UMAP CLASSES OF GROUPS

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Abstract. We call a class \( G \) of MAP topological abelian groups a UMAP-class if it has the following property: if \( G \) is an abelian group and \( \tau_1, \tau_2 \) are distinct group topologies in \( G \) such that \((G, \tau_1)\) and \((G, \tau_2)\) are in \( G \), then \((G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge\). Several examples of UMAP-classes are discussed. In particular it is shown that the class PMAP of all Polish MAP-groups is a UMAP-class.

The note is based on [? , ?].

1. Definitions

For groups \( X, Y \) the set of all group homomorphisms from \( X \) to \( Y \) is denoted by \( \text{Hom}(X, Y) \).

For topological groups \( X, Y \) the set of all continuous group homomorphisms from \( X \) to \( Y \) is denoted \( \text{CHom}(X, Y) \).

A set \( \Gamma \subset \text{Hom}(X, Y) \) will be called separating, if

\[(x_1, x_2) \in X \times X, x_1 \neq x_2 \Rightarrow \exists \gamma \in \Gamma, \gamma(x_1) \neq \gamma(x_2).\]

For a group \( X \), a topological group \( Y \) and a non-empty \( \Gamma \subset \text{Hom}(X, Y) \) we denote by \( \sigma(X, \Gamma) \) the coarsest topology in \( X \) with respect to which all members of \( \Gamma \) are continuous. Note that \( \sigma(X, \Gamma) \) is a group topology in \( X \); if \( Y \) is Hausdorff, then the topology \( \sigma(X, \Gamma) \) is Hausdorff iff \( \Gamma \) is separating.

We write:

\[ \mathbb{T} := \{ t \in \mathbb{C} : |t| = 1 \}. \]

From now on all considered groups will be abelian.

For a group \( G \) we write

\[ G^a := \text{Hom}(G, \mathbb{T}). \]

A member of \( G^a \) is called character and \( G^a \) itself is the algebraic dual of \( G \).

For a topological group \( G \) we write

\[ G^\wedge := \text{CHom}(G, \mathbb{T}). \]

A member of \( G^\wedge \) is called a continuous character and \( G^\wedge \) itself is the topological dual of \( G \).

A topological group \( G \) is called maximally almost periodic for short a MAP-group, if \( G^\wedge \) is separating.

For a group \( G \) and for a group topology \( \tau \) in \( G \) we write:

\[ (G, \tau)^\wedge := \text{CHom}((G, \tau), \mathbb{T}). \]

For a topological group \( G \) the topology \( \sigma(G, G^\wedge) \) is called the Bohr topology of \( G \).

For a group \( G \) and for a group topology \( \tau \) in \( G \) we write \( \tau^+ \) for the Bohr topology of \( (G, \tau) \). Clearly \( \tau^+ \leq \tau \) and \( \tau^+ \) is a precompact group topology [?].
2. UMAP-classes

Let us begin with a statement, which justifies the definition of UMAP-class.

**Theorem 2.1.** The class MAP of all MAP-groups is not a UMAP-class.

**Proof.** Let $G$ be an abelian group and $\tau$ be the discrete topology in $G$. Then $(G, \tau) \in \text{MAP}$. Clearly, $(G, \tau)^\wedge = G^a$. It is evident that $(G, \tau^\wedge)^\wedge = G^a$. Suppose now that $G$ is infinite. Then $\tau^\wedge \neq \tau$ (this is not evident, but true). Therefore in an infinite abelian group $G$ we found two distinct group topologies $\tau$ and $\tau^\wedge$ such that $(G, \tau)^\wedge = (G, \tau^\wedge)^\wedge$. Hence MAP $\not\subseteq \text{UMAP}$. □

**Theorem 2.2.** (Glicksberg-Varopoulos) The class LCA of all locally compact Hausdorff topological abelian groups is a UMAP-class.

**Proof.** We have LCA $\subseteq$ MAP by Peter-Weil-Van Kampen theorem. Let us see that LCA $\subseteq$ UMAP. Let $G$ be an abelian group and $\tau_1, \tau_2$ be distinct locally compact Hausdorff group topologies in $G$. Let us see that then $(G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge$. Suppose $(G, \tau_1)^\wedge = (G, \tau_2)^\wedge$. Then $\tau_1^+ = \tau_2^+$. By Glicksberg theorem the topologies $\tau_1, \tau_2$ and $\tau_1^+$ have the same collections of compact sets. From this, since a locally compact Hausdorff space is a k-space, we get that $\tau_2 = \tau_1$, a contradiction. □

**Remark 2.3.** A topological group $(G, \tau)$ respects compactness (J. Trigos-Arrieta) if $\tau^+$-compact sets are $\tau$-compact as well.

(a) Let RES - MAPK be the class of MAP-groups which respect compactness and which are k-spaces. Then RES - MAPK is a UMAP-class.

(Proof of Theorem ?? works).

(b) W. Banaszczyk and E. Martín-Peinador Let NUC be the class of nuclear groups [?] and NUCK be the class of nuclear which are k-spaces. Then NUCK is a UMAP-class.

(Use (a) and the fact that nuclear groups respect compactness [?].)

(c) L. Aussenhofer Let SCH be the class of locally quasi-convex Schwartz groups and SCHK be the class of locally quasi-convex Schwartz groups which are k-spaces. Then SCHK is a UMAP-class.

(Use (a) and the fact that locally quasi-convex Schwartz groups groups respect compactness [?].)

**Theorem 2.4.** (Comfort-Ross, [?, Corollary 1.4]) The class PCA of all precompact Hausdorff topological abelian groups is a UMAP-class.

**Theorem 2.5.** ([?] Let BTM be the class defined as follows: a topological abelian group $G$ belongs to BTM if

- $G$ is locally quasi-convex Hausdorff [?],
- $\text{Card}(G^\wedge) \leq 8_0$,
- there exists a natural number $n \geq 2$ such that $nx = 0, \forall x \in G$.

Then BTM $\subseteq$ PCA and hence, BTM is a UMAP-class.

**Theorem 2.6.** The class LPA of all locally precompact Hausdorff topological abelian groups is not a UMAP-class.

**Proof.** The same proof as that of Theorem ??.

A topological space is called Polish if it is homeomorphic to a complete separable metric space. A topological group is Polish if it as a topological space is Polish.

We need the following statement.

**Theorem 2.7.** [?, Satz 10] Let $X, Y$ be Polish abelian groups and $f : X \rightarrow Y$ be a group homomorphism whose graph $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. Then $f$ is continuous.

**Corollary 2.8.** Let $G$ be an abelian group and $\tau_1, \tau_2$ be the distinct Polish group topologies in $G$. Then $\tau_1 \cap \tau_2$ is a $T_1$-topology, which is not a Hausdorff topology.

In particular, $\tau_1 \cap \tau_2$ is not a group topology.

**Proof.** Suppose that $\sigma := \tau_1 \cap \tau_2$ is a Hausdorff topology. Let $f : G \rightarrow G$ be the identity mapping. Its graph $\Delta_G = \{(x,x) : x \in G\}$ is closed in $(G \times G, \sigma \times \sigma)$. Since $\sigma \times \sigma \leq \tau_1 \times \tau_2$, it follows that $\Delta_G = \{(x,x) : x \in G\}$ is closed in $(G \times G, \tau_1 \times \tau_2)$. Therefore, $f$ as a mapping from $(G, \tau_1)$ to $(G, \tau_2)$ has closed graph. Then by Theorem ??, we get that $f$ is $(\tau_1, \tau_2)$-continuous. Consequently $\tau_2 \leq \tau_1$. In a similar way we get that $\tau_1 \leq \tau_2$. Hence, $\tau_2 = \tau_1$, a contradiction. □
The class PMAP of all Polish MAP-groups is a UMAP class.

Proof. Let $G$ be an infinite abelian group and $\tau_1, \tau_2$ be distinct Polish MAP group topologies in $G$. Let us see that then $(G, \tau_1)^\wedge \neq (G, \tau_2)^\wedge$. Suppose $(G, \tau_1)^\wedge = (G, \tau_2)^\wedge$. Then $\sigma := \tau_1^\wedge = \tau_2^\wedge$. Clearly $\sigma \leq \tau_1$ and $\sigma \leq \tau_2$. Since $(G, \tau_1)$ is MAP, $\sigma$ is a Hausdorff topology. As $\sigma \leq \tau_1 \cap \tau_2$, we get that $\tau_1 \cap \tau_2$ is a Hausdorff topology, but this contradicts to Corollary 2.12.

Question 2.10. Let CMMAP be the class of all complete metrizable MAP groups. Is then CMMAP a UMAP class?

I have not a counterexample.

Theorem 2.11. Let $X, Y$ be complete metrizable topological vector spaces over the same non-discrete valued division ring $K$ and $f : X \to Y$ be a $K$-linear mapping whose graph $\{(x, f(x)) : x \in X\}$ is a closed subset of $X \times Y$. Then $f$ is continuous.

Corollary 2.12. Let $G$ be an abelian group and $\tau_1, \tau_2$ be the distinct complete metrizable group topologies in $G$. Assume further that $(G, \tau_1)$ and $(G, \tau_2)$ admit a structure of topological vector space over the same non-discrete valued division ring $K$. Then $\tau_1 \cap \tau_2$ is a $T_1$-topology, which is not a Hausdorff topology.

In particular, $\tau_1 \cap \tau_2$ is not a group topology.

Proof. The same as that off Corollary 2.12: use now Theorem 2.11 and take into account that the identity mapping is $K$-linear.

Theorem 2.13. Let $K$ be a non-discrete valued division ring and $K$-TVSCMMAP be the class defined as follows: $G \in K$-TVSCMMAP if $G \in$ CMMAP and $G$ admits a structure of a topological vector space over $K$.

Then $K$-TVSCMMAP is a UMAP class.

Proof. Proof is similar to that of Theorem 2.11: instead of Corollary 2.12 use Corollary 2.12.

Acknowledgements. This is the text of a talk given in March 25th, 2010 in ‘Jornada sobre grupos topológicos’ organized by Departamento Geometría y Topología of the UCM in collaboration with IMI.

The author is grateful to both Institutions for their support. Also to Professors L. Aussenhofer, E. Martín-Peinador and L. Ribes for their useful comments.

The author was also partially supported by grant MTM2009-14409-C02-02.

References