QUALITATIVE METHODS FOR NONLINEAR DIFFUSION EQUATIONS

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INTRODUCTION

A considerable effort has been devoted during the last years to the study of nonlinear diffusion equations or reaction-diffusion systems. A typical example is the following semilinear parabolic problem

$$
(0.1) \qquad \begin{aligned} u_t - a\,\Delta u &= f(u,v) \quad \text{in } \Omega \times (0,T)\ , \\ v_t - b\,\Delta v &= g(u,v) \quad \text{in } \Omega \times (0,T)\ , \end{aligned}
$$

where Ω is a smooth not necessarily bounded domain in $\mathbb{R}^N$, $a,b,T > 0$ are real numbers and f and g are sufficiently smooth functions. Of course, the above system should be complemented with suitable boundary and initial conditions. These systems, where the interaction between diffusion (measured in (0.1) by the diffusion coefficients $a, b > 0$) and the reaction terms f and g gives rise to an extremely rich phenomenology, arise in many applications, giving qualitative models for a number of physical and biological phenomena. We can mention, among many others, chemical reactions (e.g., Brusselator), combustion theory, population dynamics (predator-prey interactions, competition, etc.), nerve conduction (Fitzhugh-Nagumo system), morphogenesis (activator-inhibitor interaction), superconductivity, etc. Cf. the book by Smoller [106] for more information and references (cf. also the lecture notes by Fife [54]).

One of the main mathematical problems posed by reaction-diffusion systems is the asymptotic behavior of solutions, i.e. to investigate what happens with solutions when the time t goes to infinity. (Clearly, this implies the existence and uniqueness of solutions). From this point of view it is particularly interesting to study the existence and (maybe) multiplicity of solutions of the stationary problem associated to (0.1), namely the elliptic system

$$(0.2)\qquad \begin{aligned} -a\,\Delta u &= f(u,v) \quad \text{in } \Omega \,, \\ -b\,\Delta v &= g(u,v) \quad \text{in } \Omega \,, \end{aligned}$$

together with the corresponding boundary conditions. Indeed, in many interesting cases, the solutions of the evolution problem converge to one of the solutions (maybe unique) of (0.2). Therefore, the study of the stability of stationary solutions is one of the most important problems in the theory. This is not the only possibility, and the reader can found a good survey of many possible situations (travelling waves, spiral patterns, λ-ω systems, periodic solutions, etc.) in a paper by Fife [55]. However, a great deal of work has been devoted to the stability of solutions of systems as (0.2). It is well-known that exchanges of stability when a parameter varies are closely related with bifurcation phenomena, i.e., with changes in the structure of the solution set of (0.2) when the parameter crosses a critical value. We do not intend to treat this question here, but the reader can found some references in the Comments.

Our aim in these notes is to show how some methods of Nonlinear Functional Analysis can be applied to the study of existence and multiplicity of some systems like (0.2). We do not pretend to give an exhaustive survey of the different mathematical tools, and a similar consideration can be made concerning the variety of problems we consider. On the contrary, our intention is to exhibit how some mathematical tools (as sub and supersolutions, degree theory, critical point theory, etc.) work in a few examples which seem interesting and useful for further extensions. However, this overview of qualitative methods

(comparison methods, topological methods, variational methods) is by no means complete: to mention only two important instances, local bifurcation methods (as, e.g., the Liapunov-Schmidt method [34], [64], [88]) and Conley index ([38], [106]) are not considered. On the other hand, it is very well-known that the methods in these notes have other interesting applications: for example, comparison arguments involving sub and supersolutions (cf., e.g., [78], [79], [82], [86]), topological degree [95], local inversion techniques [47] and the fixed point index [10] have applications in the study of the stability of steady-state solutions, and sub and supersolutions [56] and degree theory [23] have been used in the study of travelling waves.

A particular attention is payed to the case of a single equation. Of course, it seems reasonable to deal with this case before to treat full systems. But there is also another reason for that: by using a decoupling technique, some systems can be reduced to a single equation with a nonlocal perturbation, and some of the proofs are very similar to the case of a single equation. Cf. Comments for more details.

Very often, we are only interested in positive solutions when dealing with problems arising in physical applications. This interest shall appear in several ways along these notes, in particular we will use positivity arguments concerning the natural order in the corresponding function spaces and some related tools, as the Maximum Principle. We also emphasize the use of a cone of positive functions, replacing then the topological degree by the fixed point index. Cf. the excellent survey by Amann [10] (and [74]) for a very complete exposition of this kind of arguments.

The first chapter contains an exposition of the method of sub and supersolutions and some applications. Section 1.1 treats the case of a semilinear elliptic equation, and the method is applied in Section 1.2 to the nonlinear eigenvalue problem

$$(1.1)\qquad \begin{aligned} -\Delta u + f(u) &= \lambda u \quad \text{in } \Omega \,, \\ u &= 0 \quad \text{on } \partial\Omega \,, \end{aligned}$$

where λ is a real parameter and f behaves, roughly speaking, as u^3, to obtain existence of positive solutions. A uniqueness theorem is also given. Section 1.3 includes an extension of the method to systems without monotonicity assumptions, and some applications.

Chapter 2 is devoted to topological methods, especially the Leray-Schauder degree and some of its applications. Section 2.1 gives a brief survey (without proofs) of the main properties of the degree, which are applied in Section 2.2 to obtain the exact number of solutions of (0.3) for some values of λ. Section 2.3 includes the global bifurcation theorem by Rabinowitz, and some applications, and Section 2.4 treats the case without bifurcation. The remainder of this chapter deals with positive solutions: Section 2.5 gives some global bifurcation theorems for positive solutions, Section 2.6 contains a brief survey of the main properties of the fixed point index, and in Section 2.7 a global existence theorem for positive solutions is used, together with *a priori* estimates, to prove existence of positive solutions of a system arising in chemical reactions.

Two different methods are introduced in Chapter 3. The model example (0.3) is revisited, this time by using a continuation method based on a local inversion theorem by Crandall-Rabinowitz. A few variants of (0.3) are also considered. Some indications about the application of similar methods to the study of 'bending points' in the case without bifurcation are given in Section 3.2. Finally, in Section 3.3 we use variational methods to improve under additional hypotheses the results in Section 3.2.

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1. THE METHOD OF SUB AND SUPERSOLUTIONS AND APPLICATIONS

This chapter is mainly concerned with the method of sub and supersolutions and some of its applications.

The first paragraph treats the case of a nonlinear elliptic equation. Even if these results are very well-known, we find convenient to treat them here with some detail not only because they are the best illustration for the general case of systems, but also because, as it was indicated in the Introduction, existence and multiplicity results for some systems can be obtained by reformulating the problems in such a way that they become mild perturbations of the nonlinear eigenvalue problem we consider in Section 1.2. For this problem we study existence and uniqueness of positive solutions, together with some interesting additional qualitative properties, mostly concerning the smoothness of the branch of positive solutions we obtain. Finally, the last paragraph considers the extension of the method of sub and supersolutions to systems, and includes a rather general existence theorem. Some applications are also given.

1.1. AN EXISTENCE THEOREM FOR SEMILINEAR ELLIPTIC EQUATIONS.

Our main interest here is to show how the method of sub and supersolutions works in a concrete but very illustrative instance, namely the case of a mildly nonlinear elliptic equation. This method, which is in some sense reminiscent of Perron's method for solving Dirichlet's problem by using subharmonic functions, not only gives an existence proof but also provides an iterative scheme to approximate solutions by sequences converging monotonically to these solutions.

Now we consider the nonlinear elliptic problem

(1.1) $$-\Delta u = f(x,u) \quad \text{in } \Omega$$

(1.2) $$u = 0 \quad \text{on } \partial\Omega ,$$

where Ω is, as in all which follows, a connected bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$ with a very smooth boundary $\partial\Omega$; in particular, we assume that $\partial\Omega$ is $C^{2,\alpha}$, $0 < \alpha < 1$, but the reader may think about $\partial\Omega$ as being C^3 or even C^∞.

We also assume that the function $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies

(1.3) $\quad f \in C^1(\bar{\Omega} \times \mathbb{R})$,

(1.4) $\quad$ For any $x \in \bar{\Omega}$, $f(x,u)$ is increasing in u .

Remark 1.1: The assumption (1.4) can be weakened, it is sufficient that there exists $M > 0$ such that $\left|\frac{\partial f(x,u)}{\partial u}\right| \leq M$. Indeed, in this case problem (1.1), (1.2) is equivalent to the problem

$$- \Delta u + M u = \tilde{f}(x,u) \quad \text{in } \Omega ,$$
$$u = 0 \qquad \partial\Omega ,$$

where $\tilde{f}(x,u) = f(x,u) + M u$ satisfies (1.3), (1.4). Moreover, we shall see below that (1.4) should only be satisfied on some bounded interval in u , and then (1.4) will follow from (1.3) (Cf. Remark 1.5). On the other hand, (1.3) can also be replaced (cf. [2],[103]) by the more general condition that f is locally C^α, $0 < \alpha < 1$, and there is a $M > 0$ such that (1.4) is satisfied for $f(x,u) + M u$ instead of f .

First, we give the definition of sub and supersolutions for our problem.

Definition: A sufficiently smooth function (i.e., in $C^2(\Omega) \cap C(\bar{\Omega})$) u_0 (resp. u^0) is called a subsolution (resp. a supersolution) of (1.1) (1.2) if we have

(1.5) $\quad - \Delta u_0 - f(x,u_0) \leq 0 \leq - \Delta u^0 - f(x,u^0) \quad$ in Ω ,

(1.6) $\quad u_0 \leq 0 \leq u^0 \quad$ on $\partial\Omega$,

where inequalities are to be understood pointwise.

Our aim is to prove the following existence result.

Theorem 1.1: Let u_0 (resp. u^0) be a subsolution (resp. a supersolution) of (1.1) (1.2) such that $u_0 \leq u^0$ (in Ω). Then there exist $\underline{u}$ and $\bar{u}$ solutions of (1.1) (1.2) satisfying $u_0 \leq \underline{u} \leq \bar{u} \leq u^0$. Moreover, $\underline{u}$ (resp. $\bar{u}$) is minimal (resp. maximal) in the sense that if u is a solution such that $u_0 \leq u \leq u^0$, then $\underline{u} \leq u \leq \bar{u}$.

A few remarks seem to be in order before giving the proof of Theorem 1.1. First, this is not the most general version of this kind of existence results, it is even the contrary, so as to say. Our main interest here is to demonstrate how the method works, and it seems convenient from this point of view to get rid of irrelevant technical difficulties. In fact, the theorem can be extended to much more general differential operators and boundary conditions (Cf. Comments). There is also an "abstract" version in ordered Banach spaces (cf. [6],[18]).

First, let us point out that both the minimal and the maximal solutions $\underline{u}$ and $\bar{u}$ may coincide, and this is obviously the case when the solution is unique. (By the way, here solution means classical solution, i.e., a function in $C^2(\Omega) \cap C(\bar{\Omega})$). But here, contrarily to other different situations (as, e.g., Banach's fixed point theorem), Theorem 1.1 has no implications concerning uniqueness, which should be proved (when there is !) by a different argument, exploiting the particular features of the corresponding problem, Cf. below, e.g., Theorem 1.3.

On the other hand, this method does not provide all the solutions of (1.1) (1.2). One way to show this is to exhibit an equation with three or more solutions. But there are also quite general, and interesting, results due to Sattinger [100],[102] and Amann [10] saying that if the solution obtained in Theorem 1.1 is unique, then it is stable or, otherwise stated, that an unstable solution cannot be obtained by using our iterative scheme. Cf. Comments.

Remark 1.2: The fact that u_0 is a subsolution and u^0 a supersolution does not imply $u_0 \leq u^0$. As it is obvious that any solution is also a subsolution (and a supersolution), any problem having more than one solution shows that this is not the case. But it is not difficult (cf., e.g., below paragraph 1.2) to find examples of strict subsolutions above a (may be strict) supersolution. (Here strict means strict inequality somewhere in (1.5) (1.6)).

Remark 1.1: The theorem is false for a subsolution u_0 and a supersolution u^0 such that $u_0 > u^0$ as shows the counterexample in [36, pp. 653-654]. Moreover, in this case, the problem has no solution at all.

Theorem 1.1 is proved by reducing the problem of finding solutions of (1.1) (1.2) to the equivalent problem of showing the existence of fixed points of a suitable nonlinear operator T defined on a function space adequately chosen. What we intend to show is that in this case we should take care not only of the topological (continuity, compactness) properties of T but also use the fact that T preserves the natural order in the corresponding function space.

Now we define our nonlinear operator T in such a way that its fixed points are (classical) solutions of (1.1) (1.2). For that, if Ω is a $C^{2,\alpha}$ domain with $0 < \alpha < 1$ we define a nonlinear operator

$$T : C^{1,\alpha}(\bar{\Omega}) \longrightarrow C^{1,\alpha}(\bar{\Omega}) ,$$
$$u \longmapsto Tu ,$$

where, for $u \in C^{1,\alpha}(\bar{\Omega})$, Tu is the (unique) solution of the linear problem

(1.7) $$-\Delta Tu = f(x,u) \quad \text{in } \Omega ,$$

(1.8) $$Tu = 0 \quad \text{on } \partial\Omega .$$

(We employ the usual notations for the spaces of Hölder continuous functions, $C^{k,\alpha}(\bar{\Omega})$, $k \geq 0$ integer, $0 < \alpha < 1$, and Sobolev spaces $W^{k,p}(\Omega)$, $k \geq 0$, $1 \leq p \leq \infty$, and the corresponding norms; we also use the space $L^{\infty}(\Omega)$ and its norm. Cf. [58]).

First, the operator T is well-defined. Indeed, if $u \in C^{1,\alpha}(\bar{\Omega})$, $f(x,u) \in C^{1}(\bar{\Omega})$ and by the well-known C^{α} theory for linear equations

(Schauder Theory) (cf., e.g. [58],[75]) there is a unique solution $Tu \in C^{2,\alpha}(\bar{\Omega})$ for (1.7) (1.8). Moreover, it is an easy task to prove, by using Schauder's estimates, that T is compact, i.e., that T is continuous and that, for B bounded, $T(B)$ is relatively compact. We shall return after to the links between order, compactness and the choice of the function space: for the moment, let us say only that, for this problem, it is still possible to work in $C(\bar{\Omega})$, $C^{1}(\bar{\Omega})$ or even $L^{2}(\Omega)$.

The following results, which follow easily from the Maximum Principle, show that T behaves "nicely" with respect to the natural order in the function space. Here $u \leq v$ means $u(x) \leq v(x)$ for any $x \in \Omega$.

Lemma 1.1: T is order-preserving, i.e., $u \leq v$ implies $Tu \leq Tv$.

Proof: From (1.7) (1.8) for u and v it follows by (1.4)

$$-\Delta(Tv - Tu) = f(x,v) - f(x,u) \geq 0 \quad \text{in } \Omega ,$$
$$Tv - Tu = 0 \quad \text{on } \partial\Omega ,$$

and by the Maximum Principle, $Tv - Tu \geq 0$.

Lemma 1.2: If u_0 is a subsolution then $u_0 \leq Tu_0$.

Proof: By the definition of T

$$-\Delta Tu_0 = f(x,u_0) \quad \text{in } \Omega ,$$
$$Tu_0 = 0 \quad \text{on } \partial\Omega ,$$

and this, together with (1.5) (1.6) gives

$$-\Delta(Tu_0 - u_0) = f(x,u_0) + \Delta u_0 \geq 0 \quad \text{in } \Omega ,$$
$$Tu_0 - u_0 \geq 0 \quad \text{on } \partial\Omega ,$$

and again by the Maximum Principle, $Tu_0 - u_0 \geq 0$.

Lemma 1.3: If u^0 is a supersolution, then $u^0 \geq Tu^0$.

<u>Proof</u>: As for Lemma 1.2.

<u>Proof of Theorem 1.1</u>: We define by iteration the sequences

$$u_1 = Tu_0, \quad \ldots, \quad u_n = Tu_{n-1}, \ldots$$
$$u^1 = Tu^0, \quad \ldots, \quad u^n = Tu^{n-1}, \ldots$$

It follows from Lemmas 1.1-1.2 that u_n is increasing, u^n is decreasing, and

(1.9) $$u_0 \le u_1 \le \ldots \le u_n \le \ldots \le u^n \le \ldots \le u^1 \le u^0 .$$

Hence, from (1.3) we obtain

(1.10) $$\|u_n\|_{L^\infty(\Omega)} \le c \qquad \|u^n\|_{L^\infty(\Omega)} \le c$$

where c denotes (as in all which follows, if not otherwise stated) a constant depending on Ω, N, f, etc. but not on n. Now from (1.10) and the L^p estimates of Agmon-Douglis-Nirenberg [1] we have, for any $1 \le p < \infty$

$$\|u_n\|_{W^{2,p}(\Omega)} = \|Tu_{n-1}\|_{W^{2,p}(\Omega)} \le c\,\|f(x,u_{n-1})\|_{L^p(\Omega)} \le c$$

and analogously

$$\|u^n\|_{W^{2,p}(\Omega)} \le c .$$

Hence there exist two subsequences, which we again denote by u_n and u^n, such that for $p > 1$

$$u_n \longrightarrow \underline{u} \quad \text{in} \quad W^{2,p}(\Omega) \quad \text{weakly}$$
$$u^n \longrightarrow \bar{u} \quad \text{in} \quad W^{2,p}(\Omega) \quad \text{weakly}$$

and by Morrey's Lemma (the immersion $W^{2,p}(\Omega) \longleftrightarrow C^{1,\gamma}(\bar{\Omega})$ is compact

for $p > N$, $\gamma = 1 - \frac{N}{p}$)

$$u_n \longrightarrow \underline{u} \quad \text{in} \quad C^{1,\gamma}(\bar{\Omega})$$
$$u^n \longrightarrow \bar{u} \quad \text{in} \quad C^{1,\gamma}(\bar{\Omega}) .$$

(Actually they converge in $C^{1,\delta}(\bar{\Omega})$ for any $0 < \delta < 1$). On the other hand, the monotonicity in (1.9) implies that the full sequences also converge. Since T is continuous

$$\lim T u_{n-1} = \lim u_n = \underline{u} = T\underline{u}$$

and hence $\underline{u}$ is a fixed point of T. We prove in the same way that $\bar{u} = T\bar{u}$, and it follows immediately from (1.9) that $\underline{u} \le \bar{u}$.

Now, let z be a solution such that $u_0 \le z \le u^0$. Clearly $z = Tz$, and by Lemmas 1.1-1.2, $u_n \le z \le u^n$. Hence $\underline{u} \le z \le \bar{u}$, and this ends the proof.

<u>Remark 1.4</u>) The monotone sequences u_n and u^n in the proof of Theorem 1.1 converge, respectively, to $\underline{u}$ and $\bar{u}$ in $C^{1,\gamma}(\bar{\Omega})$. In fact, they converge in $C^{2,\alpha}(\bar{\Omega})$. From

$$-\Delta(T\underline{u} - Tu_n) = f(x,\underline{u}) - f(x,u_n) \quad \text{in} \quad \Omega \; ,$$
$$T\underline{u} - Tu_n = 0 \qquad\qquad \text{on} \quad \partial\Omega \; ,$$

it follows by Schauder's estimates

$$\|T\underline{u} - Tu_n\|_{2,\alpha} \le c\,\|f(x,\underline{u}) - f(x,u_n)\|_{\alpha} .$$

Since $u_n \longrightarrow \underline{u}$ in $C^{1,\alpha}(\bar{\Omega})$, the result follows easily.

<u>Remark 1.5</u>) The proof of Theorem 1.1 shows clearly that all we need is (1.4) on the bounded interval $[\min u_0, \max u^0]$, and this is an easy corollary of (1.3). (In fact, we can assume f locally Lipschitz in u). Hence (1.4) is superfluous. This clarifies the observations in Remark 1.1 by showing that we do not need to assume f increasing.

This makes a big difference with respect to the case of systems (cf. Section 1.3).

It is clear that the method relies, in an essential way, on the Maximum Principle from one side (order properties) and on the classical C^{α} (Schauder Theory, cf. [58],[75]) and L^p (Agmon-Douglis-Nirenberg [1]) regularity results on the other. This implies that in the case of elliptic equations, its scope is reduced to second order differential operators, roughly speaking. Cf. the Comments for some of the possible extensions of the method.

1.2. POSITIVE SOLUTIONS FOR A NONLINEAR EIGENVALUE PROBLEM.

In this section it is shown how the ideas of the method of sub and supersolutions can be applied to the study of the existence of positive solutions of the nonlinear eigenvalue problem

(1.11) $$-\Delta u + f(u) = \lambda u \quad \text{in } \Omega ,$$

(1.12) $$u = 0 \quad \text{on } \partial\Omega ,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a real parameter, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the assumptions

(1.13) f is C^2, increasing, and $f(0) = f'(0) = 0$,

(1.14) $\frac{f(u)}{u}$ is strictly increasing for $u > 0$ and strictly decreasing for $u < 0$,

(1.15) $$\lim_{|u| \to \infty} \frac{f(u)}{u} = + \infty .$$

Problems of this kind arise in the theory of nuclear reactors (cf. [36],[108]) and have been studied by several authors (cf. [16],[108],[81],[20],[13]).

First, let us introduce a very useful notation. If the function c is smooth enough (this means $C(\bar{\Omega})$ or even $L^{\infty}(\Omega)$), the eigenvalue problem

(1.16) $$-\Delta w + cw = \lambda w \quad \text{in } \Omega$$

(1.17) $$w = 0 \quad \text{on } \partial\Omega$$

possesses an infinite sequence of eigenvalues $\lambda_n(c)$, which can be ordered according to their multiplicity, going to $+\infty$ when $n \to +\infty$, and depending in a increasing continuous way on c (cf. [41],[106]). The classical book by Courant-Hilbert [41] is a good reference for all which concerns the properties of eigenvalues and eigenfunctions, in particular the continuous and monotone dependence on the coefficient c and the domain Ω and the variational characterization of the eigenvalues, etc, which will be very useful in the following.

In the particular case $c \equiv 0$, we will use the notation $\lambda_n = \lambda_n(0)$ for the eigenvalues of

(1.18) $$-\Delta w = \lambda w \quad \text{in } \Omega ,$$

(1.19) $$w = 0 \quad \text{on } \partial\Omega .$$

It is well known that $\lambda_1(c)$ is simple, i.e., it has multiplicity one, and the corresponding eigenfunctions do not change sign in Ω. For $c \equiv 0$ we will denote by φ_1 the eigenfunction of λ_1 such that $\varphi_1 > 0$ and $\|\varphi_1\|_{L^{\infty}(\Omega)} = 1$ (normalization condition). Hence we have

(1.20) $$-\Delta\varphi_1 = \lambda_1\varphi_1 , \quad \varphi_1 > 0 \quad \text{in } \Omega ,$$

(1.21) $$\varphi_1 = 0 \quad \text{on } \partial\Omega .$$

A preliminary result concerning the existence of solutions of (1.11),(1.12) can be proved by using a comparison argument.

<u>Lemma 1.4</u>: Problem (1.11),(1.12) has no nontrivial solution for $\lambda \leq \lambda_1$.

<u>Proof</u>: Suppose u is a nontrivial solution. We can define the

function

$$\text{(1.22)} \qquad h(x) = \begin{cases} \dfrac{f(u(x))}{u(x)} & \text{for } u(x) \neq 0 \,, \\ 0 & \text{for } u(x) = 0 \,. \end{cases}$$

(After, this function will be denoted by $\left(\frac{f(u)}{u}\right)$). It is not very difficult to show that h is continuous, and (1.14) yields $h \geq 0$, $h \not\equiv 0$. Hence

$$\text{(1.23)} \qquad -\Delta u + hu = \lambda u \quad \text{in } \Omega \,,$$

$$\text{(1.24)} \qquad u = 0 \quad \text{on } \partial\Omega \,,$$

and by the above mentioned comparison results

$$\lambda = \lambda_1(h) > \lambda_1(0) = \lambda_1 \,.$$

<u>Corollary 1.1</u>: If u is a nontrivial positive solution of (1.11), (1.12), then $u > 0$ in Ω.

<u>Proof</u>: If $u \geq 0$ is a solution of our problem, then u is a solution of (1.23),(1.24) with $h \geq 0$ continuous. By the Strong Maximum Principle, $u \equiv 0$ or $u > 0$ in Ω.

Our next aim is to use Theorem 1.1 to prove the existence of positive solutions of (1.11),(1.12). (A completely similar argument could be used to study negative solutions, but this is not interesting for our purposes).

Thus, we need a subsolution u_0 and a supersolution u^0 satisfying $0 < u_0 \leq u^0$. For the supersolution we may choose $u^0 \equiv M$, with $M > 0$ sufficiently large. Indeed, M is a supersolution if $f(M) - \lambda M \geq 0$ and (1.15) implies that such a M exists. More precisely, if we put $k(x) = \frac{f(x)}{x}$, $x \neq 0$, and $g = k^{-1}$ (which exists and is strictly increasing by (1.14)), we may choose as M

$$\text{(1.25)} \qquad M_\lambda = g(\lambda) \,.$$

This will be useful below.

As a subsolution we pick $u_0 = \varepsilon \varphi_1$, with φ_1 given by (1.20), (1.21) and $\varepsilon > 0$ small enough. Indeed,

$$-\Delta u_0 + f(u_0) - \lambda u_0 = \varepsilon \varphi_1 \left[\lambda_1 - \lambda + \frac{f(\varepsilon\varphi_1)}{\varepsilon \varphi_1} \right]$$

and for $\lambda > \lambda_1$ fixed it is easy to see that for $\varepsilon > 0$ sufficiently small, u_0 is a subsolution.

<u>Theorem 1.2</u>: Problem (1.11),(1.12) has at least a nontrivial positive solution for any $\lambda > \lambda_1$.

<u>Proof</u>: The assertion follows immediately from Theorem 1.1 and the obvious fact that $u_0 < u^0$ for ε (resp. M) small (resp. large) enough.

The following theorem states that for every $\lambda > \lambda_1$ our problem has a unique nontrivial positive solution; this result will follow essentially from (1.14).

<u>Theorem 1.3</u>: If $\lambda > \lambda_1$, then problem (1.11),(1.12) has a unique nontrivial positive solution.

<u>Proof ([37],[108],[61])</u>: Take $\lambda > \lambda_1$ fixed. Existence has been already proved in Theorem 1.2. Suppose that u and v are nontrivial positive solutions of (1.11),(1.12). Hence, by Corollary 1.1, $u > 0$ and $v > 0$ in Ω.

We consider first the case, e.g., $u \leq v$. We multiply (1.11) by v and integrate over Ω by using Green's formula and (1.12). Doing the same thing with (1.11),(1.12) for v, and u, we get

$$\int_\Omega -\Delta u \cdot v + f(u)v = \lambda \int_\Omega uv = \int_\Omega \nabla u \cdot \nabla v + f(u)v =$$

$$\int_\Omega -\Delta v \cdot u + f(v)u = \int_\Omega \nabla u \cdot \nabla v + f(v)u$$

and this yields

$$\int_\Omega f(u)v - f(v)u = 0 \,,$$

If we put

$$A = \{x \in \Omega \mid 0 < u(x) < v(x)\}$$

$$B = \{x \in \Omega \mid 0 < u(x) = v(x)\}$$

it is clear that the integral over B is 0, and hence

(1.28) $$\int_\Omega f(u)v - f(v)u = \int_A f(u)v - f(v)u = \int_A uv\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right) = 0$$

and by (1.14) and the continuity of u and v, $u \equiv v$.

Suppose now that u and v are not ordered, i.e., $u \not\leq v$ and $v \not\leq u$. In this case, as it was shown earlier, it is possible to find a supersolution $M > 0$ such that $u < M$, $v < M$. By applying Theorem 1.1 with this $M > 0$ and the same subsolution as before, we prove the existence of a maximal solution $\tilde{u}$ such that $u \leq \tilde{u}$, $v \leq \tilde{u}$. But now, by the first part of the proof $u \equiv v \equiv \tilde{u}$.

<u>Proof 2 ([20])</u>: It follows easily from the Maximum Principle that if $u > 0$ is a solution, then $u \leq M_\lambda$, with M_λ given by (1.25). Hence there is a <u>maximum</u> solution, which we will call u_λ, and it is the unique positive solution. Indeed, if $u \geq 0$ is a solution, then $u \leq u_\lambda$, $u \not\equiv u_\lambda$ in Ω, and by (1.14), $\frac{f(u)}{u} \geq \frac{f(u_\lambda)}{u_\lambda}$. Reasoning as in Lemma 1.4 we get

$$\lambda = \lambda_1\left(\frac{f(u)}{u}\right) > \lambda_1\left(\frac{f(u_\lambda)}{u_\lambda}\right) = \lambda ,$$

a contradiction.

<u>Proof 3 ([17])</u>: By assumption (1.14), it follows that the function $\lambda u - f(u)$ is concave, and then the uniqueness is an easy consequence of the results in [17].

<u>Remark 1.6</u>: The first part of Proof 1 works equally if u and v are two ordered positive solutions (i.e., $u \leq v$ or $v \leq u$) and f satisfies, instead of (1.14), which means strict convexity, a strict concavity assumption, namely $\frac{f(u)}{u}$ strictly decreasing for $u > 0$ and

strictly increasing for $u < 0$. Then $u \equiv v$. But the second part, involving sub and supersolutions, does not work any more if they are not available. We will return to this problem in Section 3.1.

<u>Remark 1.7</u>: The same method can be applied if (1.15) is replaced by the assumption that f is asymptotically linear (cf. (3.4) below). The main difference in this case is that a different trick should be employed to find supersolutions (cf. [47]).

We have obtained some results concerning existence and uniqueness of nontrivial positive solutions of (1.11) (1.12) depending on the values of the real parameter λ. It is possible to obtain, in exactly the same way, the corresponding results for nontrivial negative solutions.

The next propositions provide some complementary results concerning qualitative properties of the "branch" of positive solutions. These matters will be considered again in these notes (cf. in particular Section 3.1). In fact, the following proposition says that if $u(\lambda)$ is the only nontrivial positive solution for the value λ of the parameter, then $u(\lambda)$ is an increasing continuous "branch".

<u>Proposition 1.1</u>: The mapping $\lambda \longmapsto u(\lambda)$ from $(\lambda_1, +\infty)$ into $C(\bar{\Omega})$ is continuous. Moreover, if $\lambda \leq \mu$, then $u(\lambda) \leq u(\mu)$ (pointwise).

<u>Proof</u>: It is an easy exercise by using sub and supersolutions and the continuity of f. Cf., e.g., [41],[20].

<u>Remark 1.8</u>: This is by no means the best smoothness result for the branch $u(\lambda)$. We will show below (Section 3.1) that $u(\lambda)$ is actually C^2 as a consequence of (1.13) (1.14). On the other hand, the monotonicity result can be made more precise by using the Maximum Principle.

<u>Remark 1.9</u>: It is also possible to prove that λ_1 is the only bifurcation point for positive solutions, cf. [13],[12],[41].

Another useful result is contained in the following proposition. Its proof, which uses sub and supersolutions and comparison arguments, can be found in [28].

Proposition 1.2: If $\lambda > 0$, $u(\lambda)(x) \xrightarrow[\lambda \to +\infty]{} +\infty$. Moreover

(1.27) $$g(\lambda - \lambda_1) \leq \|u(\lambda)\|_{L^\infty(\Omega)} \leq g(\lambda) .$$

Finally, a very simple comparison argument (cf. [20]) shows that, in the interval $\lambda_1 < \lambda < \lambda_2$, the three solutions we know, namely the trivial solution, the unique positive solution and the unique negative solution, are the only ones. A more complicated proof of this result was given before in [13] by using topological methods, cf. Section 2.2. Cf. also [76].

Proposition 1.3: If $\lambda_1 < \lambda < \lambda_2$, then problem (1.11),(1.12) with f satisfying (1.13)-(1.15) has exactly 3 solutions.

Proof: If u is a nontrivial solution, then u satisfies

$$- \Delta u + \left(\frac{f(u)}{u}\right) u = \lambda u \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega .$$

Hence $\lambda = \lambda_i\left(\frac{f(u)}{u}\right)$ for some $i \geq 1$. But $\frac{f(u)}{u} > 0$, and this gives

$$\lambda_2 > \lambda = \lambda_i\left(\frac{f(u)}{u}\right) > \lambda_i(0) = \lambda_i ,$$

Hence $i = 1$, and $u > 0$ or $u < 0$.

The results in this section can be summarized in the following diagram.

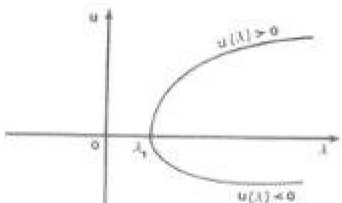

Fig. 1

1.3. THE METHOD OF SUB AND SUPERSOLUTIONS FOR SYSTEMS; APPLICATIONS.

We study in this Section the existence of solutions for some reaction-diffusion systems. For the sake of simplicity we only treat the case of systems with two equations with homogeneous Dirichlet boundary conditions. Let the system

(1.28) $$- \Delta u = f(u,v) \quad \text{in } \Omega ,$$

(1.29) $$- \Delta v = g(u,v) \quad \text{in } \Omega ,$$

(1.30) $$u = v = 0 \quad \text{on } \partial\Omega ,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and the nonlinear terms f and g satisfy the smoothness assumption

(1.31) $$f, g \in C^1 .$$

Now it seems natural, when trying to extend the method of sub and supersolutions to systems, to add some monotonicity assumptions on the growth of f and g as functions of u and v.

The first (apparently) reasonable extension is to assume that f and g are both, separately, increasing in the variables u and v. In this case, if we adopt an (apparently) reasonable definition of sub and supersolutions for systems, the method extends easily to the system (1.28)-(1.30). More precisely, if f and g satisfy (1.31) and are separately increasing in u and v, and if we define a subsolution (u_0,v_0) and a supersolution (u^0,v^0) of (1.28)-(1.30) as pair of smooth functions satisfying

(1.32) $$- \Delta u_0 - f(u_0,v_0) \leq 0 \leq - \Delta u^0 - f(u^0,v^0) \quad \text{in } \Omega ,$$

(1.33) $$- \Delta v_0 - g(u_0,v_0) \leq 0 \leq - \Delta v^0 - g(u^0,v^0) \quad \text{in } \Omega ,$$

(1.34) $$u_0 \leq 0 \leq u^0 , \quad v_0 \leq 0 \leq v^0 \quad \text{on } \partial\Omega ,$$

then, if $u_0 \leq u^0$, $v_0 \leq v^0$, the corresponding version of Theorem 1.1

is true and its proof can be carried out in a completely similar way, as was already pointed out in [300].

Moreover, it is always possible to make f (resp. g) increasing in u (resp. in v) by considering (cf. Remark 1.1) the equivalent problem

$$(1.35) \qquad -\Delta u + Mu = f(u,v) + Mu \quad \text{in } \Omega ,$$

$$(1.36) \qquad -\Delta v + Mv = g(u,v) + Mv \quad \text{in } \Omega ,$$

$$(1.37) \qquad u = v = 0 \quad \text{on } \partial\Omega ,$$

where $M > 0$ is such that the right-hand side in (1.35) (resp. in (1.36)) is increasing in u (resp. in v).

However, if f is not increasing in v, or if g is not increasing in u, then the nonlinear operator T defined (as in the proof of Theorem 1.1) by means of the system (1.35)-(1.37) is no more increasing, separately, in u and v, and the method of monotone iterations does not work. Here is the reason for the relevance of those monotonicity assumptions: if f (resp. g) is increasing in v (resp. in u) or, more generally, if each nonlinear term is increasing in the "off-diagonal" variables, the system is called quasi-monotone. There is a large literature on this subject.

Another possibility is to use, when monotonicity properties of f and g allow it, different tricks (changes of variable, particular iterative schemes, etc.) to obtain monotone sequences converging to maximal and minimal solutions. Cf., e.g., [78],[104],[86].

On the other side, if f and g do not satisfy suitable monotonicity assumptions, with the above definition of sub and supersolutions (1.32)-(1.34) the natural extension of Theorem 1.1 is false, as shows the following counterexample: the system we consider is

$$-\Delta u = u - v + 1 \quad \text{in } \Omega ,$$

$$-\Delta v = u \quad \text{in } \Omega ,$$

$$u = v = 0 \quad \text{on } \partial\Omega .$$

It is easy to see that, for this system, $(u_0,v_0) \equiv (0,0)$ is a subsolution and $(u^0,v^0) \equiv (0,1)$ is a supersolution. Hence, if the theorem is still valid in this case, there is a solution $(\tilde{u},\tilde{v})$ with $\tilde{u} \equiv 0$ and $0 \le \tilde{v} \le 1$. But if $\tilde{u} \equiv 0$, then $\tilde{v} \equiv 0$ on the one side and $\tilde{v} \equiv 1$ on the other, a contradiction.

This counterexample seems to indicate that the above definition of sub and supersolution (1.32)-(1.34) is too weak, at least without additional monotonicity assumptions. An idea about how to modify the definition of sub and supersolution can be obtained by looking, from a somewhat different point of view, at the proof of Theorem 1.1. Indeed, by the order-preserving properties of the operator T (Lemmas 1.1-1.3) it follows immediately that the interval

$$K = [u_0,u^0] = \{z \mid u_0(x) \le z(x) \le u^0(x) \quad \text{in } \Omega\}$$

is invariant by T, i.e., $T(K) \subset K$, and this suggests to use, e.g., Schauder's fixed point theorem if the corresponding assumptions on T and K are satisfied. Of course, this result would be weaker than Theorem 1.1, but its application to systems could be interesting.

We remark that, even if the result is weaker, it supposes some additional restrictions on the function space we employ. The proof of Theorem 1.1 was carried out in $C^{1,\alpha}(\bar{\Omega})$, but many other function spaces, as it was pointed out, were equally well-adapted, e.g., $C^{\alpha}(\bar{\Omega})$, $C(\bar{\Omega})$ or $L^2(\Omega)$. To apply Schauder's fixed point theorem we need a compact operator T and a closed, bounded, convex subset K. There is no problem with T. But the interval K is not bounded in $C^{1,\alpha}(\bar{\Omega})$ (and even in $C^{\alpha}(\bar{\Omega})$, cf., e.g. [10, p. 629]), while it is in $C(\bar{\Omega})$ or $L^2(\Omega)$. (The use of these spaces would imply, as usual, some regularity argument to show that the weak solutions we obtained are actually classical solutions). The "reason" for these difficulties is that the norms involve the derivatives in some cases, while the order involves only the functions. But, with a suitable choice of the function space and the definition of sub and supersolution, it is possible to prove an existence theorem for systems generalizing Theorem 1.1 without monotonicity assumptions on f and g.

__Definition__: The pair $(u_0,v_0)-(u^0,v^0)$ is called a __sub-supersolution__ (or a coupled sub-supersolution) of system (1.28)-(1.30) if $u_0, u^0, v_0, v^0 \in H^2(\Omega) \cap L^\infty(\Omega)$, $u_0 \le u^0$, $v_0 \le v^0$ and

(1.38) $$-\Delta u_0 - f(u_0,v) \le 0 \le -\Delta u^0 - f(u^0,v) \quad \forall v \in [v_0,v^0]$$

(1.39) $$-\Delta v_0 - g(u,v_0) \le 0 \le -\Delta v^0 - g(u,v^0) \quad \forall u \in [u_0,u^0]$$

(1.40) $$u_0 \le 0 \le u^0 , \qquad v_0 \le 0 \le v^0 ,$$

where we use the notation

$$[w,\hat{w}] = \{z \in L^2(\Omega) \mid w(x) \le z(x) \le \hat{w}(x) \quad \text{a.e. on } \Omega\} .$$

__Remark 1.10__: It is clear that there is a coupling between the components of a sub-supersolution, contrarily to the case of definition (1.32)-(1.34). However, both coincide when f and g are, separately, increasing in u and v; if not, the last is much more stringent than the preceding one (cf. Remark 1.12 for an illuminating geometrical interpretation).

__Remark 1.11__: The idea of this definition comes from some classical work by Müller for systems of ordinary differential equations. Some different versions of it have appeared in works concerning both parabolic and elliptic reaction-diffusion systems, often in a simplified form due to the monotonicity properties of the nonlinear terms. Cf. [79],[82], [86].

__Remark 1.12__: A very important particular instance of sub-supersolution is the case in which every component is a constant function; this is equivalent to say that the sub-supersolution determines an __invariant rectangle__, where the vector field (f,g) is always pointed to the interior of the rectangle along its boundary. The same geometrical interpretation in the case of definition (1.32)-(1.34) says that only the vectors corresponding to the four vertices are pointed to the interior. This notion of invariant rectangle is very important for the study of reaction-diffusion systems, cf. [35],[illegible].

__Theorem 1.4 [62]__: Suppose that $(u_0,v_0)-(u^0,v^0)$ is a sub-supersolution of the system (1.28)-(1.30) with f and g satisfying (1.31). Then there exists at least a (classical) solution (u,v) of the system such that $u_0 \le u \le u^0$, $v_0 \le v \le v^0$.

__Proof__: Let $E = (L^2(\Omega))^2$ and $K = [u_0,u^0] \times [v_0,v^0]$. Then K is closed, bounded, convex subset of E. We define a nonlinear operator

$$T : K \longrightarrow E ,$$
$$(\bar{u},\bar{v}) \longmapsto (w,z) = T(\bar{u},\bar{v}) ,$$

where (w,z) is the (unique) solution of the linear system

(1.41) $$-\Delta w + Mw = f(\bar{u},\bar{v}) + M\bar{u} \quad \text{in } \Omega ,$$

(1.42) $$-\Delta z + Mz = g(\bar{u},\bar{v}) + M\bar{v} \quad \text{in } \Omega ,$$

(1.43) $$w = z = 0 \qquad \text{on } \partial\Omega ,$$

with $M > 0$ such that the right-hand side in (1.41) (resp. (1.42)) is increasing in $\bar{u}$ (resp. in $\bar{v}$).

First, T is well-defined. Indeed, by the classical linear theory, as $f(\bar{u},\bar{v}) + M\bar{u} \in L^\infty(\Omega)$, there is a unique $w \in W^{2,p}(\Omega)$, for any $p \ge 1$, solution of (1.41) (1.43). The same for z. By the classical regularity theory, it follows in a quite straightforward way that T is compact.

It remains to prove that $T(K) \subset K$, i.e., that $u_0 \le w \le u^0$ and $v_0 \le z \le v^0$. We only show that $u_0 \le w$, the other three inequalities can be proved in a completely analogous way. By (1.38) with $v = \bar{v}$ and (1.41)

$$0 \ge -\Delta(u_0 - w) - f(u_0,\bar{v}) + f(\bar{u},\bar{v}) + M\bar{u} - Mw$$
$$= -\Delta(u_0 - w) + f(\bar{u},\bar{v}) - f(u_0,\bar{v}) + M(\bar{u} - u_0) + M(u_0 - w) .$$

We multiply this inequality by $(u_0-w)^+$ (we employ the usual notation $u^+ = \max(u,0)$, $u = u^+ - u^-$) and integrate over Ω using Green's Formula, and (1.40),(1.41):

$$0 \geq \int_\Omega -\Delta(u_0-w)(u_0-w)^+ + \int_\Omega [f(\bar{u},\bar{v}) - f(u_0,\bar{v}) + M(\bar{u}-u_0)](u_0-w)^+$$
$$+ M\int_\Omega (u_0-w)(u_0-w)^+ = \int_\Omega |\nabla(u_0-w)^+|^2 - \int_{\partial\Omega} (u_0-w)^+ \frac{\partial(u_0-w)}{\partial n}$$
$$+ M\int_\Omega |(u_0-w)^+|^2 = \int_\Omega |\nabla(u_0-w)^+|^2 + M\int_\Omega |(u_0-w)^+|^2 ,$$

and hence $(u_0-w)^+ = 0$ taking into account that the second integral is positive by monotonicity and the boundary integral is zero. (We have used a classical result by Stampacchia, namely that if $u \in H^1(\Omega)$, then $u^+, u^- \in H^1(\Omega)$ and $\int_\Omega \nabla u^+ \cdot \nabla u^- = 0$). By Schauder's fixed point theorem, there is at least a fixed point of T, which is a weak solution of the system.

Finally, we show that (u,v) is a classical solution if and only if it is a fixed point of T. It is obvious that any classical solution is a fixed point of T. Conversely, if (u,v) is a fixed point, then $u, v \in W^{2,p}(\Omega)$ for any $1 \leq p < \infty$, and, by Morrey's Lemma, belong to $C^{1,\delta}(\bar{\Omega})$ for any $0 < \delta < 1$. Hence $f(u,v), g(u,v) \in C^1(\bar{\Omega})$ and by Schauder's Theory, $u, v \in C^{2,\delta}(\bar{\Omega})$ for any $0 < \delta < 1$ and (u,v) is a classical solution.

Remark 1.13: The existence result in Theorem 1.4 can be extended so as to cover the case where f and g are not locally Lipschitz, and even maximal monotone graphs. For these generalizations, which are motivated by free boundary problems arising, e.g., in chemical reactions and mathematical biology (cf. Comments) cf. [53],[45].

Remark 1.14: Concerning uniqueness, it is not difficult to prove by a direct argument, that if f and g are globally Lipschitz and the corresponding Lipschitz constants are "small", then the solution is unique. It is also possible, by using nonlinear semigroups [3] or sub and supersolutions [62], to prove simultaneously that the solution

is unique and globally asymptotically stable.

Finally, we give an application of Theorem 1.4. Consider the system

$$-\Delta u + f(u)\,g(v) = 0 \quad \text{in } \Omega ,$$
$$-\Delta v - h(u)\,k(v) = 0 \quad \text{in } \Omega ,$$
$$u = v = 1 \quad \text{on } \partial\Omega ,$$

where f, g, h, k are C^1. Assume that they satisfy

(1.44) $\qquad f(0) = 0 , \quad f(1) > 0 ,$

(1.45) $\qquad g(0) = 0 , \quad g(v) > 0 \quad \text{if } v > 0 ,$

(1.46) $\qquad h(u) > 0 \quad \text{if } 0 < u \leq 1 ,$

(1.47) $\qquad$ There is $M > 0$ such that $0 < k(v) \leq M$ for $v \geq 0$.

Then it is easily seen that $(u_0,v_0) = (u^0,v^0)$, where $u_0 = v_0 = 0$, $u^0 \equiv 1$ and v^0 is the unique solution of the linear problem

$$-\Delta w = M \max_{0 \leq u \leq 1} h(u) \quad \text{in } \Omega ,$$
$$w = 1 \quad \text{on } \partial\Omega ,$$

is a sub- supersolution and then there is at least one solution (u,v) of our problem such that $0 \leq u \leq 1$, $0 \leq v \leq v^0$. In particular, (1.44)-(1.47) are satisfied for $f(u) = \alpha u^p$, $h(u) = \alpha\beta u^p$, $g(v) = k(v) = e^{\frac{\gamma(v-1)}{v}}$, $\alpha, \beta, \gamma > 0$, $p \geq 1$, and for $f(u) = h(u) = u^p$, $g(v) = \alpha_1 e^{\frac{\gamma_1(v-1)}{v}} + \alpha_2 e^{\frac{\gamma_2(v-1)}{v}}$,

$k(v) = \alpha_1\delta_1 e^{\frac{\gamma_1(v-1)}{v}} + \alpha_2\delta_2 e^{\frac{\gamma_2(v-1)}{v}}$, $\alpha_1,\alpha_2,\delta_1,\delta_2,\gamma_1,\gamma_2 > 0$, $p \geq 1$.

These systems arise in chemical reactions ([6],[40],[62]). They can be extended by using, e.g., the existence results in [40], as was pointed out in [53].

II. TOPOLOGICAL METHODS. THE LERAY-SCHAUDER DEGREE. GLOBAL BIFURCATION THEOREMS AND APPLICATIONS.

This chapter is devoted to topological methods, in particular the Leray-Schauder degree, and some of their applications to the nonlinear elliptic problems we are studying. The topological degree in infinite dimensional spaces was introduced in a very famous paper by Leray-Schauder [77] in 1934, generalizing the topological degree in finite dimension due to Brouwer.

The Leray-Schauder degree is one of the more general and powerful methods to prove existence theorems for nonlinear equations. Contrarily to the method of sub and supersolutions in the first chapter, comparison arguments do not play an important role here; in particular this implies that applications to nonlinear partial differential equations are not limited to second order equations where the Maximum Principle holds. An important example of this situation are (fourth order) Von Karman's equations in nonlinear elasticity, where comparison results are not available.

The topological degree is a very useful tool to prove existence results for nonlinear problems; from this point of view the main difficulty is usually to obtain *a priori* estimates on the solutions in order to be able to apply the theory. But the degree can also be used to "count" the number of solutions, obtaining lower bounds, or even giving the exact number of solutions (cf. Theorem 2.2 below). Moreover, the degree is the main tool to prove the so-called global bifurcation theorems, which are very useful, together with *a priori* estimates and maybe some additional information, to prove existence results.

Section 2.1 gives an informal account of the definition of topological degree, followed by a list of its main properties. These properties are used in Section 2.2 to compute the exact number of solutions of the nonlinear eigenvalue problem in Section 1.2. Section 2.3 includes (without proofs) a global bifurcation theorem by Rabinowitz and

some applications, in particular to the same problem in Section 1.2. The next section gives similar results for the case without bifurcation.

The second part of this Chapter is especially concerned with positive solutions. Section 2.5 is devoted to global bifurcation theorems for positive solutions, Section 2.6 treats the fixed point index, and finally Section 2.7 contains an application to reaction-diffusion systems.

2.1. THE TOPOLOGICAL DEGREE. MAIN PROPERTIES.

In this Section we intend to give only a very general and informal idea of the definition of the topological degree, first in finite dimension, and then in the infinite dimensional case. For a systematic exposition, including full proofs, cf. [97],[85],[91],[22].

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, and let $\varphi : \overline{\Omega} \longrightarrow \mathbb{R}^N$, $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$. We define $S = \{x \in \Omega \mid J_\varphi(x) = 0\}$, where $J_\varphi(x)$ denotes the Jacobian of φ at the point $x \in \Omega$, which is called the set of singular points of φ. If $b \notin \varphi(\partial\Omega) \cup \varphi(S)$, it is easy to see that $\varphi^{-1}(b)$ is a discrete compact subset, and hence finite. Consequently the number

$$d(\varphi,\Omega,b) = \sum_{x \in \varphi^{-1}(b)} \operatorname{sg} J_\varphi(x) ,$$

where sg denotes the sign, is well-defined since the sum in the right-hand side is finite. The integer $d(\varphi,\Omega,b)$ is called the degree of the map φ relative to the point b with respect to Ω. Defined in this way, the degree can be considered as a kind of "algebraic" counting of the number of solutions of the equation $\varphi(x) = b$ in Ω.

We would like to eliminate two restrictions in this definition. The first one, $b \notin \varphi(S)$, is easy to overcome by using Sard's theorem ($\varphi(S)$ has Lebesgue measure 0). The second is more important; we want to define the degree for every continuous map φ. Then φ is approximated (uniformly) by C^1 functions, and it is necessary to check that the degree is independent of the approximation.

Before to generalize the degree to the infinite dimensional case, it is convenient to remark that it is impossible to define there a topological degree for every continuous map with the same properties as in finite dimension. Indeed, if not, Brouwer's fixed point theorem (which can be proved by using degree theory, cf., e.g., [9]) would be still valid in infinite dimension, and well-known counterexamples show that this is not the case. Schauder's fixed point theorem (or theorems), which, in some sense, play its role, require some compactness assumption: if $T(K) \subset K$ and K is a closed, bounded, convex subset, T should be compact; if T is only continuous, then K should be compact (and convex).

It seems then necessary, in order to have an extension to infinite dimension of the topological degree, to work in some subclass of the class of all continuous maps involving some compactness condition: the set of all mappings of the form $I + T$, with T compact, usually called compact perturbations of the identity or "compact vector fields" in the terminology of [1].

Let E be a real Banach space, and let $T : \bar{\Omega} \longrightarrow E$, where Ω is a bounded open subset of E, with T compact. Then it is possible to define the degree of the map $\Phi = I - T$ by using, roughly speaking, the fact that compact operators can be approximated by operators with finite dimensional range. Then, if $b \notin \Phi(\partial\Omega)$, it is possible to define an integer $d(\Phi,\Omega,b)$, the <u>Leray-Schauder degree</u> of $\Phi = I - T$ relative to the point b with respect to Ω, with the following properties:

1. <u>Continuity with respect to T</u>: There exists a neighborhood V of T in $K(\bar{\Omega},E)$ (space of compact mappings from $\bar{\Omega}$ into E with the norm $\|T\| = \sup_{x \in \bar{\Omega}} \|Tx\|$) such that for any $S \in V$, $b \notin (I-S)(\partial\Omega)$ and

$$d(I - S,\Omega,b) = d(\Phi,\Omega,b) .$$

2. <u>Homotopy invariance</u>: Let $H \in C(\bar{\Omega} \times [0,1],E)$, where $H(u,t) = u - S(u,t)$, with $S \in C(\bar{\Omega} \times [0,1],E)$. If $b \notin H(\partial\Omega \times [0,1])$, then for any $t \in [0,1]$

$$d(H(\cdot,t),\Omega,b) = \text{const.} \; .$$

3. The degree is constant on connected components of $E - \Phi(\partial\Omega)$.

4. <u>Additivity</u>: Let $\Omega = \Omega_1 \cup \Omega_2$, with Ω_1 and Ω_2 disjoint bounded open subsets of E. If $b \notin \Phi(\partial\Omega_1) \cup \Phi(\partial\Omega_2)$, then

$$d(\Phi,\Omega,b) = d(\Phi,\Omega_1,b) + d(\Phi,\Omega_2,b) .$$

5. $$d(I,\Omega,b) = \begin{cases} 1 & \text{if } b \in \Omega \\ 0 & \text{if } b \notin \bar{\Omega} \end{cases} .$$

6. If $b \notin \Phi(\bar{\Omega})$, then $d(\Phi,\Omega,b) = 0$;

7. <u>Existence</u>: If $d(\Phi,\Omega,b) \neq 0$, then there exists a $x_0 \in \Omega$ such that $\Phi(x_0) = b$;

8. <u>Excision</u>: If $K \subset \bar{\Omega}$ is closed and $b \notin \Phi(K)$, then

$$d(\Phi,\Omega,b) = d(\Phi,\Omega - K,b) .$$

We are now in the position to introduce another topological tool, easily derived from the degree, which will also be useful in the following, the index or local degree.

Let $\Phi = I - T$ as above and let u_0 be an isolated solution of the equation $\Phi(u) = b$, i.e., a solution such that it is the unique solution in some neighborhood of u_0. By the excision property

$$d(\Phi,B_r(u_0),b) = d(\Phi,B_{r_0}(u_0),b)$$

for every $0 < r < r_0$, r_0 sufficiently small, and the <u>index</u> of the isolated solution u_0 is defined by

$$i(\Phi,u_0,b) = d(\Phi,B_r(u_0),b) .$$

The following results are very useful, since they allow to calculate the index, and then the degree, in some situations, and this is very convenient.

Theorem 2.1: Let $L : E \longrightarrow E$ be a compact linear operator and let $\Phi = I - L$. If 1 is not a characteristic value of L, then

$$i(\Phi,0,0) = (-1)^{\beta} ,$$

where β is the sum of the (algebraic) multiplicities of the characteristic values of L lying in $(0,1)$.

Remark that $i(\Phi,0,0)$ is well-defined, since $\Phi(u) = 0$ implies $u = 0$.

Lemma 2.1: Let $T : \Omega \longrightarrow E$, $T0 = 0$, and let T compact and Fréchet-differentiable at 0. Then $T'(0)$ is a compact linear operator.

Corollary 2.1: Let $\Phi = I - T$, $T \in K(\Omega,E)$, where Ω is a neighborhood of 0. Suppose that $T0 = 0$ and that T is Fréchet-differentiable at 0. If 1 is not a characteristic value of $T'(0)$, then 0 is an isolated solution of $\Phi(u) = 0$ and $i(\Phi,0,0) = (-1)^{\beta}$, where β is the sum of the (algebraic) multiplicities of the characteristic values of $T'(0)$ lying in $(0,1)$.

Corollary 2.2: Suppose that T satisfies the assumptions in Corollary 2.1 and let $\Phi_{\lambda}(u) = u - \lambda Tu$. Then $i(\Phi_{\lambda},0,0)$ is well-defined if λ is not a characteristic value of $T'(0)$ and it changes by $(-1)^{m_j}$ when λ crosses a characteristic value μ_j of (algebraic) multiplicity m_j.

2.2. AN APPLICATION: THE EXACT NUMBER OF SOLUTIONS OF A NONLINEAR EIGENVALUE PROBLEM.

In this Section the topological tools of the preceding will be applied to the study of the exact number of solutions of the nonlinear eigenvalue problem (1.11) (1.12) already considered in Section 1.2. This problem was settled by Proposition 1.3, whose proof was indeed very simple. We include this alternative (longer) proof to illustrate

how to use the degree in such situations. Moreover, this method can be also applied in more general problems where comparison results are not available (cf. [13]).

We consider again the problem

(2.1) $$-\Delta u + f(u) = \lambda u \quad \text{in } \Omega ,$$

(2.2) $$u = 0 \quad \text{on } \partial\Omega ,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a real parameter, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the assumptions

(2.3) f is C^2, increasing, and $f(0) = f'(0) = 0$,

(2.4) $\frac{f(u)}{u}$ is strictly increasing for $u > 0$ and strictly decreasing for $u < 0$,

(2.5) $$\lim_{|u| \to +\infty} \frac{f(u)}{u} = +\infty .$$

We restate Proposition 1.3.

Theorem 2.2: If $\lambda_1 < \lambda < \lambda_2$, then problem (2.1) (2.2), with f satisfying (2.3)–(2.5), has exactly 3 solutions.

Roughly speaking, the proof can be described as follows: for λ fixed, $\lambda_1 < \lambda < \lambda_2$, problem (2.1) (2.2) can be reformulated, once again, in an equivalent form, namely to find the fixed points of an associated compact nonlinear operator T in $C(\bar{\Omega})$ (or in $L^2(\Omega)$, cf. [76]). Suppose that we claim that the solutions of $u - Tu = 0$ have the properties:

i) Every solution is isolated,

ii) If $u = 0$, $i(I - T,0,0) = -1$,

iii) If $u \neq 0$, $i(I - T,u,0) = +1$,

iv) There is an a priori estimate for solutions of (2.1), (2.2) and $d(I - T, B_R(0), 0) = +1$ for $R > 0$ sufficiently large.

Now the theorem follows readily from these claims by a simple counting.

First, we reformulate our problem as a fixed point equation $u - Tu = 0$. The problem can be written as

$$(2.6)\qquad -\Delta u = \lambda u - f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega .$$

We are led to define the Nemitskii operator associated with the right-hand side in (2.6), $F : C(\bar\Omega) \longrightarrow C(\bar\Omega)$, where for $u \in C(\bar\Omega)$, Fu is defined by $Fu(x) = \lambda u(x) - f(u(x))$. It is well-known (cf., e.g., [19]) that F is continuous and bounded (i.e., A bounded implies $F(A)$ bounded). If we write $L = (-\Delta)^{-1}$, then (2.6) is equivalent to

$$(2.7)\qquad u = LF(u) = Tu ,$$

where $T = LF : C(\bar\Omega) \longrightarrow C(\bar\Omega)$ is compact since L is compact.

The following series of lemmas proves the claims (i)-(v) and this gives the proof of Theorem 2.2.

Lemma 2.2: There exists $R > 0$ such that if u is a solution of (2.1),(2.2) with $\lambda_1 < \lambda < \lambda_2$, then $\|u\|_{L^\infty(\Omega)} < R$.

The proof, which is not very difficult, can be found in [28],[16], [30]. For the case of positive solutions, cf. Section 3.1 below.

Lemma 2.3 ([16]): If $u \neq 0$ is a solution of (2.1),(2.2), then, for any $w \in H_0^1(\Omega)$, $w \neq 0$

$$0 < \int_\Omega (|\nabla w|^2 - \lambda w^2 + f'(u)w^2)\, dx .$$

Remark 2.1: Lemma 2.3 means that, for $\lambda_1 < \lambda < \lambda_2$, the linearized operator along a nontrivial solution is invertible or, in other words, that these solutions are not degenerated.

Lemma 2.4: There exists $R > 0$ such that $u - Tu = 0$ implies $\|u\|_{L^\infty(\Omega)} < R$. Moreover, $d(I - T, B_R(0), 0) = 1$.

Proof: The first part is simply Lemma 2.2. For the second, let $t \in [0,1]$ and let u satisfying $u - t\,Tu = 0$. Consequently

$$-\Delta u = t(\lambda u - f(u)) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega ,$$

and hence it is clear that the estimate for $t = 1$ (Lemma 2.2) yields the estimate for any $t \in [0,1]$. By the homotopy invariance for the degree

$$d(I - tT, B_R(0), 0) = d(I - T, B_R(0), 0) = d(I, B_R(0), 0) = 1 .$$

Lemma 2.5: If $u_0 - Tu_0 = 0$, $u_0 \neq 0$, then u_0 is isolated and $i(I - T, u_0, 0) = 1$.

Proof: By Corollary 2.1 it is sufficient to show that $\gamma \geq 1$ and γ eigenvalue of $T'(u_0)$, i.e., $T'(u_0)w = \gamma w$, imply $w = 0$. Indeed, for $\gamma = 1$ this means that 1 is not a characteristic value, and for $\gamma > 1$ that no characteristic value is in $(0,1)$. Hence $i(I - T, u_0, 0) = 1$.

Let us check this. The equation $T'(u_0)w = LF'(u_0)w = \gamma w$ is equivalent to

$$-\gamma\Delta w = \lambda w - f'(u_0)w \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega .$$

Multiplying by w and integrating over Ω we obtain ($\gamma \geq 1$) by Lemma 2.3

$$\int_\Omega \gamma|\nabla w|^2 - \lambda w^2 + f'(u_0)w^2 = 0 \geq \int_\Omega |\nabla w|^2 - \lambda w^2 + f'(u_0)w^2 > 0$$

if $w \neq 0$, a contradiction. Hence $w = 0$.

Lemma 2.6: The solution $u = 0$ is isolated and $i(I - T, 0, 0) = -1$.

Proof: In a similar way, since $f'(0) = 0$, the equation $w = T'(0)w$ is equivalent to $-\Delta w = \lambda w$ in Ω, $w = 0$ on $\partial\Omega$. Since $\lambda_1 < \lambda < \lambda_2$, $w = 0$. By Corollary 2.1 $u = 0$ is isolated and $i(I - T, 0, 0) = (-1)^\beta$, where β is the sum of the multiplicities of the characteristic values in $(0,1)$. Then $\gamma w = T'(0)w$, $w \neq 0$, $\gamma \geq 1$, is equivalent to $\frac{\lambda}{\gamma} = \lambda_1$

and $i = \frac{1}{\lambda_2} > 1$. Hence $j = 1$, $k = 1$ and finally $i(I - T, 0, 0) = -1$.

2.3. GLOBAL BIFURCATION THEOREMS.

Let E be a real Banach space and let $F : \mathbb{R} \times E \longrightarrow E$ smooth. Assume that $F(\lambda,0) = 0$ for every $\lambda \in \mathbb{R}$. The problem $F(\lambda,u) = 0$ has, for all values of the real parameter λ, the solution $u = 0$, which is usually called the trivial solution. We only consider problems of the form

(2.8) $$F(\lambda,u) = u - G(\lambda,u) = 0 ,$$

where

(2.9) $$G(\lambda,u) = \lambda Lu + H(\lambda,u) ,$$

with L and H satisfying

(2.10) L is a compact linear operator,

(2.11) H is compact and $H(\lambda,u) = o(\|u\|)$ as $u \to 0$, uniformly for λ bounded, i.e., $\lim_{u \to 0} \frac{H(\lambda,u)}{\|u\|} = 0$, uniformly for λ on bounded intervals.

In this situation it is well-known that to ensure the point λ_0 (or $(\lambda_0,0)$) is a bifurcation point of (2.9), a necessary condition is that λ_0 is a characteristic value of L (cf., e.g., [73]). Very simple counter-examples show that this condition is not sufficient. There are some rather general sufficient conditions: one of them is that λ_0 is a characteristic value of L of odd (algebraic) multiplicity (cf. [73]). But this result has a purely local character: all that it means is that there is a (maybe "small") neighborhood of $(\lambda_0,0)$ in $\mathbb{R} \times E$ such that every "sub-neighborhood" contains a solution (λ,u) of (2.9) with $u \neq 0$. The following theorem, due to Rabinowitz, shows that the fact that λ_0 has odd multiplicity has much more deep implications, concerning in particular the global structure of the solution set of (2.9).

Theorem 2.3 ([91],[92]): Assume (2.10) (2.11). If λ_0 is a characteristic value of odd (algebraic) multiplicity of L, then there exists a connected component C of S (the closure in $\mathbb{R} \times E$ of the set of nontrivial solutions of (2.9)) such that C contains $(\lambda_0,0)$ and either C is unbounded either contains $(\hat{\lambda}_0,0)$, where $\hat{\lambda}_0$ is a characteristic value of L, $\hat{\lambda}_0 \neq \lambda_0$.

Roughly speaking, this is the plan of the proof. If the theorem is false, then it follows by some topological arguments the existence of a one-parameter family of mappings such that, on the one side they have the same degree by homotopy invariance and, on the other, satisfy the assumptions of Corollary 2.2. This implies that the index should change from 1 to -1 (or from -1 to 1) when crossing an odd characteristic value, a contradiction.

It is very easy to see that both cases in Theorem 2.3 actually occur. If $H \equiv 0$, the problem is reduced to $u = \lambda Lu$, and the connected components are simply the corresponding eigenspaces. As an example of the second ([97]) we can take $E = \mathbb{R}^2$ and

$$u = \lambda Lu + \lambda Hu \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad H(u) = \begin{pmatrix} -u_2^3 \\ u_1^3 \end{pmatrix} .$$

L has the characteristic values 1 and 2, both simple, which are bifurcation points. A very simple calculation shows that the set of nontrivial solutions is a closed loop in the interval $1 \leq \lambda \leq 2$.

Remark 2.2: Similar results can be proved for bifurcation at infinity, cf. [95].

The main difficulty in order to apply Theorem 2.3 is to know which of the two above possibilities actually occurs. In two very interesting cases (a class of nonlinear Sturm-Liouville problems and a class of quasi-linear elliptic problems) it is possible to show (cf. [91], [92],[97]) that the continua of solutions bifurcating from simple eigenvalues (as it is the case for all the eigenvalues for Sturm-Liouville problems and the first for some elliptic problems) are unbounded. The proof exploits largely some qualitative properties of the solutions. More precisely, in the case of Sturm-Liouville problems, the nodal properties of the eigenfunctions: if φ_k is an eigenfunction for the eigenvalue λ_k, then φ_k has exactly $(k-1)$ simple zeroes in the corresponding interval. The same properties imply that continua arising from different eigenvalues cannot intersect. In the case of quasi-linear elliptic problems having a linearized eigenvalue problem as, e.g., (1.16) (1.17) such arguments only apply to the first eigenvalue, which is simple and has a positive eigenfunction.

Global bifurcation theorems can be used to obtain existence results for (2.8). However, we point out that, even if it is possible to prove that the continuum of solutions given by Rabinowitz's theorem is unbounded, the information on the solution set is not very precise. Indeed, all we know, in principle, about C is that C is a closed connected subset. Thus, it is very convenient to get as more additional information as possible on solutions.

A decisive step in this direction is to have a priori estimates for the solutions of (2.9). If there is a continuous function $c : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $\Phi(\lambda,u) = 0$ implies $\|u\| \le c(\lambda)$ and C is unbounded, then C can "go to infinity" "to the right" and/or "to the left", but not "vertically" (cf. Fig. 2).

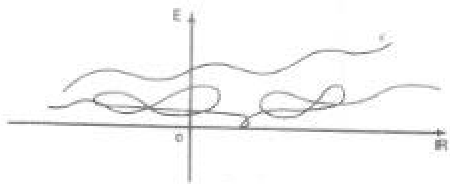

Fig. 2

Moreover, if some more information on solutions is available, then the structure of the solution set can be made more precise. Cf. Sections 1.2 and 3.1.

Let us apply this result to our problem (2.1) (2.2) with assumptions (2.3)-(2.5). The problem can be written as

$$u = \lambda Lu - L(f(u))$$

where $L = (-\Delta)^{-1}$. Since $f(0) = f'(0) = 0$ it is not difficult to check that for any "reasonable" norm (e.g. $C(\bar{\Omega})$, $C^1(\bar{\Omega})$, etc.) $\frac{\|Lf(u)\|}{\|u\|} \xrightarrow[u \to 0]{} 0$, and the linearized problem $u = \lambda Lu$ has the simple eigenvalue λ_1. Now it is possible to apply Theorem 3.3 to λ_1 and to prove that the component C is unbounded. On the other hand, recall that we have some more information on solutions of (2.1) (2.2) at our disposal: for $\lambda \le \lambda_1$ there is only the trivial solution (Lemma 1.4) and there are a priori estimates (Lemma 2.2): both imply that solutions should "go to infinity on the right". But anyway, even using these complementary results, the method of sub and supersolutions in Section 1.2 and the continuation method of Section 3.1 give much more information.

These considerations concerning our test problem are still valid in a rather general setting. When applicable, the method of sub and supersolutions seems to be preferable to global bifurcation arguments in order to prove existence results for nonlinear problems of the form (2.8): besides to give a constructive proof involving monotone iterations, they allow to get more precise information on the structure of the solution set. But if sub and supersolutions are not available, then topological methods, and in particular global bifurcation theorems, can be used to prove existence results.

3.4. EXISTENCE OF CONTINUA IN THE CASE WITHOUT BIFURCATION.

Let E be a real Banach space and let $T : \mathbb{R} \times E \longrightarrow E$ compact such that $T(0,u) = 0$ for every $u \in E$. Hence $(0,0)$ is the only solution of problem

$$u = T(\lambda, u) \tag{2.12}$$

contained in $\{0\} \times E$. The following theorem is an example of the kind of results which can be proved in the non-bifurcation case.

Theorem 2.4 ([97]): With the above assumptions, if C is the connected component of the solution set of (2.12) containing $(0,0)$, then $C = C^+ \cup C^-$, where $C^\pm \subset \mathbb{R}^\pm \times E$, $C^\pm$ are unbounded and $C^+ \cap C^- = \{(0,0)\}$.

It is also possible to obtain similar results when $T(0,u) \neq 0$ if some *a priori* estimates are available. For example we can prove the following theorem (cf. [92]).

Theorem 2.5: Let $T : \mathbb{R} \times E \longrightarrow E$ compact and suppose there exists $M > 0$ such that $u = T(0,u)$ implies $\|u\| < M$. If $d(I - T(0,\cdot), B_M(0), 0) \neq 0$, then $\mathscr{S}$ has two unbounded components in $\mathbb{R}^+ \times E$ and $\mathbb{R}^- \times E$ intersecting in $(0,\bar{u})$, where $\|\bar{u}\| < M$.

2.5. GLOBAL BIFURCATION THEOREMS FOR POSITIVE SOLUTIONS.

It is fairly well-known that positive solutions play a very important role in applications as, e.g., chemical reactions, combustion theory, population dynamics, etc., where some of the unknowns (concentrations, populations, etc.) are essentially positive.

Moreover, there are global bifurcation theorems for positive solutions excluding the second possibility in Theorem 2.3, ensuring in this way the existence of an unbounded component of (positive) solutions.

Suppose (E,P) is an ordered Banach space with positive cone P, P is generating (i.e., $E = P - P$) and let $T : \mathbb{R}^+ \times P \longrightarrow P$ compact such that $T(\lambda,0) = 0$ for every $\lambda \geq 0$ and $T(0,u) = 0$ for every $u \in P$. We study the equation

$$u = T(\lambda,u) = \lambda Lu + B(\lambda,u) \tag{2.13}$$

where L is a positive compact linear operator and B is compact and such that $B(\lambda,u) = o(\|u\|)$ at $u = 0$ uniformly for λ bounded.

Theorem 2.6 ([13]): Suppose all the above assumptions are satisfied and suppose that L possesses exactly one positive characteristic value λ_0 with a positive eigenvector. Then λ_0 is the only bifurcation point for positive solutions for (2.13). Moreover, the set of positive solutions contains a connected component C which is unbounded and such that $C \cap (\mathbb{R}^+ \times \{0\}) = \{(\lambda_0, 0)\}$.

Remark 2.3: There are similar results concerning bifurcation at infinity (cf. [10]). In particular, it is possible to show that, under hypotheses similar to those in Theorem 2.6, there is a unique asymptotic bifurcation point for positive solutions and that the corresponding continuum is also unbounded.

There are results similar to those in Section 2.4 for positive solutions in the case without bifurcation. Let us give only one example.

Theorem 2.7 ([13]): Let (E,P) be an ordered Banach space with cone P and let $T : \mathbb{R}^+ \times P \longrightarrow P$ compact such that $T(0,u) = 0$ for every $u \in P$. Then the connected component of the solution set of $u = T(\lambda,u)$ containing $(0,0)$ is unbounded.

It is clear that Theorem 2.7 becomes trivial if $T(\lambda,0) = 0$ for every λ; in this case the solution set contains the "line" of trivial solutions $\mathbb{R}^+ \times \{0\}$. It is interesting in this setting to study the existence of positive solutions bifurcating from the trivial solution.

An interesting nonlinear problem, where some of the above methods can be tested is the equation

$$\begin{aligned} -\Delta u &= \lambda e^u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.14}$$

arising in the study of some simplified models for combustion (cf., e.g. [77]). This problem has been considered by several authors. In particular, Gelfand [59] proved that in the special case of the sphere $(N = 3)$ there are infinitely many solutions for a certain value $\bar{\lambda}$ of the parameter λ, and Joseph-Lundgren [47] provide a complete and interesting study of radial solutions by using ordinary differential equations methods.

Let $L = (-\Delta)^{-1}$ and let F be the Nemitskii operator (in $C(\bar{\Omega})$) associated to e^u, i.e., $Fu(x) = e^{u(x)}$. It is clear that (2.14) is equivalent to $u = \lambda LF(u)$, or to $u = T(\lambda,u)$, where $T(\lambda,u) = \lambda LF(u)$ is compact, and that $T(0,u) = 0$ for every u. On the other hand, it follows immediately from the Maximum Principle that if u is a solution of (2.14) for $\lambda \geq 0$, then $u > 0$ in Ω. Otherwise stated, since every solution is positive, the problem can be formulated in the above framework; in particular, by Theorem 2.7 it follows the existence of an unbounded continuum of positive solutions of (2.14) containing $(0,0)$. This information, which is not as precise as we desire, can be completed by some additional considerations: e.g., there is a $\bar{\lambda} > 0$ such that (2.14) has no solution for $\lambda > \bar{\lambda}$ (cf. Lemma 3.1 and [44]).

2.5. THE FIXED POINT INDEX.

It was already remarked that the Leray-Schauder degree is one of the more powerful tools to prove existence results for nonlinear equations, and that it can be useful to compute the exact number of solutions (cf. Theorem 2.2) or at least a lower bound. However, in some situations, e.g., in problems arising in physical applications, the main interest is concentrated almost exclusively on positive solutions, and it would be nice to have at our disposal some analogous tools in order to work in a (relatively) open subset of a cone of positive functions. But if this cone has empty interior (as, e.g., in $L^2(\Omega)$), then the topological degree cannot be directly applied, since the corresponding open subset (the interior of the cone) is empty. Anyway, this difficulty can be overcomed by exploiting the fact that a cone is a retract in a Banach space: this allows us to define a fixed point index for compact operators defined on the cone.

First, we recall some definitions and theorems from general topology. Let X be a topological space and let $A \subset X$. Then A is called a retract of X if there exists a continuous map $r : X \longrightarrow A$ called retraction such that $r|_A = I_A$. It is easily seen that every retract is a closed subset.

Following a theorem by Dugundji, every closed convex subset of a Banach space E is a retract of E. This allows us to define a fixed point index (cf. [10],[85] for more details).

Let X be a retract of the Banach space E. Let U be a bounded open subset of X and let $f : \bar{U} \longrightarrow X$ compact which has no fixed points on ∂U. Then there exists an integer $i(f,U,X)$ with the properties:

1. Normalization: For every constant map $f : \bar{U} \longrightarrow U$, $i(f,U,X) = 1$.

2. Additivity: For every pair of disjoint bounded open subsets U_1 and U_2 of U such that f has no fixed points on $\bar{U} - (U_1 \cup U_2)$

$$i(f,U,X) = i(f,U_1,X) + i(f,U_2,X)$$

where $i(f,U_i,X) = i(f|_{\bar{U}_i}, U_i, X)$, $i = 1,2$.

3. Homotopy invariance: For every compact interval $I \subset \mathbb{R}$ and every compact map $H : I \times \bar{U} \longrightarrow X$ such that $H(\lambda,x) \neq x$ for $(\lambda,x) \in I \times \partial U$,

$$i(H(\lambda,.),\ U,\ X) = \text{const.}$$

for every $\lambda \in I$.

4. Permanence: If Y is a retract of X and $f(\bar{U}) \subset Y$, then

$$i(f,U,X) = i(f,U \cap Y,\ Y)$$

where $i(f,U \cap Y,\ Y) = i(f|_{\overline{U \cap Y}},\ U \cap Y,\ Y)$.

5. Excision: For $V \subset U$, V bounded and open, such that f has no fixed point in $\bar{U} - V$,

$$i(f,U,X) = i(f,V,X).$$

6. Existence: If $i(f,U,X) \neq 0$, then f has at least a fixed point in U.

This integer is called the fixed point index. In the case of an ordered Banach space (E,P), the cone P is, following a theorem by Dugundji, a retract of E, and then the corresponding fixed point index is well- defined for a bounded open subset U of P and a compact map $f : \bar{U} \longrightarrow P$ with no fixed points on ∂U. In the following the fixed point index will be denoted by $i(f,U)$, omitting the reference to the cone P.

<u>Remark 2.4</u>: If U is an open subset of the space E, then $i(f,U,E) = d(I-f,U,0)$. This shows that the fixed point index is a rather natural generalization of the degree. Cf. [10].

2.7. AN APPLICATION: POSITIVE SOLUTIONS FOR A REACTION-DIFFUSION SYSTEM.

This Section is devoted to an application of the fixed point index to the existence of positive solutions of a nonlinear system arising in applications, more precisely in the study of chemical reactions (cf. [4]). A variant of the preceding global bifurcation theorems is needed for our problem, namely the following.

<u>Lemma 2.7 ([4])</u>: Let (F,P) be an ordered Banach space with cone P and let $f : \mathbb{R}^+ \times P \longrightarrow P$ compact. Suppose that there is a constant $M > 0$ such that $w = f(0,w)$ implies $\|w\| < M$ and that $i(f(0,.), B_M(0) \cap P) \neq 0$. Then there exist a continuum of solutions of $x = f(\lambda,x)$, which is unbounded in $\mathbb{R}^+ \times P$ and such that C contains $(0,\tilde{w})$ where $\tilde{w} \in P$ and $\|\tilde{w}\| < M$.

We shall apply this abstract result to our example. Consider the system

$$(2.15) \qquad -a_1 \Delta u + \frac{\lambda u}{u+1} + Auv = 0 \quad \text{in } \Omega ,$$

$$(2.16) \qquad -a_2 \Delta v + Auv = 1 \quad \text{in } \Omega ,$$

$$(2.17) \qquad u = v = 0 \quad \text{on } \partial\Omega ,$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $a_1, a_2, A > 0$ are real

numbers and λ is a real parameter. Let $f(u) = \frac{u}{u+1}$, it is clear that for $u \geq 0$, $0 \leq f(u) \leq u$. We fix a_1, a_2 and A and take λ as a parameter.

We consider the function space $E = (C^{1+\alpha}(\bar{\Omega}))^2$ with the positive cone $K = \{(w,z) \in E \mid w,z \geq 0 \text{ in } \Omega\}$. We define a map

$$G : \mathbb{R}^+ \times K \longrightarrow K$$
$$(\lambda,u,v) \longrightarrow (w,z) = G(\lambda,u,v)$$

in the following way: for each $(\lambda,u,v) \in \mathbb{R}^+ \times K$, (w,z) is the (unique) solution of the system

$$(2.18) \qquad -a_1 \Delta w + \lambda w = Auv + \lambda(u-f(u)) \quad \text{in } \Omega ,$$

$$(2.19) \qquad -a_2 \Delta z + Auv = 1 \quad \text{in } \Omega ,$$

$$(2.20) \qquad w = z = 0 \quad \text{on } \partial\Omega .$$

Indeed, for $u,v \geq 0$, the system is decoupled in two linear equations. Moreover, by the Maximum Principle, $w,z \geq 0$ in Ω (recall that $f(u) \leq u$ if $u \geq 0$). The operator G is compact, and the proof is rather straightforward.

<u>Lemma 2.8</u>: There is a constant $c > 0$ which is independent of λ, such that if (u,v) is a solution of (2.15)-(2.17) with $\lambda \geq 0$ and $u,v \geq 0$, then

$$(2.21) \qquad \|u\|_{L^\infty(\Omega)} \leq c , \qquad \|v\|_{L^\infty(\Omega)} \leq c .$$

<u>Proof</u>: Let q be the solution of the problem

$$(2.22) \qquad -\Delta q = 1 \quad \text{in } \Omega ,$$

$$(2.23) \qquad q = 0 \quad \text{on } \partial\Omega .$$

By adding equations (2.15)-(2.17) and (2.22),(2.23) we obtain

$$-\Delta(a_1u + a_2v + q) = -\lambda f(u) \le 0 \quad \text{in } \Omega ,$$

$$a_1u + a_2v + q = 0 \qquad \text{on } \partial\Omega .$$

Hence, by the Maximum Principle, $a_1u + a_2v + q \le 0$ in Ω, and then $0 \le a_1u + a_2v \le -q$, which gives the result.

<u>Lemma 2.9</u>: There is a constant $M > 0$ such that if $(u,v) \in P$ is a solution of (2.15)-(2.17) for $\lambda = 0$, then

(2.24) $$\|u\|_{1,\alpha} < \frac{M}{2} , \quad \|v\|_{1,\alpha} < \frac{M}{2} .$$

<u>Proof</u>: It follows easily from (2.27), the L^p estimates in [1] and Morrey's Lemma.

<u>Theorem 2.3</u>: There exists an unbounded component of positive solutions of (2.15)-(2.17) such that its projection on the real axis is all $\mathbb{R}^+$. In particular, for any $\lambda \ge 0$ there is at least a positive solution of (2.15)-(2.17).

<u>Proof</u>: We apply Lemma 2.7 with $F = K$, $P = K$, and $f = 0$. G is compact and $G(0,u,v) = (u,v)$ implies (2.24) (Lemma 2.9). We claim that $i(G(0,.), B_M(0) \cap K) \ne 0$. Indeed, recall that $G(0,.)$ is given by (2.18)-(2.20) with $\lambda = 0$ and define the homotopy $(w,z) = H(t,u,v)$ as the solution of the (linear) system

$$-a_1 \Delta w = tNuv \quad \text{in } \Omega ,$$

$$-a_2 \Delta z + tNuz = 1 \quad \text{in } \Omega ,$$

$$w = z = 0 \quad \text{on } \partial\Omega .$$

Since $H(1,.) = G(0,.)$ and, moreover, (2.24) implies $H(t,u,v) \ne (u,v)$ for any $t \in [0,1]$ and any $(u,v) \in \partial(B_M(0) \cap K)$, the homotopy invariance of the index yields

$$i(G(0,.), B_M(0) \cap K) = i(H(1,.), B_M(0) \cap K) = i(H(0,.), B_M(0) \cap K) = 1$$

since $H(0,.)$ is a constant map, by the normalization property.

Finally, we remark that <u>a priori</u> bounds for any $\lambda > 0$ can be easily obtained.

<u>Remark 2.3</u>: We point out that in the example (2.15)-(2.17), the discontinuity of the function $\frac{u}{u+1}$ at $u = -1$ raises serious problems concerning the associated Hanitzki operator. This is no more the case if we are restricted to $u \ge 0$. On the other hand, the fact that we only consider positive solutions makes easier to obtain <u>a priori</u> bounds (cf. the proof of Lemma 2.8). Similar arguments, but working on the whole space and not on a cone, were used in [92] for the Brusselator.

III. CONTINUATION METHODS. EXISTENCE OF A SECOND SOLUTION; VARIATIONAL METHODS.

Continuation and variational methods are presented in this third chapter. Section 3.1 treats our test problem (the nonlinear eigenvalue problem (1.11),(1.12)) by following a different approach involving the use of a local inversion theorem by Crandall-Rabinowitz and a continuation argument which is an application of the Implicit Function Theorem. Once again, a priori estimates play an important role. This method is particularly well-suited for the study of bifurcation from simple eigenvalues, as it is the case here. We also study some variants of problem (1.11), (1.12), which led to some open problems.

Moreover, the same kind of methods are still useful in problems without bifurcation. In Section 3.2 we consider a class of problems including equation (2.14). We only sketch the arguments, sending to [44] for a careful study of the "bending points". Finally, Section 3.3 introduces a new method, namely Lyusternik-Schnirelman critical point theory, which, maybe combined with some other tools, is very useful to prove existence and multiplicity results.

3.1. LOCAL INVERSION THEOREMS AND CONTINUATION: THE BIFURCATION CASE.

We will consider here, for the third time, the nonlinear eigenvalue problem (1.11), (1.12) which was treated by using sub and supersolutions in Section 1.2 and by using degree theory and global bifurcation theorems in Section 2.2 and 2.3, respectively. We remark that, even if the idea of using local (and global) inversion theorems to prove existence results is not very recent (Hadamard, Caccioppoli, etc.), we follow here the approach by Rabinowitz [90], cf. also [42].

Consider once again the problem

(3.1) $$-\Delta u + f(u) = \lambda u \quad \text{in } \Omega ,$$

(3.2) $$u = 0 \quad \text{on } \partial\Omega$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a real parameter, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

(3.3) f is C^3, increasing, and $f(0) = f'(0) = 0$,

(3.4) $\dfrac{f(u)}{u}$ is strictly increasing for $u > 0$ and strictly decreasing for $u < 0$,

(3.5) $$\lim_{|u| \to +\infty} \frac{f(u)}{u} = +\infty$$

or

(3.6) $$\lim_{|u| \to +\infty} \frac{f(u)}{u} = f'(\infty) < +\infty .$$

Lemma 3.1: Assume that f satisfies (3.2), (3.4) and (3.5) (resp. (3.6)). Then, if u is a nontrivial positive solution of (3.1), (3.2), $\lambda_1 < \lambda$ (resp. $\lambda_1 < \lambda < \lambda_1 + f'(\infty)$).

Proof: The first part is just Lemma 1.4. The second follows by a similar comparison argument.

The following theorem is the main tool for the results in this Section. Cf. also [42],[14],[4],[10].

Theorem 3.1 ([42]): Let X and Y be real Banach spaces, let $I \subset \mathbb{R}$ a bounded interval and let $F : I \times X \longrightarrow Y$, $F \in C^2$. Let $\lambda_0 \in I$ and assume that F satisfies

i) $F(\lambda, 0) = 0$ for every $\lambda \in I$,

ii) $\dim \operatorname{Ker} (F_x(\lambda_0, 0)) = \operatorname{codim} R(F_x(\lambda_0, 0)) = 1$,

iii) $F_{\lambda x}(\lambda_0, 0)x_0 \notin R(F_x(\lambda_0, 0))$, where x_0 spans $\operatorname{Ker} F_x(\lambda_0, 0)$ (this will be denoted by $[x_0] = \operatorname{Ker} F_x(\lambda_0, 0)$).

Let Z be a complementary subspace of $[x_0]$ in X. Then there exist an interval J containing the origin and two C^1 functions $\lambda : J \longrightarrow \mathbb{R}$ and $\psi : J \longrightarrow Z$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$ and

and $x(s) = s\varphi_0 + s\psi(s)$ implies $F(\lambda(s), x(s)) = 0$. Moreover, $F^{-1}(0)$ is uniquely formed (in a neighborhood of $(\lambda_0, 0)$) by the curves $x = 0$ and $(\lambda(s), x(s))$, $s \in J$.

Theorem 3.2 ([43]): Suppose that f satisfies (3.3), (3.4), (3.5) (resp. (3.6)). Then, for any λ such that $\lambda_1 < \lambda$ (resp. $\lambda_1 < \lambda < \lambda_1 + f'(\infty)$) there exist a unique nontrivial positive solution $u(\lambda)$ of (3.1), (3.2). The mapping $\lambda \longrightarrow u(\lambda)$ from $(\lambda_1, +\infty)$ (resp. from $(\lambda_1, \lambda_1 + f'(\infty))$) into $C_0^{2,\alpha}(\bar{\Omega})$ is C^1. Moreover, if f satisfies (3.6), then

$$\lim_{\lambda \to \lambda_1 + f'(\infty)} \|u(\lambda)\|_\infty = +\infty .$$

Proof: First, we recall that the uniqueness was already proved in Theorem 1.3.

In the first part of the proof, the Local Inversion Theorem 3.1 is applied to show the existence of a "small" branch of positive solutions bifurcating to the right from $\lambda = \lambda_1$. Moreover, this branch can be parametrized by λ.

We define the function space

$$C_0^{2,\alpha}(\bar{\Omega}) = \{z \in C^{2,\alpha}(\bar{\Omega}) \mid z = 0 \text{ on } \partial\Omega\}$$

and the mapping

$$F : \mathbb{R} \times C_0^{2,\alpha}(\bar{\Omega}) \longrightarrow C^\alpha(\bar{\Omega})$$

by

$$F(\lambda, u) = -\Delta u + f(u) - \lambda u .$$

It is clear that $F(\lambda, 0) = 0$ for any λ and F is C^2 by (3.3). It is easily calculated that

$$F_u(\lambda, u)w = -\Delta w + f'(u)w - \lambda w$$

and

$$F_{\lambda u}(\lambda, u)w = -w .$$

In particular

$$F_u(\lambda_1, 0)w = -\Delta w - \lambda_1 w$$

and hence

$$\operatorname{Ker} F_u(\lambda_1, 0) = [\varphi_1] ,$$

with $\varphi_1 > 0$ on Ω (recall (1.20), (1.25)). It is easily seen (cf. [90]) that we can take

$$Z = R(F_u(\lambda_1, 0)) = \{z \in C^\alpha(\bar{\Omega}) \mid \int_\Omega z\,\varphi_1 = 0\} .$$

Since $\varphi_1 > 0$ on Ω, $F_{\lambda u}(\lambda_1, 0)\varphi_1 = -\varphi_1 \notin Z$ and we apply Theorem 3.1 with $X = C_0^{2,\alpha}(\bar{\Omega})$, $Y = C^\alpha(\bar{\Omega})$ and $\lambda_0 = \lambda_1$. Hence there exist an interval $J \subset \mathbb{R}$ containing the origin and C^1 functions $\lambda : J \longrightarrow \mathbb{R}$, $\psi : J \longrightarrow Z$ such that $\lambda(0) = \lambda_1$, $\psi(0) = 0$, $u(s) = s\varphi_1 + s\psi(s)$ implies $F(\lambda(s), u(s)) = 0$ for $s \in J$; moreover, in a neighborhood of $(\lambda_1, 0)$ all the nontrivial solutions are of this form.

We claim that these solutions are positive and that they can be parametrized by λ. If $\lambda(s) = \lambda_1 + \xi(s)$, where obviously $\xi(s) = \lambda(s) - \lambda_1$, then it is possible to see that, for $s > 0$ sufficiently small, $u(s) > 0$ on Ω, and by Lemma 3.1, $\xi(s) > 0$. Hence there exists a $s_0 > 0$ such that the map $\lambda : (0, s_0) \longrightarrow \mathbb{R}$ is one-to-one. Indeed, if $\lambda(s) = \lambda(s')$, then by uniqueness $u(s) = u(s')$ and then $s = s'$ by the form of $u(s)$. This means that the curve of positive solutions can be parametrized by λ in a neighborhood on the right of λ_1.

In the second part we extend to the right this branch by using a continuation argument. Let (μ, v) be a solution such that $v > 0$. Then

$$-\Delta v + \frac{f(v)}{v} v = \mu v \quad \text{in } \Omega , \qquad v = 0 \quad \text{on } \partial\Omega . \tag{3.7}$$

On the other hand, $F_u(\mu, u)$ is an isomorphism. Indeed, if not there exists a $w \neq 0$ such that

$$(3.8) \qquad \begin{aligned} -\Delta w + f'(v)w &= \mu w \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega . \end{aligned}$$

But, by (3.3),(3.4), $\frac{f(v)}{v} < f'(v)$ and hence $(v > 0)$ we obtain by usual comparison arguments

$$\mu = \lambda_1(f'(v)) > \lambda_1\left(\frac{f(v)}{v}\right) = \mu ,$$

which is a contradiction. Hence $F_u(\mu, u)$ is an isomorphism and we apply the Implicit Function Theorem to obtain the existence of an $\varepsilon > 0$ and a C^2 mapping $\lambda \longmapsto u(\lambda)$ defined for $|\lambda - \mu| < \varepsilon$ such that $F(\lambda, u(\lambda)) = 0$.

We have to prove that the solutions obtained prolongating in this way our initial branch are actually positive. By the continuity of $\lambda \longmapsto u(\lambda)$ it follows that if $u(\lambda)$ is not in $K = \{w \in C_0^{2,\alpha}(\bar{\Omega}) | w > 0$ in $\Omega\}$ for all the values of λ, then there should be a $\bar{\lambda}$ minimal such that $\lambda_1 < \bar{\lambda}$ and $u(\bar{\lambda}) \in \partial K$, i.e., $u(\bar{\lambda})(a) = 0$ for some $a \in \Omega$. By Corollary 1.1, $u(\bar{\lambda}) \equiv 0$ and moreover $\|u(\lambda)\|_\alpha \longrightarrow 0$ when $\lambda \uparrow \bar{\lambda}$. This implies (cf. Theorem 2.6 and [10]) that $\bar{\lambda}$ is the only bifurcation point for positive solutions and $\bar{\lambda} = \lambda_1$, which is impossible.

Hence we have showed the existence of a branch of positive solutions for every λ in a maximal interval of the form (λ_1, λ^+), where $\lambda^+ \leq +\infty$.

We claim that (3.3), (3.4), (3.6) imply $\lambda^+ = \lambda_1 + f'(\infty)$. By Lemma 3.1, $\lambda^+ \leq \lambda_1 + f'(\infty)$. On the other hand

$$(3.9) \qquad \lim_{\lambda \uparrow \lambda^+} \|u(\lambda)\|_\alpha = +\infty .$$

Indeed, if not there are a constant c and a sequence λ_n such that $\lambda_n \uparrow \lambda^+$ and $\|u(\lambda_n)\|_\alpha \leq c$. A rather straightforward argument involving the compactness of the Green's operator (or solution operator) of (3.1),(3.2) shows that it is possible to apply the Implicit Function

Theorem in λ^+, contradicting the maximality of λ^+. By (3.9) and the results in [61] it follows that λ^+ is an asymptotic bifurcation point, and then $\lambda^+ = \lambda_1 + f'(\infty)$. Cf. [11],[12] for details.

If f satisfies (3.3), (3.4), (3.5), then it is clear that a necessary and sufficient condition for $\lambda^+ = +\infty$ is the existence of a priori estimates for solutions of (3.1), (3.2), i.e., the existence of a continuous function $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|u(\lambda)\|_\alpha \leq \varphi(\lambda) \ .$$

The existence of such a φ follows easily from Lemma 2.2 and the usual regularity theory. (Cf. also Section 1.2). However, as we only need a priori estimates for positive solutions, we use an alternative method. Suppose that $w > 0$ is a solution for the value λ of the parameter. Let $M = \max_{\bar{\Omega}} w = w(a) > 0$ for some $a \in \Omega$. Hence

$$-\Delta w(a) = \lambda w(a) - f(w(a)) \geq 0 ,$$

or, equivalently, $f(M) \leq \lambda M$, and (with the notation of Section 1.2) then $M \leq g(\lambda)$. This L^∞ estimate, together with the L^p and C^α regularity estimates, yields C^α (and even $C^{2,\alpha}$) estimates.

The proof of Theorem 3.2 gives an alternative approach to the study of positive solutions of our problem. (By the way, the same method can be used for the branch of negative solutions). In particular, we improve substantially the results concerning the smoothness of the branch $u(\lambda)$ as a function of λ: it was claimed in Proposition 1.1 that $u(\lambda)$ is continuous, but it is actually C^2 (and C^∞ if f is C^∞).

Now we study the same problem but with f strictly concave instead of strictly convex (assumption (3.4)), and try to apply the same method.

More precisely, we assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the assumptions

$$(3.10) \qquad f \text{ is } C^3, \text{ increasing}, \quad f(0) = 0, \quad f'(0) = 0 ,$$

$$(3.11) \qquad \frac{f(u)}{u} \text{ is strictly decreasing for } u > 0 \text{ and strictly increasing for } u < 0 ,$$

$$(3.12)\qquad \lim_{|u|\to+\infty} \frac{f(u)}{u} = f'(\infty) \ .$$

It is easily seen that in this case

$$(3.13)\qquad f'(\infty) < f'(0) \ .$$

A first result concerning non-existence of positive solutions can be proved by using the same comparison arguments as in Lemma 1.4.

Lemma 3.2: Assume that f satisfies (3.10)-(3.12). Then, if u is a nontrivial solution of (3.1), (3.2), $\lambda_1 + f'(\infty) < \lambda < \lambda_1 + f'(0)$.

A first, and nontrivial difficulty arises in the study of the uniqueness of nontrivial positive solution, which is essential to apply this continuation method, since none of the proofs of Theorem 1.3 works. Indeed, now the function $\lambda u - f(u)$ is convex and sub and supersolutions are no more available. But (recall Remark 1.4), the first part of Proof 1 of Theorem 1.3 still works: if u and v are nontrivial positive solutions of (3.1), (3.2) such that, e.g., $u \le v$, then by (3.11) the same argument yields $u \equiv v$. On the other hand, the second part, involving sub and supersolutions, does not work. However, if it is possible to prove that for two nontrivial positive solutions u and v, $u \le v$ or $v \le u$, then we have uniqueness. This will be a consequence of the following additional assumption

$$(3.14)\qquad \lambda_1 + f'(0) \le \lambda_2 + f'(\infty)$$

which can be written equivalently as $f'(0) - f'(\infty) \le \lambda_2 - \lambda_1$. This assumption is exactly of the same kind of those in [4] or [7].

Lemma 3.3: Assume that f satisfies (3.10)-(3.12) and (3.14). If u and v are nontrivial positive solutions of (3.1), (3.2), then $u \le v$ or $v \le u$.

Proof: If u and v are solutions we have

$$-\Delta(u-v) + f(u) - f(v) = \lambda(u-v) \quad \text{in } \Omega \ ,$$
$$w = 0 \quad \text{on } \partial\Omega \ .$$

and if we put $w = u - v$, this can be rewritten as

$$(3.15)\qquad -\Delta w + aw = \lambda w \quad \text{in } \Omega \ ,$$
$$w = 0 \quad \text{on } \partial\Omega \ ,$$

where a is defined as follows

$$a(x) = \begin{cases} \dfrac{f(u(x)) - f(v(x))}{u(x) - v(x)} & \text{if } u(x) \ne v(x) \ , \\ f'(u(x)) & \text{if } u(x) = v(x) \ . \end{cases}$$

Then a is continuous and (3.11) and the definition of a imply $f'(\infty) \le a \le f'(0)$, $a \ne f'(\infty)$.

We claim that if λ satisfies

$$\lambda_1 + f'(\infty) < \lambda \le \lambda_2 + f'(\infty) \ ,$$

then λ is the first eigenvalue of problem (3.15). Indeed, if not the usual comparison results imply

$$\lambda \ge \lambda_2(a) > \lambda_2(f'(\infty)) = \lambda_2 + f'(\infty) \ ,$$

a contradiction. Hence the corresponding eigenfunction $w = u - v$ does not change sign in Ω and, e.g., either $w \ge 0$, i.e., $u \ge v$, or either $w \le 0$, $u \le v$.

Theorem 3.1 ([6]): Suppose that f satisfies (3.10)-(3.12) and (3.14). Then, for any λ such that

$$(3.16)\qquad \lambda_1 + f'(\infty) < \lambda < \lambda_1 + f'(0)$$

there exists a unique nontrivial positive solution $u(\lambda)$ of (3.1), (3.2). The mapping $\lambda \longmapsto u(\lambda)$ from $]\lambda_1 + f'(\infty), \lambda_1 + f'(0)[$ into $C_0^{2,\alpha}(\bar\Omega)$ is C^1. Moreover

$$(3.17)\qquad \lim_{\lambda\downarrow\lambda_1+f'(\infty)} \|u(\lambda)\|_0 = +\infty .$$

<u>Proof</u>: The uniqueness follows easily from Lemmas 3.2 and 3.3. The existence proof is quite similar to that in Theorem 3.2, so we only sketch it.

First, by a completely similar application of Theorem 3.1 we obtain the existence of a "small" branch of nontrivial positive solutions in a left-neighborhood of $\lambda_1 + f'(0)$. This branch can be parametrized by λ, and can be continued to the left by using again the Implicit Function Theorem: the proof that $F_u(u,v)$, where (u,v) is a solution with $v > 0$, is an isomorphism is very similar, but this time $f'(v) < \frac{f(v)}{v}$ by (3.11). The proofs that the solutions on this continuated branch are positive, that there is a minimal λ^- for this continuation, and that $\lambda^- = \lambda_1 + f'(\infty)$ are analogous.

The existence results in Theorems 3.2 and 3.3 can be summarized in the following diagrams (cf. Figures 3 to 5). We point out that $u(\lambda)$ is an increasing function of λ in the first case (cf. Proposition 1.7) but we do not know if the same is true, i.e., if $u(\lambda)$ is decreasing in λ, under the assumptions of Theorem 3.3, in Fig. 5.

Figure 3 corresponds to Theorem 3.2 with assumption (3.5) and Figure 4 to the same theorem with assumption (3.6). Figure 5 illustrates the situation for Theorem 3.3.

It remains the case of a function f satisfying the same hypotheses (3.10)-(3.12), but not the additional hypothesis (3.14), which play a decisive role in the uniqueness proof. Suppose that f satisfies (3.10)-(3.12) and

$$\lambda_2 + f'(\infty) < \lambda_1 + f'(0) .$$

In this situation, the first part of the argument is untouched, and we prove exactly as before the existence of a "small" branch of nontrivial positive solutions, parametrized by λ, in a left-neighborhood of $\lambda_1 + f'(0)$. Moreover, if (u,v) is a solution with $v > 0$ and $\lambda < \lambda_2 + f'(\infty)$, then $F_u(u,v)$ is an isomorphism (the proof is the same) and the Implicit Function Theorem can be applied. In this way it can be proved, by using the fact $\lambda_1 + f'(\infty)$ is an asymptotic bifurcation

point for positive solutions, that for any λ in the interval $(\lambda_1 + f'(\infty), \lambda_2 + f'(\infty))$ there is a unique nontrivial positive solution. However, we do not know how to connect this branch with the "small" branch of positive solutions bifurcating from $\lambda_1 + f'(0)$. (Cf. Figure 6). On the other hand, it was proved in [67] by using the results in [3],[74] that there is <u>at least</u> a nontrivial positive solution for any λ in the interval $(\lambda_2 + f'(\infty), \lambda_1 + f'(0))$.

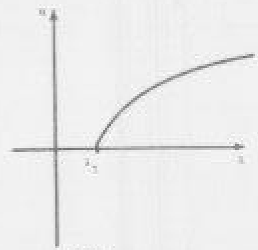

Fig. 3

Fig. 4

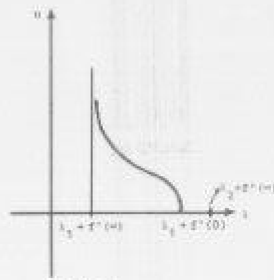

Fig. 5

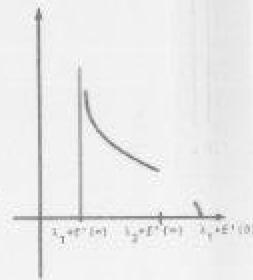

Fig. 6

3.2. LOCAL INVERSION THEOREMS AND CONTINUATION: THE CASE WITHOUT BIFURCATION.

We consider the nonlinear eigenvalue problem

$$(3.18) \qquad -\Delta u = \lambda f(x,u) \quad \text{in } \Omega ,$$

$$(3.19) \qquad u = 0 \quad \text{on } \partial\Omega ,$$

where the function $f : \bar{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

(3.20) f is C^3, $f(x,0) > 0$ for any $x \in \bar{\Omega}$,

(3.21) $f_u(x,0) > 0$ for any $x \in \bar{\Omega}$,

(3.22) $f_{uu}(x,s) > 0$ for any $x \in \bar{\Omega}$ and any $s \geq 0$.

This kind of problems arise in several physical applications (cf. [59],[77]). An interesting particular case, namely $f(x,u) = e^u$, was considered in Section 2.5. But in some respects the case $f(x,u) = (1 + \gamma u)^\beta$, $\beta,\gamma > 0$, is more difficult to treat (cf. [67],[44]).

It was pointed out in Section 2.5 that for $\lambda > 0$ every solution u is strictly positive on Ω, and that there is an unbounded component of positive solutions containing $(0,0)$. Moreover, by using sub and supersolutions and the obvious fact that 0 is a subsolution for (3.18), (3.19), it follows the existence of a minimal positive solution for some range of values of the parameter λ (cf. [18],[44]). But we are interested in getting some more information about solutions of (3.18),(3.19).

Let $\mu_1 > 0$ be the first eigenvalue of the linearized problem

$$-\Delta w = \mu f_u(x,0) w \quad \text{in } \Omega ,$$

$$w = 0 \quad \text{on } \partial\Omega .$$

By using comparison and continuation arguments it is not difficult to prove the following result (cf. [69],[44]).

<u>Theorem 3.4</u>: If u is a solution of (3.18),(3.19) with $\lambda > 0$, then $\lambda < \mu_1$. Moreover, there is a number $0 < \bar{\lambda} \leq \mu_1$ which is maximal with respect to the existence of a curve $\lambda \longrightarrow u(\lambda)$ of positive solutions with the properties

i) the mapping $\lambda \longrightarrow u(\lambda)$ from $[0,\bar{\lambda})$ into $C_0^{2,\alpha}(\bar{\Omega})$ is continuous,

ii) the operator $v \longrightarrow -\Delta v - \lambda f_u(x,u(\lambda)(x))v$ is invertible as a map from $C_0^{2,\alpha}(\bar{\Omega})$ into $C^\alpha(\bar{\Omega})$ for $0 \leq \lambda < \bar{\lambda}$.

Finally, $u(\lambda)$ is increasing as a function of λ and, for fixed λ, $0 \leq \lambda < \bar{\lambda}$, is the minimal positive solution of (3.18),(3.19).

It is interesting to see the behaviour of the curve of solutions near the critical value $\bar{\lambda}$ of the parameter. The special cases e^u and $(1+\gamma u)^\beta$ were treated in [67] for spheres by using phase plane techniques and some of these results were extended to general bounded domains Ω in [44]. This is the main theorem in [44].

<u>Theorem 3.5</u>: Suppose there exists $M > 0$ such that $\|u(\lambda)\|_{L^\infty(\Omega)} \leq M$, where $u(\lambda)$ is the curve of minimal positive solutions given by Theorem 3.4. Then there exists $\bar{u} = \lim_{\lambda \uparrow \bar{\lambda}} u(\lambda)$ in the topology of $C_0^{2,\alpha}(\bar{\Omega})$, and there exists $\varepsilon > 0$ such that the solutions of (3.18),(3.19) near $(\bar{\lambda},\bar{u})$ are on a curve $(\lambda(s),\tilde{u}(s))$, $|s| < \varepsilon$, with λ and $\tilde{u}$ satisfying

i) the map $s \longrightarrow (\lambda(s),\tilde{u}(s))$ from $(-\varepsilon,\varepsilon)$ into $\mathbb{R} \times C_0^{2,\alpha}(\bar{\Omega})$ is C^2,

ii) $\lambda(0) = \bar{\lambda}$, $\tilde{u}(0) = \bar{u}$, $\tilde{u}'(0) = v$, where v satisfies $-\Delta v = \bar{\lambda} f_u(x,\bar{u}(x))v$, $v = 0$ on $\partial\Omega$, $v > 0$ in Ω,

iii) $\lambda'(0) = 0$, $\lambda''(0) < 0$.

It is clear that this theorem implies the existence of exactly two solutions in a deleted left neighborhood of the critical value $\bar{\lambda}$. The proof can be divided in two parts. Suppose that the curve of minimal positive solutions given by Theorem 3.4 has a limit point of the form $(\bar{\lambda},\bar{u})$ (e.g., in the topology of $\mathbb{R} \times C_0^{2,\alpha}(\bar{\Omega})$). Then, by local inversion arguments as those in Theorem 3.1 it follows that $\lim_{\lambda \uparrow \bar{\lambda}} u(\lambda) = \bar{u}$ and that the curve $u(\lambda)$ can be continuated "smoothly" through $(\bar{\lambda},\bar{u})$ "bending back" precisely in $(\bar{\lambda},\bar{u})$ in such a way that there are exactly two positive solutions in a (deleted) left neighborhood of $\bar{\lambda}$. It is possible to prove that, under assumptions (3.20)–(3.22), this is equivalent to the existence of *a priori* estimates for the solutions $u(\lambda)$, $0 < \lambda < \bar{\lambda}$. This is not very easy for $f(x,u) = e^u$ and even more difficult for $f(x,u) = (1 + \gamma u)^p$ (cf.[44]).

3.3. EXISTENCE OF A SECOND SOLUTION. VARIATIONAL METHODS.

Our aim in this section is to improve the results of the preceding. For that, we make some supplementary assumptions which allow the use of a different kind of methods. Under these assumptions we shall prove that (3.18),(3.19) has at least two solutions for any λ in $(0,\bar{\lambda})$. Assume that f satisfies in addition

(3.23) $\quad f(x,z) \le c_1 + c_2 |z|^p \quad$ for $z > 0$, $1 < p < \frac{N+2}{N-2}$ if $N \ge 3$ or

$f(x,z) \le e^{\varphi(z)}$ with $\lim_{z \to +\infty} \frac{\varphi(z)}{z^2} = 0$ if $N = 2$.

(3.24) $\quad \lim_{|z| \to +\infty} \frac{f(x,z)}{z} = +\infty$,

(3.25) $\quad$ There exist $0 < \theta < \frac{1}{2}$ and $\bar{z} > 0$ such that for $z > \bar{z}$

$$F(x,z) = \int_0^z f(x,t)\,dt \le \theta z f(x,z) \ .$$

<u>Theorem 3.6 ([44])</u>: If f satisfies (3.20)–(3.25), then (3.18),(3.19) has at least two positive solutions for every λ in $(0,\bar{\lambda})$.

Theorem 3.6 follows from a very well-known abstract result, the "Mountain Pass Lemma", due to Ambrosetti-Rabinowitz [15] (cf. also [96], [84]).

<u>Theorem 3.7 ("Mountain Pass Lemma")</u>: Let E be a real Banach space and let $I \in C^1(E,\mathbb{R})$, with $I(0) = 0$. Suppose that I satisfies

(I_1) There exist $\alpha, \rho > 0$ such that $I(u) > 0$ in $B_\rho(0) - \{0\}$ and $I(u) \ge \alpha$ in $\partial B_\rho(0)$.

(I_2) There exists $e \in E - \{0\}$ such that $I(e) = 0$.

(I_3) (Palais-Smale condition): If (u_n) is a sequence in E such that $I(u_n) > 0$, $I(u_n)$ is bounded and $I'(u_n) \longrightarrow 0$, then u_n possesses a convergent subsequence.

Then I has a critical point $\bar{u} \in E$ such that $I(\bar{u}) \ge \alpha$.

<u>Proof of Theorem 3.6 (sketch)</u>: We redefine f for $z < 0$ in such a way that $f > 0$ and $f(x,-z)$ satisfies (3.23)–(3.25). Then the functional

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega F(x,u)\,dx$$

is $C^1(H_0^1(\Omega),\mathbb{R})$ by the continuity of f and (3.23).

We define another functional

$$I(u) = J(u + u(\lambda)) - J(u(\lambda))$$

for $0 < \lambda < \bar{\lambda}$ fixed, where $u(\lambda)$ is the minimal positive solution. Then $I(0) = 0$ and 0 is the critical point corresponding to $u(\lambda)$. The existence of a second solution follows from the application of Theorem 3.7 to the functional I; it is clear that (I_1)–(I_3) will follow from assumptions (3.20)–(3.25).

COMMENTS AND BIBLIOGRAPHICAL REMARKS

For a general overview of nonlinear diffusion equations we send to the recent book by Smoller [106], and also to Fife [54][55] and Henry [60]. A classical reference for bifurcation theory is the book by Krasnoselski [73]. The recent book by Chow-Hale [34] covers many related topics, especially local bifurcation methods. Other useful references are the books by Berger [22], Deimling [51] and Prodi-Ambrosetti [88], the lecture notes by Ioos [66], Nirenberg [83], Rabinowitz [97] and Sattinger [102], and the surveys by Stakgold [107], Marsden [81] and Nirenberg [84]. For positive solutions there is the classical book by Krasnoselski [74] and the excellent surveys by Amann [16] and P.L. Lions [80], where many more references can be found.

Needless to say, the topological methods considered here are applicable to a very large variety of nonlinear problems. Among them, we want to mention another interesting topic, namely Ambrosetti-Prodi type problems (cf. the survey by de Figueiredo [48], and also [14][28][63]).

The iterative scheme in Section 1.1 can be found in the book by Courant-Hilbert [41]. However, the method of sub and supersolutions was only developed at the end of the sixties with papers by Cohen [36], Cohen-Laetsch [37], Keller-Cohen [70], Simpson-Cohen [105], etc (cf. the bibliography of [103]. A more general and systematic presentation was given by Amann [2] and Sattinger [100]. An abstract version of the method in the framework of ordered Banach spaces was given some years after by Amann [6][10].

As it was pointed out above, we do not try to offer a very general version of this method, but only a simplified one in order to show clearly how it works. The operator $-\Delta$ can be replaced by more general second order linear differential operators with sufficiently smooth coefficients satisfying the Maximum Principle (cf. [2][101]). The assumption that the nonlinear term f is C^1 is not necessary, the method still works if f is locally C^α $(0 < \alpha < 1)$ and there exists $M > 0$ such that for any x, $f(x,u) + Mu$ is increasing in u (cf. again [2][10]). The method is also applicable to nonlinear boundary conditions ([5][2]) (even including unilateral constraints [65]), to

nonlinearities depending on the gradient (cf. [11] and its references), and to problems with discontinuous linear terms (cf., e.g., the work by Chang [33] and Stuart [109]). There are many other applications: integral and integro-differential equations, variational inequalities, etc. For all which concerns our main tool, the Maximum Principle, cf. the books by Protter-Weinberger [89] and Gilbarg-Trudinger [58].

Some interesting applications of this method to nonlinear elliptic problems can be found in the paper by Kazdan-Warner [69] (cf. also [48]). Sub and supersolutions are also applied in [29] to obtain S-shaped bifurcation curves and in [30] for the study of perturbed bifurcation problems.

The problem considered in Section 1.2 seems to arise in the theory of nuclear reactors (cf. [36][108]) and has been studied in [100][102][61][20], etc. (Of course, many of the results in, e.g., [10] or [80] are also applicable to this problem). A basic tool are the classical results contained in the book by Courant-Hilbert [41] (cf. also [106]) concerning the properties of the eigenvalues and the eigenfunctions of (1.16)(1.17), continuous and monotone dependence on the domain and the coefficient ρ, etc. The results in this Section are taken from [61][20] (cf. also [102][106]). The first uniqueness proof is a slight improvement of [37] and [100] (cf. the generalization in [67]), the second is included in [20]. Moreover, a fourth uniqueness proof, using Serrin's Sweeping Principle, can be found in [102]. Cf. [27] for a related result. The case of asymptotically linear f is treated in [61]. The exact multiplicity result of Proposition 1.3 was first proved in [31] by reasoning as in Section 3.2 below.

The results in Section 1.3 are contained in [62]. A first existence theorem for systems by using sub and supersolutions was already given by Sattinger [100]. Cf. [78][79][86][104], and the references therein for related results. Concerning the Maximum Principle for systems cf. [89][50]. An extension of the existence theorem in Section 1.3 was given in [53]; this generalization was motivated by the study of some free boundary problems for reaction-diffusion systems, cf. [16][52][57][46]. Cf. also [45][87] for related existence results. Uniqueness for systems is a rather difficult question: some more or less partial results can be found in, e.g., [9][45][62][78][79] and the remarks at the end of these Comments. Alternative existence proofs can be given by finding stationary points of the associated semigroups ([8][9]) or by global bifurcation arguments (cf. Section 2.7 and [82]).

We do not include here the application of these methods, mainly invariant regions, to the associated parabolic problems. Cf. the book by Smoller [106] and the work by Sattinger [100][102], Amann [8][9], Chueh-Conley-Smoller [35], Sabarnes et al. [17]-[19]. For an application to a system arising in combustion theory cf. [103][68].

Concerning the stability of solutions of the stationary problem, it was proved by Sattinger [100][102] (cf. also [10]) in the case of a single equation that if the solution obtained by sub and supersolutions is unique, then it is stable and all solutions with initial data in the interval $[u_0, u^0]$ converge to this unique solution (when time goes to infinity). If there are multiple solutions, then the problem becomes more involved (cf., e.g., [107]). For stability results for systems with sub and supersolutions cf. [42][78][79][84].

A very nice presentation of topological degree theory is given in the lecture notes by Rabinowitz [97], where many applications are also included. A detailed and systematic treatment of the degree can also be found in the books by Berger [22] and Deimling [31] (cf. also [73][83][34][106]).

The results of Section 2.2 are due to Lazer-McKenna [76], cf. also [13]. For other multiplicity results cf. [10][80][14][20][48] and the corresponding bibliographies.

The global bifurcation theorems in Section 2.3 are due to Rabinowitz (cf. [91]-[93][97]), where applications to nonlinear Sturm-Liouville problems and quasilinear elliptic equations are also included. (Cf. also [83]). For bifurcation at infinity, cf. [95]. The theorems in Section 2.4 are equally due to Rabinowitz ([91][92]). Global bifurcation theorems for positive solutions were obtained independently by Dancer [46] and Turner [110], cf. also [12][72].

More information on the fixed point index is given in [10] and [85]. In particular, [10] contains a number of applications, cf. also [3] (where some results by Krasnoselski [74] are extended) and [6].

Section 2.7 is a simplified version of [63], where the case of

nonlinear boundary conditions was treated. A variant of a predator-prey system considered by Conway-Smoller [39] is also studied in [61]. A related system (the Brusselator) was studied in [62] by using Theorem 2.5. As it was already pointed out in Remark 2.5 it is advantageous to employ the fixed point index, mainly because in this case it is sufficient to obtain a priori estimates only for positive solutions.

For other applications of the degree, cf. [20][22][48][80][83][84], etc. The Leray-Schauder degree can also be used to get stability results, cf. [99][101]. It is possible to combine the method of sub and supersolutions with the degree (or the fixed point index) to obtain multiplicity results, cf., e.g., [6][10][48][49][83].

The technique in Section 3.1 was used by Rabinowitz, cf. [92] (and [42]). The main tool (Theorem 3.1) is contained in [41], but there are similar results in the literature ([42][14][43][90]). For inversion theorems for mappings with singularities and applications cf. [13][14][88]. The theorems in Section 3.1 are in [41], cf. [4][7][10][20][89] for related work. An interesting application to a predator-prey system is given in [40]. The assumption (1.14) for the "concave" case is of the same kind of [14] or [7]. We remark that in the open problem of Fig. 6 it is possible to prove uniqueness for positive solutions in the one-dimensional case by using phase plane methods.

Sections 3.2 and 3.3 contain results by Crandall-Rabinowitz [44] (cf. also [69] for Section 3.2). These problems were treated by Gelfand [59] and Joseph-Lundgren [67] in the case of a sphere. Other results for this "forced" case are in [4][7][10]. Similar problems have been studied by Brezis-Nirenberg (cf., e.g., [26]). For some results on S-shaped curves by using sub and supersolutions, cf. [29].

A good reference for the critical point theory used in Section 3.3 is the survey [96] by Rabinowitz, which includes some applications. The Mountain Pass Lemma is due to Ambrosetti-Rabinowitz [15], cf. also [84]. For other applications combining critical point theory with the previous methods (sub and supersolutions, degree, etc), cf. [83][94]. The critical exponent of assumption (3.23) arises also in other contexts, cf. [48][49].

Our main reason to consider nonlinear eigenvalue problems of the form

$$- \Delta u + f(u) = \lambda u \quad \text{in } \Omega \ , \qquad u = 0 \quad \text{on } \partial\Omega$$

instead of nonlinear problems of the type

$$- \Delta u = \lambda f(x,u) \quad \text{in } \Omega \ , \qquad u = 0 \quad \text{on } \partial\Omega,$$

which are more frequently studied in the literature, relies on the fact that the problem of existence of positive solutions for some reaction-diffusion systems with homogeneous Dirichlet boundary conditions can be reduced to a nonlocal perturbation of the former problem. (Needless to say, the above problems are not independent, and many of the results of, e.g., [18] apply to both of them).

More precisely, a system arising in the activator-inhibitor interaction in morphogenesis can be reduced by a decoupling technique (cf. [98][76][103]) to the equation

$$- \Delta u + Ku + f(u) = \lambda u \quad \text{in } \Omega \ , \qquad u = 0 \quad \text{on } \partial\Omega,$$

where K is the solution operator of a linear elliptic equation and f satisfies the usual assumptions. This problem was treated by Rothe [98] for odd f by using critical point theory. After, Lazer-McKenna [76] used degree theory for the same problem with f not necessarily odd: actually the results in Section 2.2 are the specialization of those in [76] to $K = 0$.

The principal difficulty to apply the method of sub and supersolutions to this problem is the existence of some kind of Maximum Principle for the nonlocal linear operator $- \Delta + K$. A theorem of this type, together with applications to some nonlinear equations (including the previous one) was given by de Figueiredo-Mitidieri [50]. The same paper contains also an application of the Mountain Pass Lemma to related problems (cf. [71][72] for other results in the same direction). The local inversion and continuation methods in Section 3.1 can also be applied in this situation (cf. [45]), improving some of the previous results.

Another interesting example is the predator-prey system

$$- \Delta u = au - f(u) - uv \quad \text{in } \Omega \ ,$$
$$- \Delta v = bv - g(v) + uv \quad \text{in } \Omega \ ,$$
$$u = v = 0 \quad \text{on } \partial\Omega \ ,$$

where $a,b > 0$ are real parameters and f and g are as above. A slight variant of this system was studied in [40], and an open problem was existence and uniqueness of nontrivial positive solutions for some range of the parameters. Existence was proved by Blat-Brown [23] by using global bifurcation and by Dancer [47], this time by using degree for cones. Here again continuation methods can be applied [45]. Related problems arising in the study of the spread of a bacterial infection (cf. [31][32]) were treated by Blat-Brown (cf. [24][25]), again by global bifurcation methods and sub and supersolutions.

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REACTION-DIFFUSION PROBLEMS IN CHEMICAL ENGINEERING

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Introduction

In this set of lectures, we shall investigate some reaction-diffusion problems arising in chemical engineering. The range of phenomena that can occur is quite remarkable and we shall confine ourselves to a few problems that illustrate the rich variety of the field. Some important areas such as periodic oscillations and flame propagation will not be touched upon at all. Among the important books that deal with reaction-diffusion problems in chemical engineering, we mention here those of Aris [1], Buckmaster and Ludford [7], and Bischoff and Froment [14]. I have also relied on a number of valuable review papers, including those of Aris [2], Chandra and Davis [8], Boddington, Gray and Wake [6], and Pao [19]. Additional references on specific points will appear in the text.

1. The basic equations and some elementary applications

The basic equations governing chemical reactions are formulations of the conservation of mass and heat. The conservation of scalar physical quantities can be expressed in general as follows. Let Ω be a domain with boundary $\partial\Omega$ and let D be a bounded subdomain of Ω with boundary B. Let $H_D(t)$ be the accumulation or total amount in D at time t of the physical quantity under consideration; let $G_D(t)$ be the production in D per unit time, that is the amount of the quantity being generated (perhaps through a chemical process) per unit time in D at time t; let $F_D(t)$ be the amount of the quantity flowing into D through B per unit time.

Our basic conservation equation then takes the form

$$\frac{dH_D}{dt} = G_D + F_D. \tag{1.1}$$

We shall assume that G_D and H_D can be expressed as densities

$$G_D(t) = \int_D g(x,t)\,dx\,, \quad H_D(t) = \int_D h(x,t)\,dx\,.$$

We shall also assume that F_D can be written as a surface integral of