INTRODUCTION

General questions: given an open domain Ω in $\mathbf{R}^N$, a partial differential operator P in Ω (usually non-linear) and a solution u of

(1) $$P(u) = 0 \text{ in } \Omega,$$

then

Q1- Can we define in a natural way the restriction or extended restriction of u to the boundary of Ω ?

Q2- If this extended restriction exists in some class of objects defined on this boundary, can we reconstruct the solution u from an element of this class.

This program is obviously too ambitious to be completed, but in the case of second order elliptic or parabolic equations it has been partially fulfilled, namely for the non-linear Laplace and heat equations with strong absorption:

(2) $$\Delta u = u^q$$

and

(3) $$\frac{\partial u}{\partial t} - \Delta u + u^q = 0$$

where $\Delta = \sum_{1 \leq j \leq N} \frac{\partial^2}{\partial x_j^2}$ and $q > 1$. We shall not speak of the parabolic equation is this series of conferences and shall concentrate on the elliptic equation. Moreover we shall not go very far in the trace theory but shall give a framework of what has been done recently by analytical means in that field. Equation (2) plays a key role in the understanding of super-processes in the range $1 < q \leq 2$ and in scalar curvature questions in conformal differential geometry when $q = (N+2)/(N-2)$. In physics the study of this equation was initiated by R. Emden in 1897 (problems in meteorology) and later on by Thomas-Fermi in the 20-ies (theory of atoms and electronic potential with $q = 3/2$, $N = 3$) and Chandrashekar in 1937 (astrophysics and stars equilibrium problems).

In order to understand the problem, we shall divide our intervention in three parts. We shall also always assume that Ω is a bounded open domain with a regular boundary, say C^2.

Section I The linear Dirichlet Problem

Our aim will be to solve

(LDP) $$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = g \text{ on } \partial\Omega. \end{cases}$$

in a good enough framework, which means data f and g respectively in $L^1(\Omega, \rho dx)$ $(\rho(x) = \mathrm{dist}(x, \partial\Omega))$ and $L^1(\partial\Omega)$, and to prove the Brezis a priori estimates.

Section II The non-linear Dirichlet problem

We shall present Brezis, Keller-Osserman and Gmira-Véron 's results on the solvability of

(NLDP) $$\begin{cases} -\Delta u + |u|^{q-1} u = 0 \text{ in } \Omega, \\ \qquad\qquad u = \mu \text{ on } \partial\Omega. \end{cases}$$

where $q > 1$ and $\mu \in M(\partial\Omega)$, the space of Radon measures on $\partial\Omega$. We shall also consider the boundary singularity problem.

Section III The non-linear trace

Although some particular but very important results have been first obtained by Le Gall in a purely probabilistic framework we shall present Marcus-Veron's analytic of the boundary trace of any positive solution u of the non-linear elliptic equation

(NLEE) $$\Delta u = u^q.$$

This boundary trace is an outer regular not necessarily bounded Borel measure $\nu = \mathrm{Tr}(u)$. We shall study the properties of the mapping $u \mapsto \mathrm{Tr}(u)$ in some important cases, in particular in the subcritical case: $1 < q < (N+1)/(N-1)$.

I - THE LINEAR DIRICHLET PROBLEM

Given: Ω a bounded open domain of $\mathbf{R}^N$ with a C^3 boundary $\partial\Omega$ (in fact C^2 would be enough). Our aim is to solve

$$\text{(LDP)} \qquad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

In this Section we recall some basic tools in the theory of second order linear elliptic equations such as Green's functions, regularity theory, Sobolev spaces. Finally we prove the Brezis estimates.

I-1 Construction of the Green function

If u and v are two $C^2(\overline{\Omega})$ function, Green's identity gives

$$\text{(1)} \qquad \int_\Omega (v\Delta u - u\Delta v)dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right)ds$$

where ν is the unit outward normal vector to $\partial\Omega$. Fix a point y in Ω and define the Newton's kernel Γ by

$$\text{(2)} \qquad \Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} -\left(N(N-2)\omega_N\right)^{-1}|x-y|^{2-N}, & \text{if } N>2, \\ (2\pi)^{-1}\ln|x-y|, & \text{if } N=2, \end{cases}$$

($\omega_N =$ volume of unit ball in $\mathbf{R}^N$). It is well-known that $x \overset{\Gamma_y}{\mapsto} \Gamma_y(x) = \Gamma(x-y)$ is harmonic in $\mathbf{R}^N \setminus \{y\}$ and that

$$\text{(3)} \qquad \Delta\Gamma_y = \delta_y$$

in the sense of distributions in $\mathbf{R}^N$. Approximation argument shows that $v = \Gamma_y$ is admissible in (1) and that

$$\text{(4)} \qquad \int_\Omega \Delta\Gamma(x-y)u(x)dx = u(y).$$

Relations (1)-(2) yield Green's representation formula

$$\text{(5)} \qquad u(y) = \int_{\partial\Omega}\left(u\frac{\partial\Gamma}{\partial\nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial\nu}\right)ds + \int_\Omega \Gamma(x-y)\,\Delta u dx$$

$(\forall y \in \Omega)$ and, if u is harmonic,

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu} \right) ds. \tag{6}$$

If h is a $C^1(\overline{\Omega}) \cap C^2(\Omega)$ harmonic function, then (1) implies

$$\int_\Omega h \Delta u dx = -\int_{\partial\Omega} \left(u \frac{\partial h}{\partial \nu} - h \frac{\partial u}{\partial \nu} - \right) ds.$$

If $G = \Gamma + h$ (5) becomes

$$u(y) = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial \nu} - G \frac{\partial u}{\partial \nu} \right) ds + \int_\Omega G \Delta u dx$$

and if G vanishes on $\partial\Omega$,

$$u(y) = \int_{\partial\Omega} u \frac{\partial G}{\partial \nu} ds + \int_\Omega G \Delta u dx. \tag{7}$$

Formula (7) implies a representation of harmonic functions in terms of their boundary values. The function $G = G(x,y)$ is the Green's function of Ω. It is unique.

The question of finding an harmonic function in Ω with a given continuous boundary data g can be solved by the Perron's method of sub-harmonic functions. Let S_g be the set of $C^0(\overline{\Omega})$ sub-harmonic functions with respect to g that is the set of continuous functions w which satisfy

$$w(y) \le \frac{1}{\omega_N R^N} \int_{B_R(y)} w(x) dx \tag{8}$$

$(\forall y \in \Omega,\ \forall R > 0 \ /\ \overline{B}_R(y) \subset \Omega)$ and $w \le g$ on $\partial\Omega$. Then Perron's theorem asserts that the function $u(x) = \sup_{w \in S_g} w(x)$ is harmonic in Ω and satisfies $u = g$ on $\partial\Omega$ (since the regularity of $\partial\Omega$ implies the existence of barrier functions at each point).

Remark. Estimate (8) is equivalent to

$$w(y) \le \frac{1}{N\omega_N R^N} \int_{\partial B_R(y)} w(x) ds. \tag{8'}$$

Sometimes C^2 sub-harmonic functions are defined by the fact that $\Delta u \ge 0$.

From Perron's method the function G exists. The function $\frac{\partial G}{\partial \nu}$ defined in $\Omega \times \partial\Omega$ is the Poisson kernel of Ω, it is also often quoted as P. If g belongs to $L^1(\partial\Omega)$ the function

$$P_g(x) = \int_{\partial\Omega} P(x, y) g(y) ds(y) \tag{9}$$

defined for $x \in \Omega$ is called the *Poisson potential* of g. It is an harmonic function in Ω.

I-2 Regularity results

If Γ is the Newtonian potential, then it is possible to check directly the regularity of the second derivatives of $x \mapsto \int_{\mathbf{R}^N} \Gamma(x - y) f(y) dy$ when $f \in C_0^\infty(\mathbf{R}^N)$. This regularity is given in two directions:

1- The spaces $C^{2,\alpha}$ of C^2 functions with Hölder continuous second derivatives $(0 < \alpha < 1)$.

2- The Sobolev spaces $W^{2,p}$ of L^p functions whose derivatives up to the order two belong to L^p $(1 < p < \infty)$.

This scope of the regularity estimates goes far beyond the study of the Laplace equation and applies to very general elliptic operators. A particular case of interest for us is the Helmoltz operator

$$u \mapsto \Delta u - cu \tag{10}$$

where $x \mapsto c(x)$ is a bounded measurable function in Ω.

Definition 1. *Let* $1 \leq p \leq \infty$ *and* $k \in \mathbf{N}_*$, *then*

$$|\zeta|_{W^{k,p}(\Omega)} = \sum_{|\beta| \leq k} \left\| D^\beta \zeta \right\|_{L^p(\Omega)} \quad \left(\forall \zeta \in C_0^\infty(\Omega)\right) \tag{11}$$

where $\beta = (\beta_1, \beta_2, \ldots \beta_N)$, $|\beta| = \sum_{j \leq N} \beta_j$ and $D^\beta = \frac{\partial^\beta}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_N^{\beta_N}}$.
The space $W^{k,p}(\Omega)$ *is the space of distributions belonging to* $L^p(\Omega)$ *as well as all their derivatives up to the total order k. It is endowed with the structure of a Banach space with the norm defined in* (11). *The space* $W_0^{k,p}(\Omega)$ *is the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$.

Definition 2. *Let* $0 < \alpha < 1$, *then*

(12) $$[\zeta]_{\alpha,\Omega} = \sup_{\substack{x \neq y \\ (x,y)\in\Omega\times\Omega}} \frac{|\zeta(x)-\zeta(y)|}{|x-y|^{\alpha}}$$

and

(13) $$\|\zeta\|_{\alpha,\Omega} = \sup_{x\in\Omega}|\zeta(x)| + [\zeta]_{\alpha,\Omega}$$

The space $C^{\alpha}(\overline{\Omega})$ *is the space of continuous functions defined in* $\overline{\Omega}$ *for which the above norm* $\|\ \|_{\alpha,\Omega}$ *is finite. For* $k \in \mathbf{N}_*$ *denote*

(14) $$\|\zeta\|_{k,\alpha,\Omega} = \sup_{|\beta|\leq k}\sup_{x\in\Omega}|D^{\beta}\zeta(x)| + \sup_{|\beta|=k}\sup_{x\in\Omega}[D^{\beta}\zeta]_{\alpha,\Omega},$$

then the space $C^{k,\alpha}(\overline{\Omega})$ *is defined similarly to* $C^{\alpha}(\overline{\Omega})$ *from the above norm..*

Theorem 1 (Schauder). *Suppose that* $\partial\Omega$ *is of class* $C^{2,\alpha}$ (*i.e. locally represented by* $C^{2,\alpha}$ *functions*), $c \in C^{\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$ *and let* $u \in C^{2}(\overline{\Omega})$, $g \in C^{2,\alpha}(\partial\Omega)$ *and* $f \in C^{\alpha}(\overline{\Omega})$ *such that*

(15) $$\begin{cases} \Delta u - cu = f & \textit{in } \ \Omega, \\ \qquad u = g & \textit{on } \ \partial\Omega. \end{cases}$$

Then there exists a positive constant $C = C(\alpha, \Omega, \|c\|_{\alpha,\Omega})$ *such that*

(16) $$\|u\|_{2,\alpha,\Omega} \leq C\left(\|f\|_{\alpha,\Omega} + \|g\|_{2,\alpha,\Omega} + \sup_{\Omega}|u|\right).$$

If we define the space $C^{\alpha}(\Omega)$ as the space of functions ζ for which $\|\zeta\|_{\alpha,K}$ is finite for any compact subset $K \subset \Omega$, we have a local version of Theorem 1.

Theorem 1'. *Suppose that* $c \in C^{\alpha}(\overline{\Omega})$ *for some* $0 < \alpha < 1$ *and let* $u \in C^{2}(\overline{\Omega})$ *and* $f \in C^{\alpha}(\overline{\Omega})$ *such that*

(17) $$\Delta u - cu = f \ \ \textit{in } \Omega.$$

Then u *belongs to* $C^{2,\alpha}(\Omega)$*; more precisely, for any compact subset* $K \subset \Omega$*, there exists a positive constant* $C = C(\alpha, \Omega, \|c\|_{\alpha,\Omega}, \mathrm{dist}(K, \partial\Omega))$ *such that*

(18) $$\|u\|_{2,\alpha,K} \leq C\left(\|f\|_{\alpha,\Omega} + \|g\|_{2,\alpha,\Omega} + \sup_{\Omega}|u|\right).$$

Theorem 2 (Agmon-Douglis-Nirenberg). *Suppose that* $\partial\Omega$ *is of class* C^{2} *(i.e. locally represented by* C^{2} *functions)* $c \in C(\overline{\Omega})$ *and let* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$

and $f \in L^p(\Omega)$ (*for some* $1 < p < \infty$) *satisfy* (17). *Then there exists a positive constant* $C = C(p, \Omega, \sup_\Omega |c|)$ *such that*

(19) $$\|u\|_{W^{2,p}(\Omega)} \le C\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

We say that $\zeta \in W^{k,p}_{loc}(\Omega)$ if the restriction of ζ to any open subset Θ with compact closure in Ω belongs to $W^{k,p}(\Theta)$. The local A-D-N estimates are

Theorem 2'. *Suppose that c is continuous in* $\overline{\Omega}$ *and let* $u \in W^{2,p}(\Omega)$ *and* $f \in L^p(\Omega)$ *for some* $1 < p < \infty$ *such that* (17) *holds. Then for to any open subset* Θ *as above there exists a positive constant depending on p,* Ω*,* $\sup_\Omega |c|$ *and* $\mathrm{dist}(\Theta, \partial\Omega)$ *such that*

(20) $$\|u\|_{W^{2,p}(\Theta)} \le C\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right).$$

Remark. There exist very useful imbedding theorems between Sobolev spaces $W^{k,p}(\Omega)$ and $W^{l,q}(\Omega)$ for $0 \le l < k$ and $1 \le p < q \le \infty$ or Sobolev spaces $W^{k,p}(\Omega)$ and $C^{l,\alpha}(\overline{\Omega})$ for some $0 \le l < k$ and $\alpha \in (0,1)$ depending on N, k and p. Their expression is a bit technical but we shall refer to them later on.

I-3 The maximum principles

Theorem 3 (weak maximum principle). *Suppose that* c *is continuous and non-negative in* $\overline{\Omega}$ *and let* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfy*

(21) $$\Delta u - cu \ge 0 \quad (\text{resp.} = 0) \quad \text{in } \Omega.$$

Then

(22) $$\sup_\Omega u \le \sup_{\partial\Omega} u^+ \quad (\text{resp. } \sup_\Omega |u| = \sup_{\partial\Omega} |u|).$$

This theorem admits a strong form stating that under the same assumptions on c and u , the function u cannot achieve a non-negative maximum in Ω unless it is constant. This strong maximum principle is a consequence of the so called Hopf lemma.

Theorem 4 (Hopf). *Suppose that* c *is continuous and non-negative in* $\overline{\Omega}$ *and* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *satisfies inequality* (21). *Suppose also that* $x_0 \in \partial\Omega$ *is such that* $u(x_0) > u(x)$ $(\forall x \in \Omega)$ *and* $\dfrac{\partial u}{\partial \nu}(x_0)$ *exists. If* $u(x_0) \ge 0$*, then* $\dfrac{\partial u}{\partial \nu}(x_0) < 0$. *If* $c \equiv 0$ *no assumption is needed on the sign of* $u(x_0)$.

The previous result remains true if c is no longer non-negative provided it is assumed that $u(x_0) = 0$.

I-4 Dirichlet problem in L^1

We recall that $\rho(x) = \mathrm{dist}(x, \partial\Omega)$.

Theorem 5 (Brezis estimates). *Let* f *be a measurable function in* Ω *such that* $\rho f \in L^1(\Omega)$ *and* $g \in L^1(\partial\Omega)$. *Then there exists a unique function* $u \in L^1(\Omega)$ *such that*

$$(23) \qquad -\int_\Omega u\Delta\zeta dx = \int_\Omega f\zeta dx - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g ds,$$

for any $\zeta \in C_0^{1,1}(\overline{\Omega})$, *the space of functions with compact support in* $\overline{\Omega}$ *and Lipschitz continuous gradient. Moreover there exists* $C = C(\Omega) > 0$ *such that*

$$(24) \qquad \|u\|_{L^1(\Omega)} \leq C\left(\|\rho f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}\right),$$

and u *satisfies*

$$(25) \qquad -\int_\Omega |u|\Delta\zeta dx + \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu}|g| ds \leq \int_\Omega f\zeta \,\mathrm{sgn}(u) dx,$$

$$(26) \qquad -\int_\Omega u^+\Delta\zeta dx + \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g^+ ds \leq \int_\Omega f\zeta \,\mathrm{sgn}^+(u) dx,$$

for any $\zeta \in C_0^{1,1}(\overline{\Omega})$, $\zeta \geq 0$. *In these formulas* $\mathrm{sgn}(r) = 1$ *if* $r > 0$, $\mathrm{sgn}(r) = -1$ *if* $r < 0$ *and vanishes at* 0, *while* $\mathrm{sgn}^+(r) = 1$ *if* $r \geq 0$ *and vanishes if* $r < 0$.

Remark. If $\zeta \in C_0^{1,1}(\overline{\Omega})$ there exists a constant C such that $|\zeta| \leq C\rho$.

Proof. Step 1. Take $(f_n, g_n) \in C^0(\overline{\Omega}) \times C^2(\partial\Omega)$ and call u_n the unique solution of the corresponding (LDP). Set γ a smooth, odd and increasing approximation of the sgn function with $|\gamma| \leq \mathrm{sgn}$ and let $\eta = \eta_n$ be the solution of

$$(27) \qquad \begin{cases} -\Delta\eta = \gamma(u_n) & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega. \end{cases}$$

From Theorem 1 $\eta \in C^{2,\alpha}(\overline{\Omega})$ and

(28) $$\int_\Omega u_n\gamma(u_n)dx = \int_\Omega f_n\eta dx - \int_{\partial\Omega}\frac{\partial\eta}{\partial\nu}g_n ds.$$

If β is the solution of

(29) $$\begin{cases} -\Delta\beta = 1 & \text{in } \Omega, \\ \beta = 0 & \text{on } \partial\Omega. \end{cases}$$

then β is also $C^{2,\alpha}$, $-\beta \le \eta \le \beta \le C\rho$ with $C = C(\Omega)$ and $\left|\frac{\partial\eta}{\partial\nu}\right| \le \left|\frac{\partial\beta}{\partial\nu}\right| \le C$ on the boundary. Therefore

(30) $$\int_\Omega u_n\gamma(u_n)dx \le C\left(\int_\Omega |f_n|\rho dx + \int_{\partial\Omega}|g_n|ds\right),$$

which gives (24) for (u_n, f_n, g_n) by letting γ go to sgn.

Step 2. Let $\zeta \in C_0^{1,1}(\overline{\Omega})$, $\zeta \ge 0$ and $\xi = \xi_n = \zeta\gamma(u_n)$. Then

(31) $$-\int_\Omega u_n\Delta(\zeta\gamma(u_n))dx = \int_\Omega f_n\zeta\gamma(u_n)dx - \int_{\partial\Omega}\frac{\partial}{\partial\nu}(\zeta\gamma(u_n))g_n ds.$$

But

$$-\int_\Omega u_n\Delta(\zeta\gamma(u_n))dx = \int_\Omega \gamma(u_n)\nabla u_n.\nabla\zeta dx + \int_\Omega \zeta\gamma'(u_n)|\nabla\gamma(u_n)|^2 dx$$
$$-\int_{\partial\Omega}u_n\gamma(u_n)\frac{\partial\zeta}{\partial\nu}ds - \int_\Omega \zeta u_n\gamma'(u_n)\frac{\partial u_n}{\partial\nu}ds.$$

Set $j(r) = \int_0^r \gamma(s)ds$, then

$$\int_\Omega \gamma(u_n)\nabla u_n.\nabla\zeta dx = \int_\Omega \nabla j(u_n).\nabla\zeta dx,$$
$$= -\int_\Omega j(u_n)\Delta\zeta dx + \int_{\partial\Omega} j(u_n)\frac{\partial\zeta}{\partial\nu}ds.$$

But $\int_\Omega \zeta\gamma'(u_n)|\nabla\gamma(u_n)|^2 dx \ge 0$. Since $\zeta = 0$ on $\partial\Omega$, we have

$$-\int_{\partial\Omega}u_n\gamma(u_n)\frac{\partial\zeta}{\partial\nu}ds - \int_\Omega \zeta u_n\gamma'(u_n)\frac{\partial u_n}{\partial\nu}ds = -\int_{\partial\Omega}\frac{\partial}{\partial\nu}(\zeta\gamma(u_n))g_n ds$$

and

(32) $$-\int_\Omega j(u_n)\Delta\zeta dx+\int_{\partial\Omega} j(g_n)\frac{\partial\zeta}{\partial\nu}ds\le\int_\Omega f_n\gamma(u_n)\zeta dx.$$

By letting γ go to sgn we get (25) for (u_n, f_n, g_n). We prove (26) in the same way by replacing γ by a smooth increasing approximation from below of sgn^+ which vanishes on $(-\infty,0)$ and is positive on $(0,\infty)$.

Step 3. Existence and uniqueness. Assume that $\{(\rho f_n, g_n)\}\underset{n\to\infty}{\to}\{(\rho f, g)\}$ in $L^1(\Omega)\times L^1(\partial\Omega)$. Then $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega)$ from Step 1 and its limit u satisfies (23)-(26). This gives existence. For uniqueness we suppose that w is an integrable function which satisfies $\int_\Omega w\Delta\zeta dx=0$ for any $\zeta\in C_0^{1,1}(\overline{\Omega})$; for test function, we take η the solution of

(33) $$\begin{cases}-\Delta\eta=\gamma(w) & \text{in } \Omega,\\ \eta=0 & \text{on } \partial\Omega.\end{cases}$$

Although it is not a $C^2(\overline{\Omega})$ function it belongs to $\bigcap_{p<\infty}\left(W^{2,p}\cap W_0^{1,p}\right)$ by Theorem 2 which is included into $\left(\bigcap_{\alpha<1}C^{1,\alpha}(\overline{\Omega})\right)\cap C_0^1(\overline{\Omega})$ by Sobolev imbedding theorems. This is sufficient for a test function since $\Delta\eta$ is essentially bounded. Then $\int_\Omega w\gamma(w)dx=0$ and $w\equiv 0$.

Remark. Estimate (26) means that the mapping $\mathsf{P}_\Omega:(f,g)\mapsto u=\mathsf{P}_\Omega(f,g)$ solution of (LDP) in the sense of Theorem 5 is order preserving from $L^1(\Omega,\rho dx)\times L^1(\partial\Omega)$ into $L^1(\Omega)$.

II - THE NON-LINEAR DIRICHLET PROBLEM

Given Ω a C^3 bounded open domain of $\mathbf{R}^N$, our aim is to solve

$$\text{(NLDP)}\qquad \begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases}$$

where $q > 1$ and $\mu \in M(\partial\Omega)$, the space of Radon measures on $\partial\Omega$. We also are interested into the following question: does it exists a non-zero function u belonging to $C(\overline{\Omega} \setminus \{a\}) \cap C^2(\Omega)$ for some $a \in \partial\Omega$ and satisfying

$$\text{(ISP)}\qquad \begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{a\}. \end{cases}$$

II-1 The regular non-linear Dirichlet problem

If we want to consider the problem

$$(1)\qquad \begin{cases} -\Delta u + |u|^{q-1}u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $q > 1$ and $g \in C^{2,\alpha}(\partial\Omega)$, we set $\Phi(x) = \int_{\partial\Omega} P(x,y)g(y)ds$ (Φ is $C^{2,\alpha}$), and introduce

$$(2)\qquad J_g(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + \frac{1}{q+1}|v + \Phi|^{q+1} \right) dx$$

on $W_0^{1,2}(\Omega) \cap L^{q+1}(\Omega)$. The functional J_g is strictly convex and l.s.c. in $W_0^{1,2}(\Omega) \cap L^{q+1}(\Omega)$ and it satisfies

$$(3)\qquad \lim_{\|v\|_{W^{1,2} \cap L^{q+1}} \to \infty} J_g(v) = \infty.$$

A classical result from convex analysis asserts that J_g achieves its minimum in $W_0^{1,2}(\Omega) \cap L^{q+1}(\Omega)$ at a unique point w where

$$(4)\qquad \int_\Omega \left(\nabla w . \nabla\zeta + |w + \Phi|^{q-1}(w + \Phi)\zeta \right) dx = 0$$

$\left(\forall \zeta \in W_0^{1,2}(\Omega) \cap L^{q+1}(\Omega) \right)$. This implies that the function $u = w + \Phi$ satisfies

$$(5)\qquad \int_\Omega \left(\nabla u . \nabla\zeta + |u|^{q-1}u\zeta \right) dx = 0.$$

In particular if γ is a smooth increasing approximation of the function $r \mapsto r^+$ vanishing on $(-\infty,0]$ and positive on $(0,\infty)$ and if $m=\sup_{\partial\Omega} g$, then $\gamma(u-m) \in W_0^{1,2}(\Omega)\cap L^{q+1}(\Omega)$ and

(6) $$\int_\Omega \left(\gamma'(u-m)|\nabla u|^2+|u|^{q-1}u\gamma(u-m)\right)dx=0.$$

Therefore $\gamma(u-m)\equiv 0$ a.e. which implies that u is essentially bounded from above. In the same way u is bounded from below and finally $u\in L^\infty(\Omega)$. Extensions of Theorems 1-2 in Sect. I yield $u\in C^{2,\alpha}(\overline{\Omega})$. Consequently we have proved that for any $g\in C^{2,\alpha}(\partial\Omega)$ there exists a function $u\in C^{2,\alpha}(\overline{\Omega})$ satisfying (1); u is unique from Brezis theorem since, if u_1 and u_2 are solutions corresponding to boundary data g_1 and g_2, we have

(7) $$\begin{aligned}-\int_\Omega\left(|u_1-u_2|\Delta\zeta dx\right)+\int_\Omega\left(\left||u_1|^{q-1}u_1-|u_2|^{q-1}u_2\right|\operatorname{sgn}(u_1-u_2)\zeta dx\right)\\ \leq -\int_\Omega\frac{\partial\zeta}{\partial\nu}\operatorname{sgn}(u_1-u_2)ds,\end{aligned}$$

for any $\zeta\in C_0^{1,1}(\overline{\Omega})$, $\zeta\geq 0$. In particular

(8) $$\|u_1-u_2\|_{L^1(\Omega)}+\left\|\left(|u_1|^{q-1}u_1-|u_2|^{q-1}u_2\right)\rho\right\|_{L^1(\Omega)}\leq C\|g_1-g_2\|_{L^1(\partial\Omega)}.$$

In the same way $g_1\leq g_2\Rightarrow u_1\leq u_2$. From $C^{2,\alpha}$ boundary data, we can go to continuous boundary data, by using (8) and the fact that $\sup_\Omega|u|\leq\sup_{\partial\Omega}|g|$.

We shall denote $u=\mathsf{P}_\Omega^q(g)$ the solution of (1) in Ω with boundary data g.

II-2 The Keller-Osserman estimates and the large solutions

One of the striking properties of any solution of

(9) $$-\Delta u+|u|^{q-1}u=0 \quad\text{in } \Omega,$$

$(q>1)$ is the existence of an a priori estimate.

Theorem 1 (Keller-Osserman). *There exists a constant* $C=C(q,N)>0$ *such that if* u *is any* $C^2(\Omega)$ *solution of* (9),

(10) $$|u(x)|\leq C\rho(x)^{-2/(q-1)} \qquad (\forall x\in\Omega).$$

Proof. Step 1. Suppose that $v \in C^0(B_R(0))$ is C^2 on $O^+ = (x:\ v(x) > 0)$ where it satisfies

(11) $$-\Delta v + Av^q \leq B.$$

for some $A > 0$ and $B \geq 0$. Then we claim that

(12) $$v(0) \leq \left(\frac{\beta(N,q)}{AR^2}\right)^{1/(q-1)} + \left(\frac{B}{A}\right)^{1/(q-1)}.$$

Take $\rho \in (0,R)$ and set $\psi(x) = \psi(r) = \lambda(\rho^2 - r^2)^{-2/(q-1)} + \mu$, where $\lambda > 0$ and $\mu \geq 0$ have to be determined in order to have

(13) $$-\Delta\psi + A\psi^q \geq B.$$

Then

$$-\Delta\psi + A\psi^q$$
$$\geq \lambda(\rho^2 - r^2)^{-2q/(q-1)}\left(A\lambda^{q-1} - \frac{2NR^2}{q-1} + \frac{2}{q-1}\left(N - 2\frac{q+1}{q-1}\right)r^2\right) + A\mu^q$$

and if we choose $\beta = \max\left(\frac{2N}{q-1}, \frac{4(q+1)}{(q-1)^2}\right)$, $\lambda = \left(\frac{\beta}{A\rho^2}\right)^{1/(q-1)}$ and $\mu = \left(\frac{B}{A}\right)^{1/(q-1)}$ we have (13). From Kato's inequality

(14) $$\Delta(v-\psi)^+ \geq \operatorname{sgn}^+(v-\psi)\Delta(v-\psi)$$

in the sense of distributions in $B_\rho(0)$ where $(v-\psi)^+$ has compact support. Therefore $(v-\psi)^+ \equiv 0$ and $v(0) \leq \psi(0)$. Letting $\rho \uparrow R$ yields (12)

Step 2. Let $x_0 \in \Omega$ and $R = \operatorname{dist}(x_0, \partial\Omega) = \rho(x_0)$. The function u^+ is continuous in $B_R(0)$ and is C^2 on $O^+ = (x:\ u(x) > 0)$ where there holds

(15) $$-\Delta u + u^q = 0.$$

By applying Step 1 in $B_R(x_0)$, we get

(16) $$u(x_0) \leq \left(\frac{\beta}{\rho^2(x_0)}\right)^{1/(q-1)},$$

and (10) follows by replacing u by $-u$.

From the Keller-Osserman estimate it is possible to construct a positive solution u of (9) which blows up everywhere on the boundary it is called the **large solution**. In fact the for $n > 0$ set u_n the solution of

$$\text{(17)} \qquad \begin{cases} -\Delta u_n + |u_n|^{q-1} u_n = 0 & \text{in } \Omega, \\ u_n = n & \text{on } \partial\Omega, \end{cases}$$

Then $0 < u_n < u_{n'}$ for $n' > n$. Because of (10) $\{u_n\}$ is locally bounded in Ω, independently of n and therefore it converges, when n goes to infinity, to some function u_Ω which satisfies

$$\text{(18)} \qquad \lim_{\rho(x)\to 0} u_\Omega(x) = \infty.$$

Going to the limit in the relation

$$\text{(19)} \qquad \int_\Omega \left(-u_n \Delta\zeta + u_n^q \zeta\right) d\zeta = 0$$

where $\zeta \in C_0^\infty(\Omega)$ implies that u_Ω satisfies the same expression. Since it is locally bounded it is a solution of (9).

One of the main problem concerning the large solution was the question of uniqueness. This uniqueness was proved:

1- In the case $q = (N+2)/(N-2)$ by Loewner and Nirenberg (1974) in a geometric framework.

2- When Ω is star-shaped with respect to some point by Iscoe, by using a transformation which conserves the equation

$$\text{(20)} \qquad u \overset{N_k}{\mapsto} N_k(u) \quad \text{where } N_k(u)(x) = k^{2/(q-1)} u(kx) \quad (k > 0).$$

3- When $\partial\Omega$ is smooth and compact by Bandle and Marcus and separately Véron (1992). The technique uses the expansion

$$\text{(21)} \qquad \lim_{\rho(x)\to 0} \rho^{2/(q-1)}(x) u_\Omega(x) = \left(\frac{2(q+1)}{(q-1)^2}\right)^{1/q-1)}.$$

4- When $\partial\Omega$ is not regular, first results are due to Le Gall (1994) for $N = 2 = q$ in a probabilistic framework and then Marcus and Véron (1995) in the general case when $\partial\Omega$ is locally the graph of a continuous function (in that case existence may not hold if $q \geq (N-1)/(N-3)$).

5- When $1 < q < N/(N-2)$ it has been recently noticed by Véron (MSRI Oct. 1997) that no assumption on $\partial\Omega$ is needed in order to have a large solution. Moreover, if $\partial\Omega \subset \overline{\Omega}^c$, this large solution is unique.

The following result follows from the Keller-Osserman estimate.

Theorem 2. *Suppose* $g \in C(\partial\Omega, [0,\infty])$, *then there exists a positive solution u of* (1).

Proof. Considering the increasing scheme $\{v_n\}$ for $n > 0$

(22)
$$\begin{cases} -\Delta v_n + |v_n|^{q-1} v_n = 0 \ \text{ in } \ \Omega, \\ v_n = g_n = \min(n, g) \ \text{ on } \ \partial\Omega, \end{cases}$$

then v_n is positive and converges to a solution of (1) when n goes to infinity.

Remark. In 1993 Kondratiev and Nikishkin proved that, within this framework, uniqueness may not hold when $1 < q < (N+1)/(N-1)$.

Remark. This large solution is the maximal solution of (9) in Ω in the sense that any other solution is dominated by it.

II-3 The L^1 non-linear Dirichlet problem

Theorem 3 (Brezis). *Let* $g \in L^1(\partial\Omega)$, *then there exists a unique* $u \in L^1(\Omega) \cap L^q(\Omega, \rho dx)$ *such that*

(23)
$$\int_\Omega \left(-u\Delta\zeta + |u|^{q-1} u\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g ds$$

$\left(\forall \zeta \in C_0^{1,1}(\overline{\Omega})\right)$. *Moreover the mapping* $g \mapsto u = P_\Omega^q(g)$ *is increasing and*

(24)
$$\|u_1 - u_2\|_{L^1(\Omega)} + \|\rho(h(u_1) - h(u_2))\|_{L^1(\Omega)} \le C\|g_1 - g_2\|_{L^1(\partial\Omega)}$$

where $u_j = P_\Omega^q(g_j)$, $j = 1,2$ *and* $h(r) = |r|^{q-1} r$.

Proof. Let $\{g_n\} \in C^3(\partial\Omega)$ such that $g_n \to g$ in $L^1(\partial\Omega)$ when n goes to infinity and denote $u_n = P_\Omega^q(g_n)$. Then

(25)
$$\int_\Omega \left(-u_n\Delta\zeta + |u_n|^{q-1} u_n\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} g_n ds$$

for any $\zeta \in C_0^{1,1}(\overline{\Omega})$, and from estimate (8) $\{(u_n, h(u_n))\}$ is a Cauchy sequence in $L^1(\Omega) \times L^1(\Omega, \rho dx)$. Therefore $h(u_{n_k}) \to h(u)$ a.e. and in $L^1(\Omega, \rho dx)$ which implies (23). The uniqueness follows from the Brezis linear estimates and the monotonicity from the monotonicity of P_Ω^q in $C^0(\partial\Omega)$.

II-4 Measure boundary data

The Poisson formula (9)-Sect. I which expresses the Poisson potential of a function $g \in L^1(\partial\Omega)$ is extendible to a Radon measure on $\mu \in \mathcal{M}(\partial\Omega)$. We set

(26) $$P_\mu(x) = \int_\Omega P(x,y)\, d\mu(y)$$

$(\forall x \in \Omega)$, and the function P_μ is harmonic in Ω and takes the value μ on $\partial\Omega$ in the sense that

(27) $$-\int_\Omega P_\mu \Delta\zeta dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu$$

$(\forall \zeta \in C_0^{1,1}(\overline{\Omega}))$. In order to extend Theorem 3 to measure boundary data, we need the following estimates on the Poisson kernel: there exists $C = C(\Omega) > 0$ such that

(28) $$C^{-1}|x-y|^{-N} \rho(x) \le P(x,y) \le C|x-y|^{-N} \rho(x)$$

$(\forall (x,y) \in \Omega \times \partial\Omega)$. Consequently

(29) $$\|P(.\,,y)\|_{L^p(\Omega)} \le K_{p,\Omega} \left(\forall 1 \le p < N/(N-1)\,,\ \forall y \in \partial\Omega\right),$$

(30) $$\|P(.\,,y)\|_{L^p(\Omega,\rho dx)} \le K^*_{p,\Omega}$$
$$\left(\forall 1 \le p < (N+1)/(N-1),\ \forall y \in \partial\Omega\right).$$

Theorem 4 (Gmira-Véron). *Suppose that* $1 < q < (N+1)/(N-1)$, *then for any* $\mu \in \mathcal{M}(\partial\Omega)$ *there exists a unique* $u \in L^1(\Omega) \cap L^q(\Omega, \rho dx)$ *such that*

(31) $$\int_\Omega \left(-u\Delta\zeta + |u|^{q-1} u\zeta\right) dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} d\mu$$

$\left(\forall \zeta \in C_0^{1,1}(\overline{\Omega})\right)$ *and the mapping* $g \mapsto u = P_\Omega^q(g)$ *is increasing. If* $\{\mu_n\} \subset \mathcal{M}(\partial\Omega)$ *converges weakly to* $\mu \in \mathcal{M}(\partial\Omega)$ *when* n *goes to infinity,* $P_\Omega^q(\mu_n)$ *converges to* $P_\Omega^q(\mu)$*, locally uniformly in* Ω.

Proof. Uniqueness and monotonicity follow from the Brezis estimates (25)-(26) in Sect. I. Let $\{g_n\} \subset L^1(\partial\Omega)$ such that $g_n \to \mu$ weakly in $\mathcal{M}(\partial\Omega)$ and set $u_n = P_\Omega^q(g_n)$. From the maximum principle

$$-P_{g_n^-} \le u_n \le P_{g_n^+}. \tag{32}$$

Since $P_{g_n^\pm}(x) = \int_{\partial\Omega} P(x,y) g_n^\pm(y) ds$, we take $f \in L^{p'}(\Omega)$ with $p' = p/(p-1)$ and $1 \le p < N/(N-1)$, and we have

$$\begin{aligned} \int_\Omega P_{g_n^\pm}(x) f(x)\,dx &= \int_\Omega \int_{\partial\Omega} P(x,y)\, g_n^\pm(y) ds(y)\, f(x) dx, \\ &= \int_{\partial\Omega} \left(\int_\Omega P(x,y) f(x) dx \right) g_n^\pm(y) ds(y), \\ &\le K_{p,\Omega} \|f\|_{L^{p'}(\Omega)} \int_{\partial\Omega} g_n^\pm(y) ds(y). \end{aligned} \tag{33}$$

This implies

$$\left\| P_{g_n^\pm} \right\|_{L^p(\Omega)} \le K_{p,\Omega} \left\| g_n^\pm \right\|_{L^1(\partial\Omega)} \le K. \tag{34}$$

Similarly

$$\left\| P_{g_n^\pm} \right\|_{L^p(\Omega,\rho dx)} \le K^*_{p,\Omega} \left\| g_n^\pm \right\|_{L^1(\partial\Omega)} \le K \tag{35}$$

for $1 \le p < (N+1)/(N-1)$. If we take $q < p$ in (34), and $1 < p$ in (35) we deduce from (32) that $\{u_n\}$ and $\left\{|u_n|^{q-1} u_n\right\}$ are equi-integrable and therefore weakly compact in $L^1(\Omega) \times L^1(\Omega,\rho dx)$. From the Osserman-Keller estimate $\{u_n\}$ remains also locally uniformly bounded in Ω. By using Theorem 1' and 2' of Sect. I, for any Θ open with $\Theta \subseteq \overline{\Theta} \subseteq \Omega$, $\left\{ \|u_n\|_{C^{2,\alpha}(\overline{\Theta})} \right\}$ remains bounded and therefore relatively compact by Ascoli's theorem. Consequently there exist a sequence $\{u_{n_k}\}$ and a $C^2(\Omega)$-function u such that $u_{n_k} \to u$, and weakly in $L^1(\Omega)$. Moreover $|u_{n_k}|^{q-1} u_{n_k} \to |u|^{q-1} u$ weakly in $L^1(\Omega,\rho dx)$. Letting n go to infinity in (25) yields (31).

The stability result is proved by the same device. If $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{M}(\partial\Omega)$, it remains bounded in the total variation norm and therefore $\left\{P_{|\mu_n|}\right\}$ remains bounded in $L^{p_1}(\Omega) \times L^{p_2}(\Omega, \rho dx)$ for any $1 \le p_1 < N/(N-1)$ and $1 \le p_2 < (N+1)/(N-1)$. Since $\left|\mathsf{P}_\Omega^q(\mu_n)\right| \le P_{|\mu_n|}$, we have all the needed compactness to let n_k go to infinity in the weak expression of $(31)_{n_k}$. Therefore the full convergence result follows from uniqueness.

Remark. The mapping P_Ω^q is increasing from $\mathcal{M}(\partial\Omega)$ to $C^2(\Omega)$.

Remark. We shall see in next paragraph that (NLDP) may not have a solution for any $\mu \in \mathcal{M}(\partial\Omega)$ when $q \ge (N+1)/(N-1)$. For example there exists no solution if $\mu = \delta_a$ for some $a \in \partial\Omega$. The full treatment of the solvability of (NLDP) has been completed very recently by Dynkin and Kuznetsov in the case $1 < q \le 2$ and Marcus and Véron when $1 < q$. This treatment involves Bessel capacities.

Remark. A more elaborated analytic tool (weighted Marcinkiewicz spaces) allowed Gmira and Véron to prove an existence and uniqueness result for the general problem

$$(35) \qquad \begin{cases} -\Delta u + g(u) = 0 & \text{in } \Omega, \\ u = \mu & \text{on } \partial\Omega, \end{cases}$$

where g is continuous and non-decreasing, $\mu \in \mathcal{M}(\partial\Omega)$ and

$$(36) \qquad \int_\Omega \left|g(P_{|\mu|})\right| \rho dx < \infty .$$

II-5 Isolated singularities

As we have seen it above, if $1 \le q < (N+1)/(N-1)$ and $a \in \partial\Omega$, for any $n > 0$ the function $u_{a,n} = \mathsf{P}_\Omega^q(n\delta_a)$ is a solution of (9) which vanishes on $\partial\Omega \setminus \{a\}$. Moreover, when n increases, it is the same with $\left\{u_{a,n}\right\}$. From the Osserman Keller estimate, this sequence is locally uniformly bounded in Ω, therefore it converges to some positive solution $u_{a,\infty}$ of (9). By using some local estimate on the boundary it can be checked that $\left\{u_{a,n}\right\}$ is equicontinuous on any compact subset of $\overline{\Omega} \setminus \{a\}$. Therefore $u_{a,\infty}$ vanishes on $\partial\Omega \setminus \{a\}$. This solution $u_{a,\infty}$ is the maximal solution of (9) which vanishes on $\partial\Omega \setminus \{a\}$. As for the behaviour of $u_{a,n}(x)$ near a it can be obtained from perturbation theory. Actually estimates (28) and

(37) $$0 \le \mathsf{P}^q_\Omega(n\delta_a)(x) \le P_{n\delta_a}(x) = nP_{\delta_a}(x) = nP(x,a),$$

gives

(38) $$0 \le u_{a,\infty}(x) \le Cn|x-a|^{-N}\rho(x).$$

Finally it is possible to prove that the non-linear term is negligible near a in some sense and that

(39) $$\lim_{x\to a} \frac{u_{a,n}(x)}{P(x,a)} = n.$$

Always in the range $1 < q < (N+1)/(N-1)$, the function $u_{a,\infty}$ has a much stronger blow-up than $u_{a,n}$. The expression of this blow-up needs to introduce spherical coordinates centered at a. In fact there exists a functions ω defined on the half unit sphere S^{N-1}_+ the equator of which belongs to the plane $T_a\partial\Omega$ tangent to $\partial\Omega$ at the point a such that

(40) $$\lim_{x\to a} |x-a|^{2/(q-1)} u_{a,\infty}(x) = \omega\big((x-a)/|x-a|\big).$$

Moreover $u_{a,\infty}$ is the unique solution of (ISP) which satisfies (39).

The following result asserts that (39) and (40) characterise all the isolated singularities of the solutions of (9).

Theorem 5 (Gmira-Véron). *Let* $1 < q < (N+1)/(N-1)$, $a \in \partial\Omega$, $g \in C^0(\partial\Omega)$ *and* $u \in C(\overline{\Omega} \setminus \{a\}) \cap C^2(\Omega)$ *is a positive solution of* (9) *which coincides with* g *on* $\partial\Omega \setminus \{a\}$. *Then either*

(i) $u = \mathsf{P}^q_\Omega(g)$, *and* u *is regular in* $\overline{\Omega}$*; either*

(ii) *there exists* $n > 0$ *such that* $u = \mathsf{P}^q_\Omega(g + n\delta_a)$ *and* $u(x) \approx u_{a,n}(x)$ *in the sense of* (39)*; or*

(iii) $u = \mathsf{P}^q_\Omega(g + \infty\delta_a) := \lim_{n\to\infty} u = \mathsf{P}^q_\Omega(g + n\delta_a)$ *and* $u(x) \approx u_{a,\infty}(x)$ *in the sense of* (40).

When $q \ge (N+1)/(N-1)$ the situation is completely different since all the isolated boundary singularities of the solutions of (9) are removable

Theorem 6 (Gmira-Véron). *Let* $q \ge (N+1)/(N-1)$, $a \in \partial\Omega$, $g \in C^0(\partial\Omega)$ *and* $u \in C(\overline{\Omega} \setminus \{a\}) \cap C^2(\Omega)$ *is a positive solution of* (9) *which coincides with* g *on* $\partial\Omega \setminus \{a\}$. *Then* u *can be extended to* $\overline{\Omega}$ *as a continuous function and in fact* $u = \mathsf{P}^q_\Omega(g)$.

The proof of this result is rather technical, however it can be noticed that if

(41) $$u(x) = o\big(P(x,a)\big) + b(x)$$

near a, for some bounded function b, then u remains uniformly bounded in Ω. Actually for any $\varepsilon > 0$, the function $x \mapsto w_\varepsilon(x) = \varepsilon P(x,a) + \|b\|_{L^\infty}$ is a supersolution of (9) which dominates u in a neighbourhood of $\partial\Omega$. By the maximum principle $u \leq w_\varepsilon$, and $u \leq \|b\|_{L^\infty}$ by letting ε go to zero. In the same way $u \geq -\|b\|_{L^\infty}$. The boundedness of u implies its continuity. As for estimate (41) it is a consequence of the Keller-Osserman estimate when $q > (N+1)/(N-1)$. In the case $q = (N+1)/(N-1)$ the scheme of the proof is more complicated.

THE NON-LINEAR TRACE

III-1 The boundary trace of harmonic functions

It is easy to check that every positive harmonic function u in the unit N-ball B possesses a boundary trace given by a positive Radon measure $\mu \in \mathcal{M}^+(\partial B)$ which is attained in the sense of weak convergence of measures:

(1) $$\lim_{r\to 1}\int_{\partial B}\zeta(\sigma)u(r\sigma)d\sigma = \int_{\partial B}\zeta d\mu$$

for every $\zeta \in C^0(\partial B)$. By the Riesz-Herglotz theorem, for every non-negative Radon measure μ on ∂B there exists a unique harmonic function with boundary trace μ, and it is represented by the Poisson integral

(2) $$u(x) = \int_{\partial B} P(x,\sigma)d\mu(\sigma).$$

This result is still valid if harmonic is replaced by super-harmonic (Doob). Moreover, the positivity assumption on u can be replaced by an integrability condition (for example $\Delta u \in L^1(B,\rho dx)$) and in that case the boundary trace is a general Radon measure on ∂B.

III-2 The non-linear boundary trace

For simplicity we shall still consider the case where the open subset is the unit N-ball B. If q > 1 and u is a positive solution of

(3) $$-\Delta u + u^q = 0 \text{ in } B,$$

Keller-Osserman estimate gives

(4) $$u(x) \le C(N,q)(1-|x|)^{-2/(q-1)}.$$

The existence of a boundary trace for positive solutions of (3) has been discovered in the case $q = 2 = N$ by Le Gall who gave a probabilistic representation of any positive solution of (3) in that case (1993). The notion of trace that will be presented is due to Marcus and Véron (1995); it is a purely analytic presentation and is extendible to much more general nonlinearities.

Theorem 1 (Marcus-Véron). *Suppose* $q > 1$ *and* u *is a positive solution of* (3) *in* B. *Then for any* $\omega \in \partial B$ *we have the following alternative. Either*
(i) *for every relatively open neighbourhood* U of ω on ∂B

(5) $$\lim_{r\to 1}\int_U u(r\sigma)d\sigma = \infty,$$

or

(ii) *there exists a relatively open neighbourhood* U *of* ω *on* ∂B *such that for every* $\zeta \in C_0^\infty(U)$

(6) $$\lim_{r\to 1}\int_U u(r\sigma)\zeta(\sigma)d\sigma = \ell(\zeta),$$

where ℓ *is a positive linear functional on* $C_0^\infty(U)$.

We denote $(r,\sigma)\in S^{N-1}\times(0,\infty)$ the spherical coordinates in $\mathbf{R}^N$ and $\Delta_{S^{N-1}}$ the Laplace-Beltrami operator on the unit sphere S^{N-1} that we identify with ∂B. If V is an open domain of S^{N-1}, we denote by φ_V the first eigenfunction of $-\Delta_{S^{N-1}}$ in $W_0^{1,2}(V)$ with the normalisation condition

(7) $$\max_{\sigma\in V}\varphi_V(\sigma) = 1.$$

The corresponding eigenvalue is $\lambda_V > 0$, and if the boundary ∂V on S^{N-1} is C^2 Hopf Lemma applies and

(8) $$\frac{\partial\varphi_V}{\partial\nu_{S^{N-1}}} < 0 \text{ on } \partial V.$$

Lemma 1. *Let* V *be an open domain of* S^{N-1} *with a* C^2 *boundary,* u *a positive solution of* (3) *in* B *and* $\alpha > (q+1)/(q-1)$. *Then we have the following dichotomy. Either*

(i)

(9) $$\int_0^1\int_V u^q\varphi_V^\alpha(1-r)r^N d\sigma dr = \infty$$

and in that case

(10) $$\lim_{r\to 1}\int_V (u\varphi_V^\alpha)(r,\sigma)d\sigma = \infty,$$

or

(ii)

(10) $$\int_0^1\int_V u^q\varphi_V^\alpha(1-r)r^N d\sigma dr < \infty,$$

and in that case, for any C^2 *function* ζ *on* V *which satisfies*

(11) $$0 \le |\zeta| \le k\varphi_V^\alpha \quad \textit{and} \quad |\Delta\zeta| \le k\varphi_V^{\alpha-2}$$

for some $k \ge 0$, *the following limit exists*

$$\lim_{r\to 1}\int_V u(r,\sigma)\zeta(\sigma)d\sigma = \ell(\zeta) \tag{12}$$

and the mapping $\zeta \mapsto \ell(\zeta)$ *is a positive linear functional defined on the subset of* $C^2(V)$ *of functions which satisfy* (11).

Proof. Step 1. The following integral is finite

$$I = \int_V \left|\Delta_{S^{N-1}}\varphi_V^{\alpha}\right|^{q/(q-1)} \varphi_V^{-\alpha/(q-1)} d\sigma. \tag{13}$$

From Hopf boundary Lemma

$$\varphi_V(\sigma) \ge C_1\rho_S(\sigma) \tag{14}$$

for any $\sigma \in V$, where $\rho_S(\sigma) = \operatorname{dist}_{S^{N-1}}(\sigma, \partial V)$ is the geodesic distance on S^{N-1} and $C_1 > 0$. Since

$$\Delta_{S^{N-1}}\varphi_V^{\alpha} = -\alpha\lambda_V\varphi_V^{\alpha} + \alpha(\alpha-1)\varphi_V^{\alpha-2}\left|\nabla\varphi_V\right|^2, \tag{15}$$

$$\left|\Delta_{S^{N-1}}\varphi_V^{\alpha}\right|^{q/(q-1)} \le C_2\rho_S^{q(\alpha-2)/(q-1)}, \tag{16}$$

and finally

$$\left|\Delta_{S^{N-1}}\varphi_V^{\alpha}\right|^{q/(q-1)} \varphi_V^{-\alpha/(q-1)}(\sigma) \le C_3\rho_S^{(q(\alpha-2)-\alpha)/(q-1)}. \tag{17}$$

But $\alpha > (q+1)/(q-1) \Rightarrow (q(\alpha-2)-\alpha)/(q-1) = \alpha - 2q/(q-1) > -1$, and (13) follows.

Step 2. Reduction of the equation. We shall suppose $N \ge 3$, the case $N = 2$ requiring a special treatment. In spherical coordinates (3) reads

$$\frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{S^{N-1}}u - u^q = 0 \tag{18}$$

in $(0,1) \times S^{N-1}$. We set

$$s = \frac{r^{N-2}}{N-2}, \quad u(r,\sigma) = r^{2-N}v(s,\sigma), \tag{19}$$

and

(20) $$s^2 \frac{d^2 v}{ds^2} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} v - K s^{N/(N-2)-q} v^q = 0$$

in $(0,a) \times S^{N-1} = (0,(N-2)^{-1}) \times S^{N-1}$ where $K = K(N,q) > 0$. Then

(21) $$s^2 \frac{d^2}{ds^2} \int_V v \varphi_V^\alpha d\sigma + \frac{1}{(N-2)^2} \int_V v \Delta_{S^{N-1}} \varphi_V^\alpha d\sigma - s^{N/(N-2)-q} K \int_V v^q \varphi_V^\alpha d\sigma = 0.$$

We set $X(s) = \int_V v \varphi_V^\alpha d\sigma$ and $Y(s) = \left(\int_V v^q \varphi_V^\alpha d\sigma \right)^{1/q}$. From Hölder's inequality

(22) $$\begin{aligned} \left| \int_V v \Delta_{S^{N-1}} \varphi_V^\alpha d\sigma \right| &\le \left(\int_V v^q \varphi_V^\alpha d\sigma \right)^{1/q} \left(\int_V \left| \Delta_{S^{N-1}} \varphi_V^\alpha \right|^{q/(q-1)} \varphi_V^{-\alpha/(q-1)} d\sigma \right)^{1-1/q} \\ &\le I^{1-1/q} Y(s) \end{aligned}$$

and (21) becomes

$(23)_+$ $$s^2 X'' + JY - K s^{N/(N-2)-q} Y^q \ge 0$$

$(23)_-$ $$s^2 X'' - JY - K s^{N/(N-2)-q} Y^q \le 0$$

where $J = I^{1-1/q} (N-2)^{-2}$.

Step 3. End of the proof $(N \ge 3)$.
Case 1: suppose that (9) holds, then

(24) $$\int_{a/2}^a \int_V v^q \varphi_V^\alpha (1 - s a^{-1}) d\sigma ds = a^{-1} \int_{a/2}^a Y^q (a-s) ds = \infty.$$

From $(23)_+$ we deduce that there exist two constants $A, B > 0$ (and depending on a) such that

(25) $$X'' \ge A Y^q - B$$

on (a/2,a). Then

(26) $$\begin{aligned} X(s) &\ge X(a/2) + (s - a/2) X'(a/2) \\ &\quad + A \int_{a/2}^s (s-\tau) Y^q(\tau) d\tau - \frac{B}{2} (s - a/2)^2, \end{aligned}$$

and $\lim_{s \to a} X(s) = \infty$ from (24); this means that (10) holds.
Case 2: suppose that (11) holds, then

(27) $$\int_{a/2}^{a}\int_V v^q\varphi_V^\alpha(1-sa^{-1})d\sigma ds = a^{-1}\int_{a/2}^{a} Y^q(a-s)ds < \infty;$$

inequality (25) is replaced by

(28) $$X'' \leq AY^q + B$$

and

(29) $$\frac{d^2}{ds^2}\left(X(s) - A\int_{a/2}^{s}(s-\tau)Y^q(\tau)d\tau - \frac{B}{2}(s-a/2)^2\right) \leq 0.$$

From concavity and (27), $\lim_{s\to a} X(s) < \infty$. If ζ is a C^2 function which satisfies (11), we set $X_\zeta(s) = \int_V v\zeta d\sigma$ and (21) implies

(30) $$s^2X_\zeta'' + \frac{1}{(N-2)^2}\int_V v\Delta_{S^{N-1}}\zeta d\sigma - Ks^{N/(N-2)-q}\int_V v^q\zeta d\sigma = 0.$$

But

(31) $$\left|\int_V v\Delta_{S^{N-1}}\zeta d\sigma\right| \leq k\int_V v\varphi_V^{\alpha-2}d\sigma \leq k\left(\int_V v^q\varphi_V^\alpha d\sigma\right)^{1/q}\left(\int_V \varphi_V^{\alpha-2q/(q-1)}d\sigma\right)^{1-1/q},$$

and

(32) $$\left|\int_V v^q\zeta d\sigma\right| \leq k\int_V v^q\varphi_V^\alpha d\sigma.$$

Therefore it follows from (27) that

(33) $$\int_{a/2}^{a}\left|\int_V v\Delta_{S^{N-1}}\zeta d\sigma\right|(a-s)ds < \infty \text{ and } \int_{a/2}^{a}\left|\int_V v^q\zeta d\sigma\right|(a-s)ds < \infty.$$

Integrating (30) twice implies that $\lim_{s\to a} X_\zeta(s) = l(\zeta)$ exists and obviously the correspondence $\zeta \mapsto l(\zeta)$ is a positive linear functional on the set of C^2 functions ζ which satisfy (11). Therefore it can be extended as a positive measure on V.

In the case $N = 2$ the principle remains the same but the change of variable (19) is replaced by

(34) $$r = e^{-t},\ u(r,\sigma) = v(t,\sigma),$$

which transforms (18) into

(35)

on $(0,\infty)\times S^1$.

Proof of Theorem 1. If $\omega\in\partial B$ and there exists an open neighbourhood V such that (10) holds, we have the existence of a positive Radon measure μ_U such that

$$\lim_{r\to 1}\int_V u(r,\sigma)d\sigma = \int_V \zeta d\mu(\sigma) \qquad \left(\forall\zeta\in C_0^0(V)\right). \tag{36}$$

If such a neighbourhood does not exist, we are in case (i).

Let $\mathcal{R}$ be the set of the $\omega\in\partial B$ such that (ii) holds; $\mathcal{R}$ is relatively open and there exists a unique (non-negative) Radon measure μ such that $\mu_{|V}=\mu_V$. The set $\mathcal{S}=\partial B\setminus\mathcal{R}$ is closed.

Definition. *The couple* $(\mathcal{S},\mu)$ *is called the boundary trace of* u. *The set* $\mathcal{S}$ *is the singular part of this trace and the measure* μ *on* $\mathcal{R}=\partial B\setminus\mathcal{S}$, *the regular part of the trace*.

For convenience it is often useful to introduce the Borel measure framework. Actually there is a one to one correspondence between the family CM of couples $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a compact subset of ∂B and μ a positive Radon measures on $\mathcal{R}=\partial B\setminus\mathcal{S}$ and the set $\mathcal{B}_{reg}$ of outer regular, positive Borel measures on ∂B (β is outer regular if for every Borel subset E of ∂B, $\mu(E)=\inf_{\mathcal{O}\in\mathcal{V}_E}\mu(\mathcal{O})$ where $\mathcal{V}_E$ is the set of relatively open subsets of ∂B containing E). Let $\beta\in\mathcal{B}_{reg}$, we define the regular set $\mathcal{R}_\beta$ and the blow-up set $\mathcal{S}_\beta$ of β as follows

$$\mathcal{R}_\beta=\left\{\omega\in\partial B:\ \exists U,\text{rel. open neighbourhood of }\omega\text{ s.t. }\beta(U)<\infty\right\}$$

$$\mathcal{S}_\beta=\partial B\setminus\mathcal{R}_\beta=\left\{\omega\in\partial B:\ \forall U,\text{rel. open neighbourhood of }\omega\ ,\ \beta(U)=\infty\right\}.$$

The correspondence $CM\xrightarrow{\mathcal{M}}\mathcal{B}_{reg}$ is given by

$$\mathcal{M}((\mathcal{S},\mu))=\tilde{\mu}\quad\text{where }\tilde{\mu}(A)=\begin{cases}\mu(A) & \text{if } A\subseteq\mathcal{R},\\ \infty & \text{if } A\cap\mathcal{S}\neq\emptyset,\end{cases} \tag{37}$$

for every Borel subset $A\subset\partial B$, and

$$\mathcal{M}^{-1}(\beta)=(\mathcal{S}_\beta,\beta_{|\mathcal{R}_\beta}). \tag{38}$$

With this notation we shall denote

(39) $$\mathrm{Tr}(u) = \mu \in \mathcal{B}_{reg} \text{ where } \mu = \mathcal{M}(\mathcal{S}_\beta, \beta_{|\mathcal{R}_\beta}).$$

The general non-linear boundary value problem is to solve

(GNBVP) $$\begin{cases} -\Delta u + |u|^{q-1}u = 0 \text{ in } \Omega, \\ \mathrm{Tr}(u) = \mu \in \mathcal{B}_{reg}. \end{cases}$$

The problem is completely different according $1 < q < (N+1)/(N-1)$ or $q \geq (N+1)/(N-1)$. This is due to the following pointwise blow-up estimate.

Theorem 2. *Suppose* $1 < q < (N+1)/(N-1)$, u *is a non negative solution of* (3) *in* B *with* $\mathrm{tr}(u) = (\mathcal{S}, \mu)$ *and* $\omega \in \mathcal{S}$. *Then*

(40) $$\lim_{r \to 1} u(r, \omega) = \infty,$$

and more precisely

(41) $$\liminf_{r \to 1} (1 - r)^{2/(q-1)} u(r, \omega) > 0.$$

Proof. We use a scaling-concentration argument. If $\omega \in \mathcal{S}$

(42) $$\lim_{r \to 1} \int_{D_\eta(\omega)} u(r, \sigma)\, d\sigma = \infty$$

for any $\eta > 0$, where $D_\eta(\omega)$ is the geodesic ball on ∂B with center ω and radius $\eta < \pi$. If $0 < k < 1$ we set $u_k = \mathcal{N}_k(u)$ defined by $u_k(x) = k^{2/(q-1)} u(kx)$, and u_k is a solution of (3) in $B_{1/k}$. We denote

(43) $$M_{\varepsilon,\eta} = \int_{D_\eta(\omega)} u(1 - \varepsilon, \sigma)\, d\sigma$$

for $0 < \varepsilon < 1$. For $m > 0$ large enough and $\eta \in (0, \pi)$ there exists $\varepsilon = \varepsilon(\eta, m)$ such that $m = M_{\varepsilon,\eta}$. Let w_η be the solution of (3) in B with the following boundary value (in which χ_E is the characteristic function of the set E)

(44) $$w_\eta(1, \sigma) = u_{1-\varepsilon}(1, \sigma) \chi_{D_\eta(\omega)}(\sigma).$$

From comparison principle $w_\eta \leq u_{1-\varepsilon}$ in B. When η goes to 0, it is the same with ε. From Theorem 4-Sect. II, w_η converges to $u_{\omega,m} = \mathcal{P}_\Omega^q(m\delta_\omega)$ and $u_{1-\varepsilon}$ to u. Therefore $u_{\omega,m} \leq u$, for any $m > 0$, which implies

(45) $\quad u_{\omega,\infty} \le u\,.$

We get (41) from Theorem 5-Sect. 2.

The main result of the subcritical case is that the general non-linear boundary value problem is well posed in terms of boundary trace.

Theorem 3. *Suppose* $1 < q < (N+1)/(N-1)$. *Then the correspondence* $u \mapsto \mathrm{Tr}(u)$ *which assigns to each non negative solution* u *of* (3) *in* B, *its boundary trace in* $\mathcal{B}_{reg}$ *is one to one.*

This theorem was proved by Le Gall in the case $q = N = 2$ with probabilistic techniques and in the general case by Marcus and Véron by an analytic method. We present below the skeleton of their proof:

(I) The problem can be reduced to the case when μ only takes the values 0 and ∞. The advantage is that this set of boundary values is conserved by sum and positive multiplications

(II) Construction of a maximal solution. Let $\mathcal{S} = \mathcal{S}_\mu$ the singular set of μ and put $\mathcal{S}_{\varepsilon)} = \left\{\sigma \in \partial B\text{: } \mathrm{dist}(\sigma, \mathcal{S}_\mu) < \varepsilon\right\}$. If μ_ε is the Borel measure with singular set $\mathcal{S}_{\varepsilon)}$ and is zero elsewhere, there exists u_ε a solution with $\mathrm{Tr}(u_\varepsilon) = \mu_\varepsilon$. $\{u_\varepsilon\}$ increases when ε decreases to 0 and $\bar{u} = \lim_{\varepsilon \to 0} u_\varepsilon$ is the maximal solution.

(III) Construction of a minimal solution. Let $\{y_k\}$ be a dense sequence in $\mathcal{S}$. Put $\mu_n = n \sum_{1 \le k \le n} \delta_{y_k}$ and u_n the solution with $u_n = \mathrm{Tr}(\mu_n)$ (Gmira-Véron). Then $\{u_n\}$ increases with n and $\underline{u} = \lim_{n \to \infty} u_n$ is the minimal solution.

(IV) There exists $C > 0$ such that $\bar{u} \le C\underline{u}$ (scaling methods).

(V) Show that $\underline{u} - \dfrac{1}{2C}(\bar{u} - \underline{u})$ is a positive super solution which is dominated by $\underline{u}$ if $\underline{u} < \bar{u}$ and construct a solution v with same trace, strictly dominated by $\underline{u}$.

The super critical case $q \ge (N+1)/(N-1)$ is more difficult and much less is known. Too concentrated Radon measures are not admissible for being the measure part of the boundary trace of a solution, and too small sets on ∂B are removable boundary singularities. Moreover a compatibility condition between a closed set $\mathcal{S} \subset \partial B$ and a positive Radon measure μ on $\mathcal{R} = \partial B \setminus \mathcal{S}$ is needed. Those conditions are expressed in terms of Bessel capacities on ∂B. Those conditions where shown to be necessary and sufficient conditions for the existence of a maximal solution u of (3) in B with $\mathrm{tr}(u) = (\mathcal{S}, \mu)$. This was proved by

Dynkin-Kuznetsov (1997) for $(N+1)/(N-1) \leq q \leq 2$ and Marcus-Véron (1996-1997) for $(N+1)/(N-1) \leq q$. It was also noticed by Le Gall (1996) that there may exist many solutions of (3) with a given boundary trace. Actually Marcus and Véron proved that if $q > (N+1)/(N-1)$, then for any $\varepsilon > 0$ there exists a Borel subset $K_\varepsilon \subset \partial B$ with $\text{meas.}(K_\varepsilon) \leq \varepsilon$ and a positive solution u of (3) in B with $\text{tr}(u) = (\partial B, 0)$ such that

(46) $$\lim_{r \to 1} u(r,\sigma) = 0$$

a.e. on $\partial B \setminus K_\varepsilon$. An attempt to modify the definition of the trace has been recently (August 1997) by Dynkin-Kuznetsov and similar researches are also conducted by Marcus-Véron.

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