then $\partial_t u \in L^2(0, T; V')$. Moreover, $\forall \varphi \in L^2(0, T; V)$,

$$
\int_0^T \langle \langle \partial_t u, \varphi \rangle \rangle dt + \int_{Q_p} (\nabla \phi(u) - b_p f_p(u)) \cdot \nabla \varphi dxdt \\
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$$

- Method of doubling the time variable $\Rightarrow$

$$
- \int_{Q_p} |u - v| \gamma'(t) dxdt \leq M_{g_p} \int_{Q_p} |u - v| \gamma(t) dxdt \\
+ \int_{\Sigma_{hp}} \text{sgn}(u^\phi - v^\phi) b_h (f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h \gamma(t) dtd\mathcal{H}^{n-1}
$$

where $u^\phi = \phi^{-1}(\phi(u) | \Sigma_{hp})$
Lemma: interface inequality

Let $u$ be a weak entropy solution. Then a.e. in $(0, T)$, $\mathcal{H}^{n-1}$-a.e. on $\Gamma_{hp}$, for any $k \in I(u^\tau, u^\phi),$

$$sgn(u^\tau - u^\phi) b_h(f_h(u^\tau) - f_h(k)) \cdot \nu_h \geq 0$$
We have

\[- \int_Q |u - v| \gamma'(t) dx dt \leq M_g \int_Q |u - v| \gamma(t) dx dt \]

\[+ \int_{\Sigma_{hp}} b_h sgn(u^\phi - v^\phi) (f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h \gamma(t) d\mathcal{H}^{n-1} dt \]

\[- \int_{\Sigma_{hp}} b_h sgn(u^\tau - v^\tau) (f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h \gamma(t) d\mathcal{H}^{n-1} dt \]

We set

\[J = (sgn(u^\phi - v^\phi) - sgn(u^\tau - v^\tau)) b_h (f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h\]
Uniqueness : last step

- We have

\[- \int_Q |u - v| \gamma'(t) dx dt \leq M_g \int_Q |u - v| \gamma(t) dx dt\]

\[+ \int_{\Sigma_{hp}} b_h \text{sgn}(u^\phi - v^\phi)(f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h \gamma(t) dH^{n-1} dt\]

\[\quad - \int_{\Sigma_{hp}} b_h \text{sgn}(u^\tau - v^\tau)(f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h \gamma(t) dH^{n-1} dt\]

- We set

\[J = (\text{sgn}(u^\phi - v^\phi) - \text{sgn}(u^\tau - v^\tau)) b_h (f_h(u^\tau) - f_h(v^\tau)) \cdot \nu_h\]

- Interface inequality \(\Rightarrow J \leq 0\)
- Then

\[\int_{\Omega} |u(t, .) - v(t, .)| dx \leq e^{M_g t} \int_{\Omega} |u_0(.) - v_0(.)| dx\]
Existence: the viscous problem

- \( \lambda_\mu(x) = \mathbb{I}_{\Omega_p}(x) + \mu \mathbb{I}_{\Omega_n}(x) \)
- \( \phi_\mu(u_\mu) = \phi(u_\mu) + \mu u_\mu \)

Find a bounded and measurable function \( u_\mu \) such that

\[
\begin{align*}
\partial_t u_\mu + \text{div}_x (b(x)f(u_\mu)) + g(t, x, u_\mu) &= \text{div}_x (\lambda_\mu \nabla_x \phi_\mu(u_\mu)) & \text{in } Q \\
u_\mu &= 0 & \text{on } \Sigma \\
u_\mu(0, .) &= u_0 & \text{on } \Omega
\end{align*}
\]
Existence: main assumption

We introduce a nondecreasing function $M_1$ such that

$$M_1(0) \geq M, \quad \forall t \in (0, T)$$

$$M_1'(t) + \nabla b(.) \cdot f(M_1(t)) + g(t, .., M_1(t)) \geq 0 \text{ a.e. on } \Omega_L \cup \Omega_R$$

and a nonincreasing function $M_2$ such that

$$M_2(0) \leq m, \quad \forall t \in (0, T)$$

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\end{align*}
$$

**Assumption (H)**

For almost all $t \in (0, T)$, a.e. on $\Gamma_{hp}$,

$$
\begin{align*}
(b_p f_p(M_1(t)) - b_h f_h(M_1(t))) \cdot \nu_h & \geq 0 \\
(b_p f_p(M_2(t)) - b_h f_h(M_2(t))) \cdot \nu_h & \leq 0
\end{align*}
$$
The viscous problem

- \( W(0, T) = \{ v \in L^2(0, T; H^1_0(\Omega)), \partial_t v \in L^2(0, T; H^{-1}(\Omega)) \} \)

Existence and uniqueness

Under \((H)\), \( \exists! \ u_\mu \in W(0, T) \cap L^\infty(Q) \) such that

- \( \forall t \in [0, T], M_2(t) \leq u_\mu(t, .) \leq M_1(t) \) a.e. in \( \Omega \)
- \( u_\mu(0, .) = u_0 \) a.e. in \( \Omega \)
- For any \( v \in H^1_0(\Omega) \), for almost all \( t \in (0, T) \),

\[
\langle \partial_t u_\mu, v \rangle + \int_\Omega \left( (\lambda_\mu(x) \nabla \phi_\mu(u_\mu) - b(x) f(u_\mu)) \cdot \nabla v + g(t, x, u_\mu)v \right) dx = 0
\]

(Sketch of) Proof

Assumption \((H) \Rightarrow L^\infty\)-estimate
- Schauder-Tychonoff fixed-point Theorem \( \Rightarrow \) Existence
- Holmgren-type duality method \( \Rightarrow \) Uniqueness
The viscous problem

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Under \((H)\), \( \exists! u_\mu \in W(0, T) \cap L^\infty(Q) \) such that

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(Sketch of) Proof

- Assumption \((H)\) \( \Rightarrow \) \( L^\infty\)-estimate
- Schauder-Tychonoff fixed-point Theorem \( \Rightarrow \) Existence
- Holmgren-type duality method \( \Rightarrow \) Uniqueness
Maximum principle

- $\mathcal{B}(a, b, c) = \max\{a, \min\{b, c\}\}$
- $u_\mu^* = \mathcal{B}(M_2(t), u_\mu, M_1(t))$
- Study of the problem

\[
\begin{aligned}
\text{Find } u_\mu \text{ in } W(0, T) \text{ such that a.e. on } (0, T) \text{ and for all } v \in H^1_0(\Omega), \\
\langle \partial_t u_\mu, v \rangle + \int_\Omega ((\lambda_\mu(x)\phi'_\mu(u_\mu^*) \nabla u_\mu - b(x)f(u_\mu^*)) \cdot \nabla v + g(t, x, u_\mu^*)v)dx = 0 \\
u_\mu(0, .) = u_0 \text{ a.e. on } \Omega
\end{aligned}
\]
Maximum principle

- $B(a, b, c) = \max\{a, \min\{b, c\}\}$
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\[
\left\{
\begin{array}{l}
\text{Find } u_\mu \text{ in } W(0, T) \text{ such that a.e. on } (0, T) \text{ and for all } v \in H^1_0(\Omega),
\\
\langle \partial_t u_\mu, v \rangle + \int_\Omega \left((\lambda_\mu(x)\phi'_\mu(u^*_\mu)\nabla u_\mu - b(x)f(u^*_\mu)) \cdot \nabla v + g(t, x, u^*_\mu)v\right)dx = 0
\\
u_\mu(0, .) = u_0 \text{ a.e. on } \Omega
\end{array}
\right.
\]

- Test function $v_\eta = sgn_\eta(u_\mu - M_1(t))^+$
- For the convective term

\[
- \int_{Q_s} b(x)f(u^*_\mu) \cdot \nabla v_\eta dxdt = \sum_{i \in \{h, p\}} \int_{Q_{i,s}} f_i(M_1(t)) \cdot \nabla b_i(x)v_\eta dxdt
\]
\[
+ \int_{\Sigma_{hp}} (b_p f_p(M_1(t)) - b_h f_h(M_1(t))) \cdot \nu_h v_\eta dtdH^{n-1}
\]
Maximum principle

- When $\eta$ goes to 0

$$\int_\Omega (u_\mu(s,x) - M_1(s))^+ \, dx + \int_{Q_s} M'_1(t) \text{sgn}(u_\mu - M_1(t))^+ \, dx \, dt$$

$$+ \sum_{i \in \{h,p\}} \int_{Q_{i,s}} (f_i(M_1(t)) \cdot \nabla b_i + g_i(t, x, M_1(t))) \text{sgn}(u_\mu - M_1(t))^+ \, dx \, dt \leq 0$$

- By definition (of $M_1$),

$$M'_1(t) + (f_i(M_1(t)) \cdot \nabla b_i + g_i(t, x, M_1(t))) \geq 0$$
Maximum principle

- When $\eta$ goes to 0

\[
\int_{\Omega} (u_\mu(s, x) - M_1(s))^+ dx + \int_{Q_s} M'_1(t) \text{sgn}(u_\mu - M_1(t))^+ dxdt \\
+ \sum_{i \in \{h,p\}} \int_{Q_{i,s}} (f_i(M_1(t)) \cdot \nabla b_i + g_i(t, x, M_1(t))) \text{sgn}(u_\mu - M_1(t))^+ dxdt \leq 0
\]

- By definition (of $M_1$),

\[
M'_1(t) + (f_i(M_1(t)) \cdot \nabla b_i + g_i(t, x, M_1(t))) \geq 0
\]

- So

\[
\int_{\Omega} (u_\mu(s, x) - M_1(s))^+ dx \leq 0
\]
The viscous limit

A priori estimates

There exists a constant $C$ independent of $\mu$ such that

\[
\|(\lambda \mu)^{1/2} \nabla \hat{\phi}(u_\mu)\|_{L^2(Q)^n}^2 + \| (\mu \lambda \mu)^{1/2} \nabla u_\mu \|_{L^2(Q)^n}^2 \leq C
\]

\[
\| \partial_t u_\mu \|_{L^2(0,T;H^{-1}(\Omega))} \leq C
\]

where $\hat{\phi}(x) = \int_0^x \sqrt{\phi'(\tau)} d\tau$
The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

$$\lambda \mapsto b_p(x)f_p(\lambda) \cdot \xi$$

and

$$\lambda \mapsto \phi(\lambda)\xi^2$$

are not linear simultaneously on any non-degenerate intervals.
The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

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\lambda \mapsto b_p(x)f_p(\lambda) \cdot \xi \quad \text{and} \quad \lambda \mapsto \phi(\lambda)\xi^2
$$

are not linear simultaneously on any non-degenerate intervals

Consequences (E. Yu. Panov)

- The sequence $(u_\mu)_{\mu > 0}$ is precompact in $L^1(Q_p)$
The viscous limit

Assumption

For almost all $x \in \Omega_p$, for all $\xi \in \mathbb{R}^n$ with $\xi \neq 0$, the functions

$$\lambda \mapsto b_p(x)f_p(\lambda) \cdot \xi \text{ and } \lambda \mapsto \phi(\lambda)\xi^2$$

are not linear simultaneously on any non-degenerate intervals.

Consequences (E. Yu. Panov)

- The sequence $(u_\mu)_{\mu > 0}$ is precompact in $L^1(Q_p)$
- Nonlinear flux on the hyperbolic zone $\Rightarrow (u_\mu)_{\mu > 0}$ is precompact in $L^1(Q_h)$
- $u_\mu \rightarrow u \in L^\infty(Q)$ in $L^1(Q)$
Study of the convective term

- Test function: \( v^\eta_\mu = \text{sgn}_\eta(\phi(u_\mu) - \phi(k))\varphi_1\varphi_2, \varphi_1 \in \mathcal{C}_c(\mathbb{R}), \varphi_2 \in \mathcal{C}_c(\Omega). \)

\[
\int_Q -b(x)f(u_\mu) \cdot \nabla (\text{sgn}_\eta(\phi(u_\mu) - \phi(k))\varphi_1\varphi_2) \, dx\, dt
= - \sum_{i \in \{h,p\}} \int_{Q_i} b_i(x)f_i(u_\mu) \cdot \nabla \phi(u_\mu) \text{sgn}'_\eta(\phi(u_\mu) - \phi(k))\varphi_1\varphi_2 \, dx\, dt
- \sum_{i \in \{h,p\}} \int_{Q_i} b_i(x) \text{sgn}_\eta(\phi(u_\mu) - \phi(k))\varphi_1f_i(u_\mu) \cdot \nabla \varphi_2 \, dx\, dt
\]
Study of the convective term

- Test function: 
  \[ v_\mu^\eta = \text{sgn}_\eta (\phi(u_\mu) - \phi(k)) \varphi_1 \varphi_2, \quad \varphi_1 \in C^\infty_c([0, T]), \varphi_2 \in C^\infty_c(\Omega). \]

  \[
  \int_Q -b(x)f(u_\mu) \cdot \nabla (\text{sgn}_\eta (\phi(u_\mu) - \phi(k)) \varphi_1 \varphi_2) \, dx \, dt \\
  = - \sum_{i \in \{h,p\}} \int_{Q_i} b_i(x)f_i(u_\mu) \cdot \nabla \phi(u_\mu) \text{sgn}_\eta'(\phi(u_\mu) - \phi(k)) \varphi_1 \varphi_2 \, dx \, dt \\
  - \sum_{i \in \{h,p\}} \int_{Q_i} b_i(x) \text{sgn}_\eta (\phi(u_\mu) - \phi(k)) \varphi_1 f_i(u_\mu) \cdot \nabla \varphi_2 \, dx \, dt
  \]

- \( J_{\mu,\eta} = - \int_{Q_h} b_h(x)f_h(u_\mu) \cdot \nabla \phi(u_\mu) \text{sgn}_\eta'(\phi(u_\mu) - \phi(k)) \varphi_1 \varphi_2 \, dx \, dt \)

- \( J_{\mu,\eta} = - \int_{Q_h} b(x) \text{div} F_{h,\eta}(u_\mu, k) \varphi_1 \varphi_2 \, dx \, dt \)

  (with \( F_{h,\eta}(u_\mu, k) = \int_{\phi(k)}^{\phi(u_\mu)} f_h \circ \phi^{-1}(\tau) \text{sgn}_\eta'(\tau - \phi(k)) \, d\tau \))
Study of the convective term

\[ J_{\mu, \eta} = \int_{Q_h} F_{h, \eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi_2 b_h) \varphi_1 dx dt \]

\[ - \int_{\Sigma_{h,p}} b_h F_{h, \eta}(u_\mu, k) \cdot \nu_h \varphi_1 \varphi_2 d\mathcal{H}^{n-1} dt \]
Study of the convective term

\[ J_{\mu,\eta} = \int_{Q_h} \mathbf{F}_{h,\eta}(u_\mu, k) \cdot (\nabla_b \varphi_2 + \nabla \varphi_2 b_h) \varphi_1 \, dx \, dt \]

\[ - \int_{\Sigma_{h,p}} b_h \mathbf{F}_{h,\eta}(u_\mu, k) \cdot \nu_h \varphi_1 \varphi_2 \, d\mathcal{H}^{n-1} \, dt \]

- \( \phi(u_\mu) \in L^2(0, T; H^1(\Omega)) \Rightarrow (\phi(u_\mu)|\Omega_h)|\Gamma_{hp} = (\phi(u_\mu)|\Omega_p)|\Gamma_{hp} \)

- \( (\mathbf{F}_{h,\eta}(u_\mu, k))_{\mu>0} \) converges strongly towards \( \mathbf{F}_{h,\eta}(u, k) \) in \( L^q(Q_p)^n \), \( 1 \leq q < \infty \)
Study of the convective term

\[ J_{\mu,\eta} = \int_{Q_h} F_{h,\eta}(u_\mu, k) \cdot (\nabla b_h \varphi_2 + \nabla \varphi_2 b_h) \varphi_1 \, dx \, dt \]
\[ - \int_{\Sigma_{hp}} b_h F_{h,\eta}(u_\mu, k) \cdot \nu_h \varphi_1 \varphi_2 \, d\mathcal{H}^{n-1} \, dt \]

- \( \phi(u_\mu) \in L^2(0, T; H^1(\Omega)) \Rightarrow (\phi(u_\mu)|_{\Omega_h})|_{\Gamma_{hp}} = (\phi(u_\mu)|_{\Omega_p})|_{\Gamma_{hp}} \)
- \( (F_{h,\eta}(u_\mu, k))_{\mu > 0} \) converges strongly towards \( F_{h,\eta}(u, k) \) in \( L^q(Q_p)^n \), \( 1 \leq q < \infty \)
- \( (F_{h,\eta}(u_\mu, k))_{\mu > 0} \) is uniformly bounded in \( L^2(0, T; V)^n \cap L^\infty(Q)^n \)
- \( (F_{h,\eta}(u_\mu, k)\varphi_2)_{\mu > 0} \) converges weakly, up to a subsequence, towards \( F_{h,\eta}(u, k)\varphi_2 \) in \( L^2(0, T; V)^n \)
- \( (F_{h,\eta}(u_\mu, k)\varphi_2)_{\mu > 0} \) converges weakly towards \( F_{h,\eta}(u, k)\varphi_2 \) in \( L^2(\Sigma_p)^n \)
The non ”nonlinear” coupling problem

Study of the problem

\[
\begin{aligned}
\partial_t u + \text{div}_x (b(x)f(u)) + g(t, x, u) &= \text{div}_x (I_{\Omega_p}(x) \nabla \phi(u)) & \text{in } Q, \\
u &= 0 & \text{on } (0, T) \times \partial \Omega, \\
u(0, \cdot) &= u_0 & \text{on } \Omega,
\end{aligned}
\]

- $\overline{\Omega} = \overline{\Omega}_h \cup \overline{\Omega}_p$, $\Omega_p \cap \Omega_h = \emptyset$
- $b(x)f(u) = b_h(x)f_h(u)I_{\Omega_h} + b_p(x)f_p(u)I_{\Omega_p}$
- $g(t, x, u) = g_h(t, x, u)I_{\Omega_h}(x) + g_p(t, x, u)I_{\Omega_p}(x)$

Main difference

The flux $b(x)f(u)$ does not satisfy the nonlinear condition
The non "nonlinear" coupling problem

\[
\Gamma_{hp} = \partial \Omega_h \cap \partial \Omega_p = \Gamma_h \cap \Gamma_p
\]

\[
\Sigma_{hp} = ]0, T[ \times \Gamma_{hp}
\]
Assumptions

Main assumptions

- $u_0 \in L^\infty(\Omega)$
- Different nonlinearities on $\Omega_h$ and $\Omega_p$
- $b_i \in W^{1,\infty}(\Omega_i)^n$, $i = h, p$
- $\Gamma_{hp} \subset \{\bar{\sigma} \in \Gamma_h, b_h(\bar{\sigma}) \cdot \nu_h \geq 0\}$
- $f_h$ is nondecreasing
- $\phi$ is nondecreasing, $\phi^{-1}$ exists, $\phi(0) = 0$
Notion of weak entropy solution

\[ V = \{v \in H^1(\Omega_p), v = 0 \text{ a.e. on } \Gamma_p \setminus \Gamma_{hp}\} \]

**Definition**

- \( u \in L^\infty(Q), \quad \phi(u) \in L^2(0, T; V) \).
- \( \forall \varphi \in C_c^\infty(Q), \forall \varphi \geq 0, \forall k \in \mathbb{R}, \)

\[
\int_Q (|u - k| \partial_t \varphi + b(x) \Phi(u, k) \cdot \nabla \varphi) \, dx \, dt - \int_{Q_p} \nabla |\phi(u) - \phi(k)| \cdot \nabla \varphi \, dx \, dt \\
- \int_Q sgn(u - k)(g(t, x, u) + \text{div}b(x)f(k)) \varphi \, dx \, dt \\
+ \int_{\Sigma_{hp}} \{b_h f_h(k) - b_pf_p(k)\} \cdot \nu sgn(\phi(u) - \phi(k)) \varphi \, dt \, d\mathcal{H}^{n-1} \geq 0.
\]

\[ \Phi(u, k) = sgn(u - k)(f(u) - f(k)) \]
Notion of weak entropy solution

Initial and boundary conditions

\[ \lim_{t \to 0^+} \int_{\Omega} |u(t, x) - u_0(x)| = 0, \]

\[ \forall \varphi \in L^1(\Sigma_h \setminus \Sigma_{hp}), \varphi \geq 0, \forall k \in \mathbb{R}, \]

\[ \lim_{\tau \to 0^-} \int_{\Sigma_h \setminus \Sigma_{hp}} \mathcal{F}_h(u(\sigma + \tau \nu_h), k)b_h \cdot \nu_h \varphi dt d\mathcal{H}^{n-1} \geq 0, \]

where

\[ \mathcal{F}_h(\tau, k) = \frac{1}{2}(|f_h(\tau) - f_h(0)| - |f_h(k) - f_h(0)| + |f_h(\tau) - f_h(k)|). \]
Uniqueness

Study in the hyperbolic zone

Let $u$ be a weak entropy solution. $\forall k \in \mathbb{R}, \forall \varphi \in C_c^\infty([0, T[ \times \mathbb{R}^n), \varphi \geq 0,$

$$
\int_{Q_h} (|u - k| \partial_t \varphi + |f_h(u) - f_h(k)| b_h \cdot \nabla \varphi - G_h(u, k) \varphi) \, dx \, dt
\geq \begin{aligned}
&\text{ess lim } \tau \to 0^- \int_{\Sigma_h} |f_h(u(\sigma + \tau \nu_h)) - f_h(k)| b_h \cdot \nu_h \varphi(\sigma) \, dt \, d\mathcal{H}^{n-1} \\
&- \text{ess lim } \tau \to 0^- \int_{\Sigma_h \setminus \Sigma_{hp}} |f_h(u(\sigma + \tau \nu_h)) - f_h(0)| b_h \cdot \nu_h \varphi \, dt \, d\mathcal{H}^{n-1} \\
&+ \int_{\Sigma_h \setminus \Sigma_{hp}} |f_h(k) - f_h(0)| b_h \cdot \nu_h \varphi \, dt \, d\mathcal{H}^{n-1}
\end{aligned}
$$

$G_h(u, k) = sgn(u - k)(g(t, x, u) + \text{div} b_h(x) f_h(k))$
Let $u$ be a weak entropy solution. $\forall k \in \mathbb{R}, \forall \varphi \in C_c^\infty(0, T[\times \mathbb{R}^n]), \varphi \geq 0,$

$$\int_{Q_h} (|u - k| \partial_t \varphi + |f_h(u) - f_h(k)| b_h \cdot \nabla \varphi - G_h(u, k) \varphi) \, dx \, dt$$

$$\geq \text{ess lim}_{\tau \to 0^-} \int_{\Sigma_{hp}} |f_h(u(\sigma + \tau \nu_h)) - f_h(k)| b_h \cdot \nu_h \varphi(\sigma) \, dt \, d\mathcal{H}^{n-1}_{\nu_h}$$

$$- \text{ess lim}_{\tau \to 0^-} \int_{\Sigma_h \setminus \Sigma_{hp}} |f_h(u(\sigma + \tau \nu_h)) - f_h(0)| b_h \cdot \nu_h \varphi \, dt \, d\mathcal{H}^{n-1}_{\nu_h}$$

$$+ \int_{\Sigma_h \setminus \Sigma_{hp}} |f_h(k) - f_h(0)| b_h \cdot \nu_h \varphi \, dt \, d\mathcal{H}^{n-1}_{\nu_h}$$

$$G_h(u, k) = \text{sgn}(u - k)(g(t, x, u) + \text{div} b_h(x)f_h(k))$$

- Method of doubling variables $\Rightarrow$ Uniqueness (on the hyperbolic area)
Uniqueness

Study in the parabolic zone

Let $u$ be a weak entropy solution. Then $\partial_t u \in L^2(0, T; V')$, and for any $v \in L^2(0, T; V)$,

$$
\int_0^T \langle \partial_t u, v \rangle \, dt + \int_{Q_p} (\nabla \phi(u) - f_p(u)b_p) \cdot \nabla v \, dx \, dt + \int_{Q_p} g_p(t, x, u)v \, dx \, dt
$$

$$
- \text{ess lim}_{\tau \to 0^-} \int_{\Sigma_{hp}} f_h(u(\sigma + \tau \nu_h)) b_h \cdot \nu_h v \, dt \, d\mathcal{H}^{n-1} = 0
$$
Study in the parabolic zone

Let $u$ be a weak entropy solution. Then $\partial_t u \in L^2(0, T; V')$, and for any $v \in L^2(0, T; V)$,

$$\int_0^T \langle \partial_t u, v \rangle \, dt + \int_{Q_p} (\nabla \phi(u) - f_p(u) b_p) \cdot \nabla v \, dx \, dt + \int_{Q_p} g_p(t, x, u) v \, dx \, dt$$

$$- \text{ess lim}_{\tau \to 0^-} \int_{\Sigma_{hp}} f_h(u(\sigma + \tau \nu_h)) b_h \cdot \nu_h v \, dt \, d\mathcal{H}^{n-1} = 0$$

- Method of doubling the time variable $\Rightarrow$ Uniqueness (on the parabolic zone)
Existence: the viscous problem

Find a bounded and measurable function $u_\mu$ such that

\[
\begin{aligned}
\partial_t u_\mu + \operatorname{div}(b(x)f(u_\mu)) + g(t, x, u_\mu) &= \operatorname{div}(\lambda_\mu \nabla \phi_\mu(u_\mu)) & \text{in } Q, \\
u_\mu &= 0 & \text{on } \Sigma, \\
u_\mu(0, .) &= u_0 & \text{on } \Omega.
\end{aligned}
\]
Existence: the viscous problem

Find a bounded and measurable function $u_\mu$ such that

$$
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u_\mu &= 0 & \text{on } \Sigma, \\
u_\mu(0, .) &= u_0 & \text{on } \Omega.
\end{aligned}
$$

Assumption $(H)$

$$
\sum_{l \in \{h, p\}} (g_i(., ., m) + f_i(m) \text{div} b_i) \leq 0 , \quad \sum_{l \in \{h, p\}} (g_i(., ., M) + f_i(M) \text{div} b_i) \geq 0,
$$

and a.e. on $\Gamma_{hp}$,

$$
(f_p(M)b_p - f_h(M)b_h) \cdot \nu_h \geq 0,
$$

$$
(f_p(m)b_p - f_h(m)b_h) \cdot \nu_h \leq 0.
$$
The viscous problem

Existence and uniqueness

Under \((H)\), \(\exists!\ u_\mu \in W(0, T) \cap L^\infty(Q)\) such that

- \(\forall t \in [0, T], m \leq u_\mu(t, .) \leq M\) a.e. in \(\Omega\)
- \(u_\mu(0, .) = u_0\) a.e. in \(\Omega\)
- For any \(v \in H^1_0(\Omega)\),

\[
\langle \partial_t u_\mu, v \rangle + \int_\Omega ((\lambda_\mu(x) \nabla \phi_\mu(u_\mu) - b(x)f(u_\mu)) \cdot \nabla v + g(t, x, u_\mu)v)dx = 0
\]
The viscous limit

A priori estimates

There exists a constant $C$ independent on $\mu$ such that

$$\|(\lambda \mu)^{1/2} \nabla \hat{\phi}(u_\mu)\|_{L^2(Q)^n}^2 + \|(\mu \lambda \mu)^{1/2} \nabla u_\mu\|_{L^2(Q)^n}^2 \leq C,$$

$$\|\partial_t u_\mu\|_{L^2(0,T;H^{-1}(\Omega))} \leq C,$$

where $\hat{\phi}(x) = \int_0^x \sqrt{\phi'(\tau)} d\tau$.
The viscous limit

- $\phi^{-1}$ is Hölder continuous with an exponent $\tau \in (0, 1)$

**Proposition**

There exists a function $u$ in $L^\infty(Q)$ with $\phi(u)$ in $L^2(0, T; V)$ and such that up to a subsequence when $\mu$ goes to $0^+$,

- $u_\mu \rightharpoonup u$ in $L^\infty(Q)$ weakly *
- $u_\mu \rightarrow u$ in $L^q(Q_p)$ strongly for any finite $q$ and a.e. on $Q_p$

Besides we also have

- $\nabla \phi(u_\mu) \rightharpoonup \nabla \phi(u)$ weakly in $L^2(Q_p)^n$
- $\mu \nabla \phi(u_\mu) \rightarrow 0$, $\lambda_\mu \mu \nabla u_\mu \rightarrow 0$ strongly in $L^2(Q_h)^n$
The viscous limit: notion of process

Theorem (Eymard, Gallouet, Herbin)

Let \((u_m)_{m>0}\) be a sequence of measurable functions on \(\mathcal{O}\) with

\[
\exists \ M > 0, \ \forall m > 0, \ \|u_m\|_{L^\infty(\mathcal{O})} \leq M
\]

Then \(\exists (u_{\varphi(m)})_{m>0}\) and \(\exists \ \pi \in L^\infty([0, 1] \times \mathcal{O})\) such that for all continuous and bounded functions \(h\) on \(\mathcal{O} \times (-M, M)\),

\[
\forall \xi \in L^1(\mathcal{O}), \ \lim_{m \to +\infty} \int_{\mathcal{O}} h(x, u_{\varphi(m)})\xi \, dx = \int_{[0, 1] \times \mathcal{O}} h(x, \pi(\alpha, x))d\alpha \xi \, dx
\]
The viscous limit: notion of process

Theorem (Eymard, Gallouet, Herbin)

Let \((u_m)_{m>0}\) be a sequence of measurable functions on \(\Omega\) with

\[ \exists \ M > 0, \forall m > 0, \|u_m\|_{L^\infty(\Omega)} \leq M \]

Then \(\exists (u_{\varphi(m)})_{m>0}\) and \(\exists \pi \in L^\infty([0, 1] \times \Omega)\) such that for all continuous and bounded functions \(h\) on \(\Omega \times (-M, M)\),

\[ \forall \xi \in L^1(\Omega), \lim_{m \to +\infty} \int_\Omega h(x, u_{\varphi(m)})\xi dx = \int_{[0,1] \times \Omega} h(x, \pi(\alpha, x))d\alpha \xi dx \]

Consequence

\((u_\mu|_{\Omega_h})_{\mu>0} \quad \rightarrow \quad \pi \in L^\infty((0, 1) \times Q_h)\)
The viscous limit on the hyperbolic zone

The process $\pi$ fulfills

- $\forall \varphi \in C^\infty_c(Q_h), \varphi \geq 0,$

$$
\int_0^1 \int_{Q_h} (|\pi - k| \partial_t \varphi + b(x) \Phi(\pi, k) \cdot \nabla \varphi) d\alpha dx dt \\
- \int_0^1 \int_{Q_h} \text{sgn}(\pi - k)(g(t, x, \pi) + \text{div} b(x) f(k)) \varphi d\alpha dx dt \geq 0
$$

- $\text{ess lim}_{\tau \to 0^-} \int_0^1 \int_{\Sigma_h \setminus \Sigma_{hp}} F_h(\pi(\alpha, \sigma + \tau \nu_h), k) b_h \cdot \nu_h \varphi d\alpha d\sigma \geq 0$

- $\text{ess lim}_{t \to 0^+} \int_0^1 \int_{\Omega} |\pi(\alpha, t, x) - u_0(x)| d\alpha dx = 0$
The viscous limit on the hyperbolic zone

The process $\pi$ fulfills

- $\forall \varphi \in C_c^\infty(Q_h)$, $\varphi \geq 0$,\[\int_0^1 \int_{Q_h} (|\pi - k| \partial_t \varphi + b(x) \Phi(\pi, k) \cdot \nabla \varphi) d\alpha dx dt - \int_0^1 \int_{Q_h} \text{sgn}(\pi - k)(g(t, x, \pi) + \text{div} b(x) f(k)) \varphi d\alpha dx dt \geq 0\]

- $\text{ess lim}_{\tau \to 0^-} \int_0^1 \int_{\Sigma_h \setminus \Sigma_{hp}} F_h(\pi(\alpha, \sigma + \tau \mathbf{v}_h), k) b_h \cdot \mathbf{v}_h \varphi d\alpha d\sigma \geq 0$

- $\text{ess lim}_{t \to 0^+} \int_0^1 \int_{Q_h} |\pi(\alpha, t, x) - u_0(x)| d\alpha dx = 0$

Consequence

For a.e. $\alpha$ in $(0, 1)$,

$u(., .) = \pi(\alpha, ., .)$ a.e. on $Q_h$

(with $u_\mu \rightharpoonup u$ in $L^\infty(Q)$ weak − *)
About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

\[
∀ \delta > 0, \quad H_δ(\tau, k) = \left((\text{dist}(\tau, I[0, k]))^2 + \delta^2\right)^{\frac{1}{2}} - \delta
\]

\[
Q_{h,\delta}(\tau, k) = \int_{k}^{\tau} \partial_1 H_δ(\lambda, k)f'_h(\lambda)d\lambda
\]
We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

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\]

\[
Q_{h,\delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k) f_h'(\lambda) d\lambda
\]

We choose \( v = \partial_1 H_\delta(u_\mu, k) \varphi \) in the variational equality fulfilled by \( u_\mu \)

\[
\int_{(0,1) \times Q_h} \left( H_\delta(\pi, k) \partial_t \varphi + Q_{h,\delta}(\pi, k) b_h \cdot \nabla \varphi d\alpha dx dt \right) \geq 0
\]
About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)
  \[ \forall \delta > 0, \quad H_\delta(\tau, k) = \left( (\text{dist}(\tau, I[0, k]))^2 + \delta^2 \right)^{\frac{1}{2}} - \delta \]
  \[ Q_{h, \delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k) f'_h(\lambda) d\lambda \]

- We choose \( v = \partial_1 H_\delta(u_\mu, k) \varphi \) in the variational equality fulfilled by \( u_\mu \)
  \[ \int_{(0,1) \times Q_h} (H_\delta(\pi, k) \partial_t \varphi + Q_{h, \delta}(\pi, k) b_h \cdot \nabla \varphi) d\alpha dx dt \geq 0 \]

- So
  \[ \text{ess lim}_{\tau \to 0^-} \int_{(0,1) \times \Sigma_h \setminus \Sigma_{hp}} Q_{h, \delta}(\pi(\alpha, \sigma + \tau \nu), k) b_h(\bar{\sigma}) \cdot \nu_h \varphi d\alpha dt d\mathcal{H}^{n-1} \geq 0 \]
About the boundary condition

- We use the concept of "boundary entropy-entropy" flux pair (F. Otto)

\[ H_\delta(\tau, k) = \left((\text{dist}(\tau, I[0, k]))^2 + \delta^2\right)^{1/2} - \delta \]

\[ Q_{h,\delta}(\tau, k) = \int_k^\tau \partial_1 H_\delta(\lambda, k)f'_h(\lambda)d\lambda \]

- We choose \( v = \partial_1 H_\delta(u_\mu, k)\varphi \) in the variational equality fulfilled by \( u_\mu \)

\[ \int_{(0,1) \times Q_h} \left( H_\delta(\pi, k)\partial_t \varphi + Q_{h,\delta}(\pi, k)b_h \cdot \nabla \varphi \right) d\alpha dx dt \geq 0 \]

- So

\[ \text{ess lim}_{\tau \to 0^-} \int_{(0,1) \times \Sigma_h \setminus \Sigma_{hp}} Q_{h,\delta}(\pi(\alpha, \sigma + \tau \nu), k)b_h(\overline{\sigma}) \cdot \nu_h \varphi d\alpha dt \mathcal{H}^{n-1} \geq 0 \]

- When \( \delta \) goes to \( 0^+ \)

\[ \text{ess lim}_{\tau \to 0^-} \int_0^1 \int_{\Sigma_h \setminus \Sigma_{hp}} \mathcal{F}_h(\pi(\alpha, \sigma + \tau \nu_h), k)b_h \cdot \nu_h \varphi d\alpha d\sigma \geq 0 \]