UMAP CLASSES OF GROUPS

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ABSTRACT. We call a class $\mathcal{G}$ of MAP topological abelian groups a UMAP-class if it has the following property: if G is an abelian group and τ_1, τ_2 are *distinct* group topologies in G such that (G, τ_1) and (G, τ_2) are in $\mathcal{G}$, then $(G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge$. Several examples of UMAP-classes are discussed. In particular it is shown that the class PMAP of all Polish MAP-groups is a UMAP-class.

The note is based on [**?**, **?**].

1. Definitions

For groups X, Y the set of all group homomorphisms from X to Y is denoted by $Hom(X, Y)$.

For topological groups X, Y the set of all continuous group homomorphisms from X to Y is denoted $CHom(X, Y)$.

A set $\Gamma \subset Hom(X, Y)$ will be called separating, if

$$(x_1, x_2) \in X \times X,\, x_1 \neq x_2 \implies \exists \gamma \in \Gamma,\, \gamma(x_1) \neq \gamma(x_2)\,.$$

For a group X, a topological group Y and a non-empty $\Gamma \subset Hom(X, Y)$ we denote by $\sigma(X, \Gamma)$ the coarsest topology in X with respect to which all members of Γ are continuous. Note that $\sigma(X, \Gamma)$ is a group topology in X; if Y is Hausdorff, then the topology $\sigma(X, \Gamma)$ is Hausdorff iff Γ is separating.

We write:

$$\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}\,.$$

From now on all considered groups will be abelian.

For a group G we write

$$G^a := Hom(G, \mathbb{T})\,.$$

A member of G^a is called *character* and G^a itself is *the algebraic dual* of G.

For a topological group G we write

$$G^\wedge := CHom(G, \mathbb{T})\,.$$

A member of $G^\wedge$ is called *a continuous character* and $G^\wedge$ itself is *the topological dual* of G.

A topological group G is called *maximally almost periodic* for short a MAP-group, if $G^\wedge$ is separating.

For a group G and for a group topology τ in G we write:

$$(G, \tau)^\wedge := CHom\left((G, \tau), \mathbb{T}\right)\,.$$

For a topological group G the topology $\sigma(G, G^\wedge)$ is called *the Bohr topology* of G.

For a group G and for a group topology τ in G we write τ^+ for the Bohr topology of (G, τ). Clearly $\tau^+ \leq \tau$ and τ^+ is a precompact group topology [**?**].

2. UMAP-classes

Let us begin with a statement, which justifies the definition of UMAP-class.

Theorem 2.1. *The class* MAP *of all MAP-groups is not a UMAP-class.*

Proof. Let G be an abelian group and τ be the discrete topology in G. Then $(G, \tau) \in$ MAP. Clearly, $(G, \tau)^\wedge = G^a$. It is evident that $(G, \tau^+)^\wedge = G^a$. Suppose now that that G is infinite. Then $\tau^+ \neq \tau$ (this is not evident, but true). Therefore in an infinite abelian group G we found two distinct group topologies τ and τ^+ such that $(G, \tau)^\wedge = (G, \tau^+)^\wedge$. Hence MAP $\not\subset$UMAP. □

Theorem 2.2. (Glicksberg-Varopoulos) *The class* LCA *of all locally compact Hausdorff topological abelian groups is a UMAP-class.*

Proof. We have LCA $\subset$MAP by Peter-Weil-Van Kampen theorem. Let us see that LCA $\subset$UMAP. Let G be an abelian group and τ_1, τ_2 be *distinct locally compact Hausdorff* group topologies in G. Let us see that then $(G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge$. Suppose $(G, \tau_1)^\wedge = (G, \tau_2)^\wedge$. Then $\tau_1^+ = \tau_2^+$ By Glicksberg theorem the topologies τ_1, τ_2 and τ_1^+ have the same collections of compact sets. From this, since a locally compact Hausdorff space is a k-space, we get that $\tau_2 = \tau_1$, a contradiction. □

Remark 2.3. A topological group (G, τ) *respects compactness* (J.Trigos-Arrieta) if τ^+-compact sets are τ-compact as well.

(a) Let RES − MAPK be the class of MAP-groups which respect compactness and which are k-spaces. Then RES − MAPK is a UMAP-class.
(Proof of Theorem **??** works).

(b) (W. Banaszczyk and E. Martín-Peinador) Let NUC be the class of nuclear groups [**?**] and NUCK be the class of nuclear which are k-spaces. Then NUCK is a UMAP-class.
(Use (a) and the fact that nuclear groups respect compactness [**?**].)

(c) (L. Aussenhofer) Let SCH be the class of locally quasi-convex Schwartz groups and SCHK be the class of locally quasi-convex Schwartz groups which are k-spaces. Then SCHK is a UMAP-class.
(Use (a) and the fact that locally quasi-convex Schwartz groups groups respect compactness [**?**].)

Theorem 2.4. (Comfort-Ross, [**?**, Corollary 1.4]) *The class* PCA *of all precompact Hausdorff topological abelian groups is a UMAP class.*

Theorem 2.5. ([**?**]) *Let* BTM *be the class defined as follows: a topological abelian group G belongs to* BTM *if*

- *G is locally quasi-convex Hausdorff* [**?**],
- $\mathrm{Card}(G^\wedge) \leq \aleph_0$,
- *there exists a natural number $n \geq 2$ such that $nx = 0, \forall x \in G$.*

Then BTM $\subset$ PCA *and hence,* BTM *is a UMAP-class.*

Theorem 2.6. *The class* LPA *of all locally precompact Hausdorff topological abelian groups is not a UMAP-class.*

Proof. The same proof as that of Theorem **??**. Note that the discrete topology τ is locally precompact and the topology τ^+ is precompact, hence it is locally precompact. □

A topological space is called *Polish* if it is homeomorphic to a complete separable metric space. A topological group is Polish if it as a topological space is Polish.
We need the following statement.

Theorem 2.7. [**?**, Satz 10] *Let X, Y be Polish abelian groups and $f : X \to Y$ be a group homomorphism whose graph $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. Then f is continuous.*

Corollary 2.8. *Let G be an abelian group and τ_1, τ_2 be the distinct Polish group topologies in G. Then $\tau_1 \cap \tau_2$ is a T_1-topology, which is not a Hausdorff topology.*
In particular, $\tau_1 \cap \tau_2$ is not a group topology.

Proof. Suppose that $\sigma := \tau_1 \cap \tau_2$ is a Hausdorff topology. Let $f : G \to G$ be the identity mapping. Its graph $\Delta_G = \{(x, x) : x \in G\}$ is closed in $(G \times G, \sigma \times \sigma)$. Since $\sigma \times \sigma \leq \tau_1 \times \tau_2$, it follows that $\Delta_G = \{(x, x) : x \in G\}$ is closed in $(G \times G, \tau_1 \times \tau_2)$. Therefore, f as a mapping from (G, τ_1) to (G, τ_2) has closed graph. Then by Theorem **??**, we get that f is (τ_1, τ_2)-continuous. Consequently $\tau_2 \leq \tau_1$. In a similar way we get that $\tau_1 \leq \tau_2$. Hence, $\tau_2 = \tau_1$, a contradiction. □

Theorem 2.9. *The class* PMAP *of all Polish MAP-groups is a UMAP class.*

Proof. Let G be an infinite abelian group and τ_1, τ_2 be *distinct Polish* MAP group topologies in G. Let us see that then $(G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge$. Suppose $(G, \tau_1)^\wedge = (G, \tau_2)^\wedge$. Then $\sigma := \tau_1^+ = \tau_2^+$. Clearly $\sigma \leq \tau_1$ and $\sigma \leq \tau_2$. Since (G, τ_1) is MAP, σ is a Hausdorff topology. As $\sigma \leq \tau_1 \cap \tau_2$, we get that $\tau_1 \cap \tau_2$ is a Hausdorff topology, but this contradicts to Corollary **??**. □

Question 2.10. *Let* CMMAP *be the class of all complete metrizable MAP groups. Is then* CMMAP *a UMAP class?*

I have not a counterexample.

Theorem 2.11. [**?**, Corollaire 3, p.1.37] *Let X, Y be complete metrizable topological topological vector spaces over the same non-discrete valued division ring* $\mathbf{K}$ *and $f : X \to Y$ be a* $\mathbf{K}$*-linear mapping whose graph $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. Then f is continuous.*

Corollary 2.12. *Let G be an abelian group and τ_1, τ_2 be the distinct complete metrizable group topologies in G. Assume further that (G, τ_1) and (G, τ_2) admit a structure of topological vector space over the same non-discrete valued division ring* $\mathbf{K}$*. Then $\tau_1 \cap \tau_2$ is a T_1-topology, which is not a Hausdorff topology.*
In particular, $\tau_1 \cap \tau_2$ is not a group topology.

Proof. The same as that off Corollary **??**: use now Theorem **??** and take into account that the identity mapping is $\mathbb{K}$-linear. □

Theorem 2.13. *Let* $\mathbf{K}$ *be a non-discrete valued division ring and* $\mathbf{K}$-TVSCMMAP *be the class defined as follows:* $G \in \mathbf{K}$-TVSCMMAP *if* $G \in$ CMMAP *and G admits a structure of a topological vector space over* $\mathbf{K}$.
Then $\mathbf{K}$-TVSCMMAP *is a UMAP class.*

Proof. Proof is similar to that of Theorem **??**: instead of Corollary **??** use Corollary **??**. □

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