Entrainment in Gravity Mass Flows: General Considerations and Constraints, a Useful Model and Numerical Simulations

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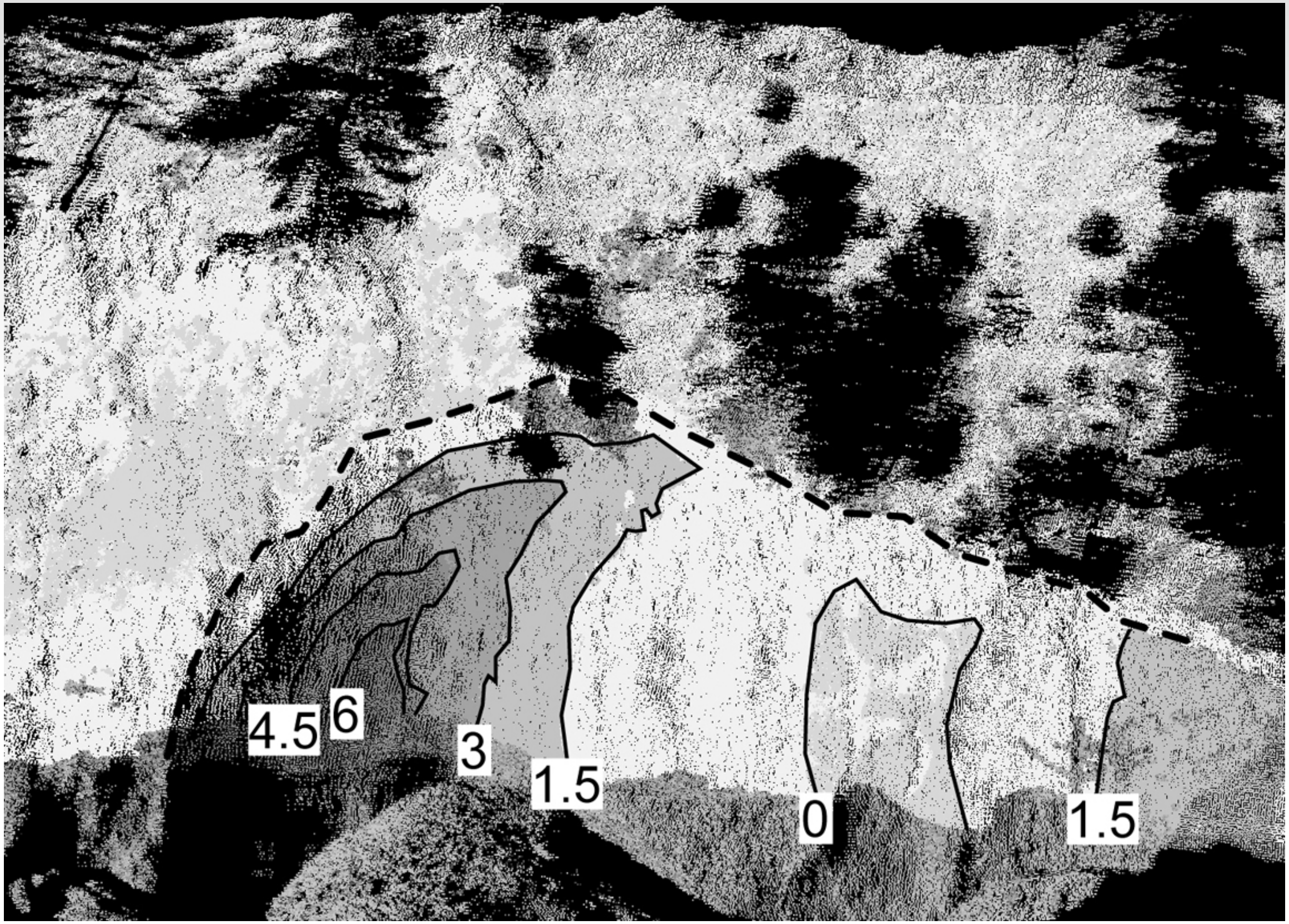
IMI, Universidad Complutense, Madrid, April 16, 2010

A debris-flow channel in Utah

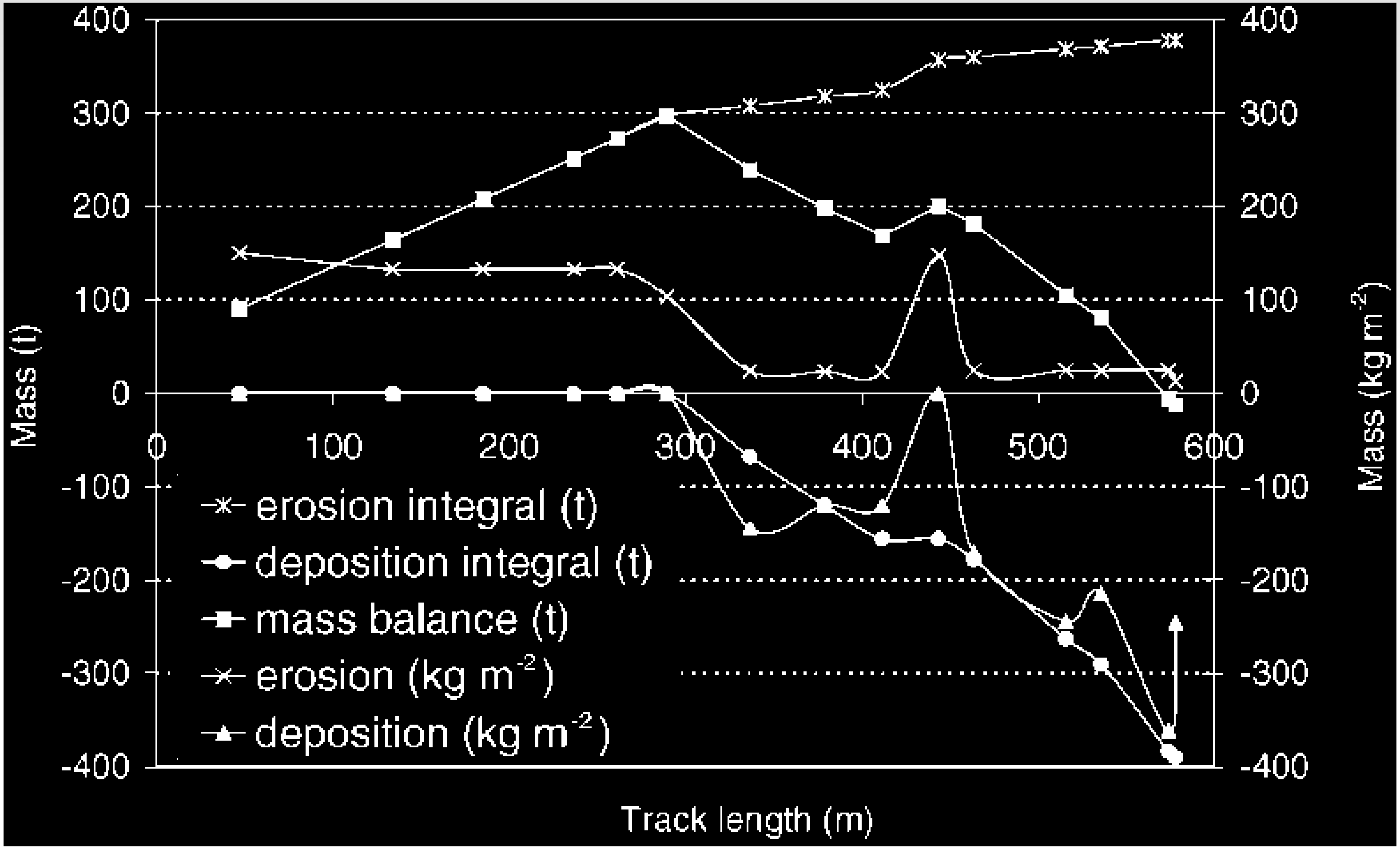
From: W. F. Case, *Debris-Flow Hazards*, Utah Geol. Surv. Pub. Info. Ser. 70, 2000

Slope erosion at a bend of the Illgraben gorge (Valais, Switzerland) in the course of 7 months (in meters)



From: Oppikofer et al., Talk at 4th Swiss Geoscience Meeting, Bern 2006

Spatial mass balance in a snow avalanche (measured at Monte Pizzac test site, Italy, in 1998)



From Sovilla et al., Annals Glaciol. 32 (2001), 230–236.

FMCW radar plot of snow avalanche at Vallée de la Sionne

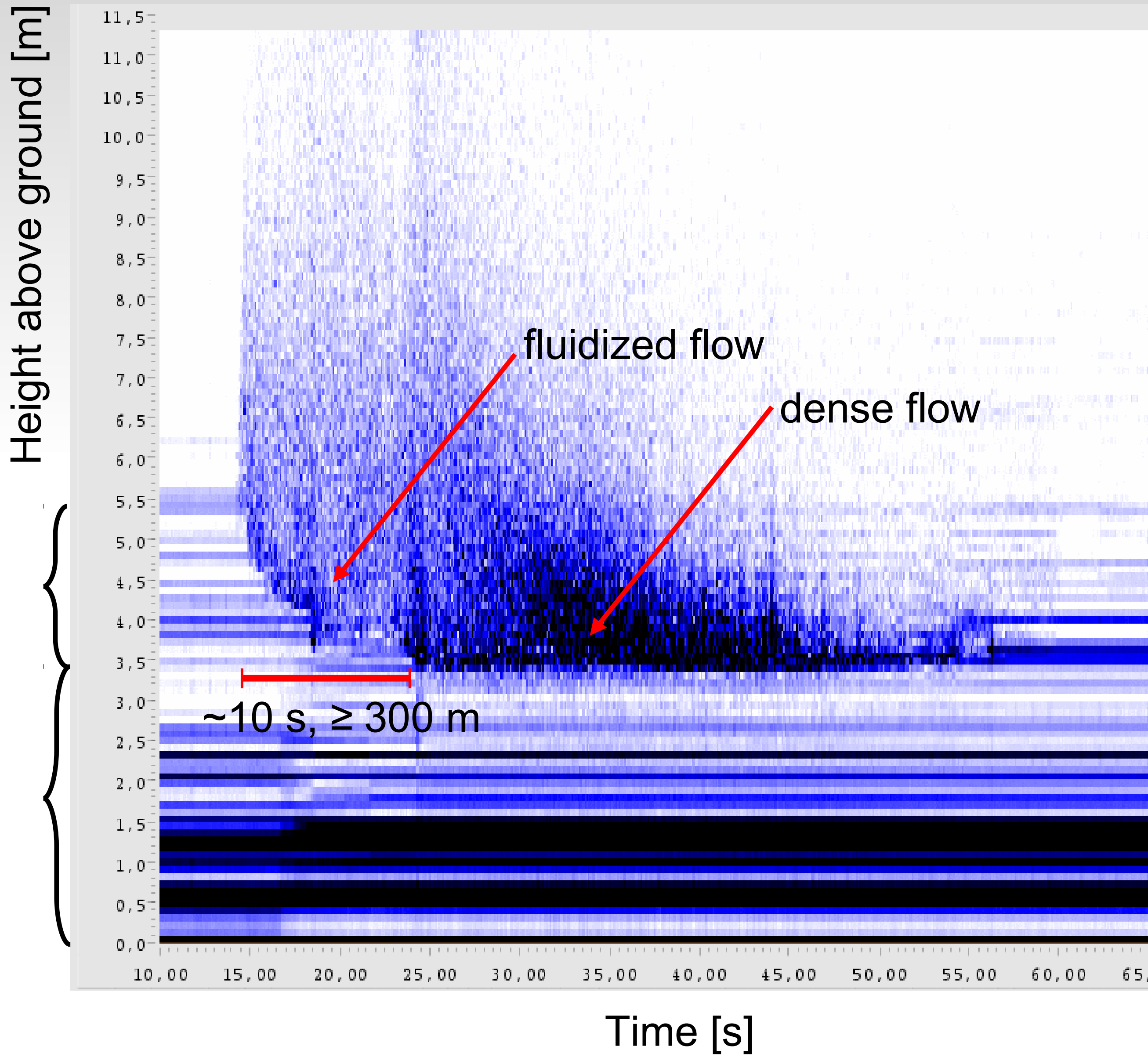
Observed entrainment rate:

10–200 kg m^{-2} s^{-1},

diminishing with time and erosion depth.

2 m of fresh snow eroded

Hard old snow not eroded



Topics of this talk:

Four key questions:

1. *How does entrainment work?*

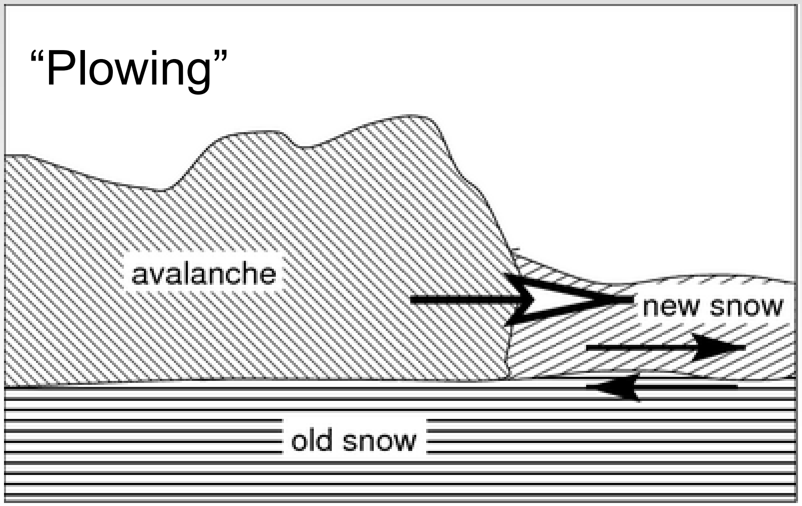
2. How is entrainment to be included in the governing equations?
   (Is there an "entrainment force" term?)

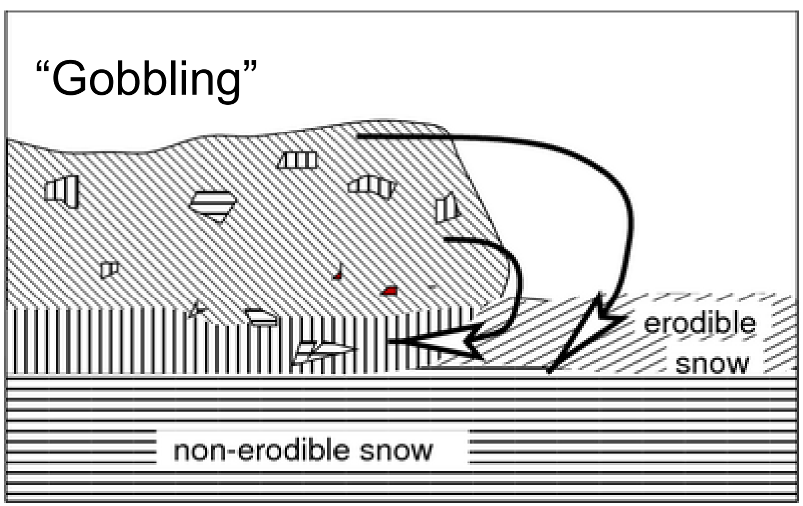
3. Are there conceptual differences between entrainment and deposition?
   (Can deposition accelerate the flow?)

4. *How can we estimate the entrainment rate?*
   (3 approaches – rigid block model, analytical toy model, numerical solution in 1D along slope-normal direction)

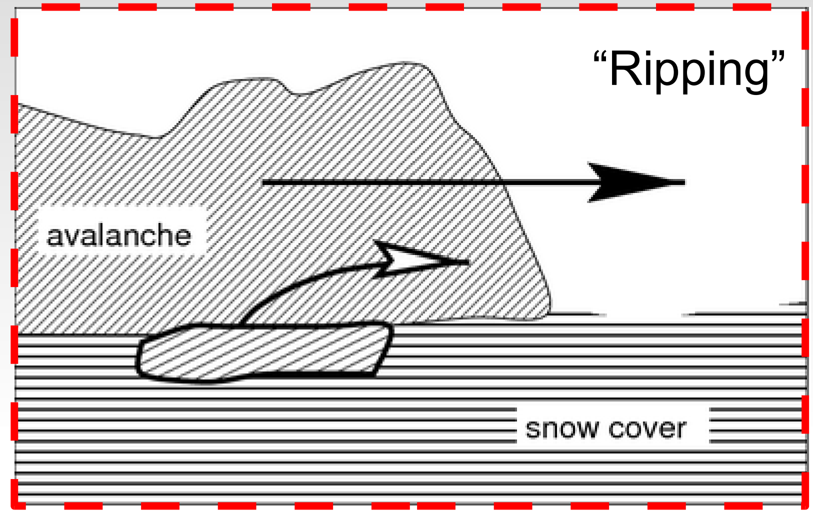
1. Erosion mechanisms in GMFs

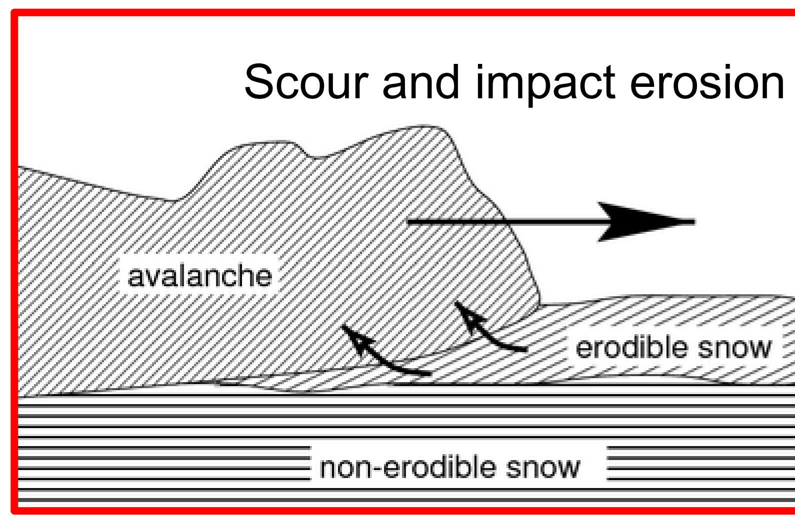
Frontal mechanisms





Mechanisms acting along bottom





Plowing:

- Dominant in wet-snow avalanches. Needs different approach

Gobbling:

- No experimental evidence so far. Disregard it in the following.

Ripping:

- Experimental evidence in dry-snow avalanches from ground-radar measurements.
- Seems to occur in strongly stratified beds if there is a weak layer underneath a strong layer.
- Can be approximated by continuous entrainment along bottom with sufficient averaging over bottom area and time.

Scour (abrasion) and impact erosion:

- Experimental evidence strong.
- Can be treated by model for continuous entrainment along bottom.

Impact traces

Ryggfonn 04/06/2003

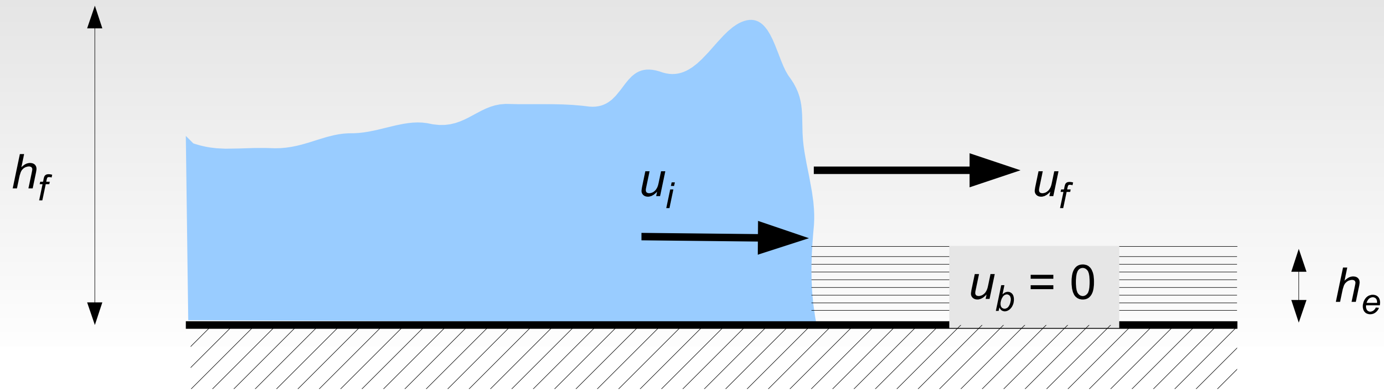
$\rho_s \approx 120$ kg m^{-3}

Abrasion traces

The plowing mechanism:

- Clearly dominant in wet-snow avalanches.
- Possibly important in dry-snow avalanches as well, but clear experimental confirmation is still lacking.
- Open question for debris flows and pyroclastic flows.
- Likely condition for plowing to be possible: Flowing material must have higher strength than bed and sufficient weight.
- In laboratory granular flows, length of plowing zone = *O*(flow height).

Early entrainment model by Eglit and coworkers (~1967) implements plowing as jump boundary condition at avalanche front:



Mass balance:

$$h_b \rho_b u_f = h_f \rho_f (u_f - u_i)$$

Momentum balance:

$$h_b \rho_b \cdot 0 \cdot u_f + k_b \rho_b g h_b^2/2 + h_b \tau_c$$
$$= h_f \rho_f u_i (u_f - u_i) + k_f \rho_f g h_f^2/2$$

Fracture strength of bed, τ_c, determines frontal dynamics.

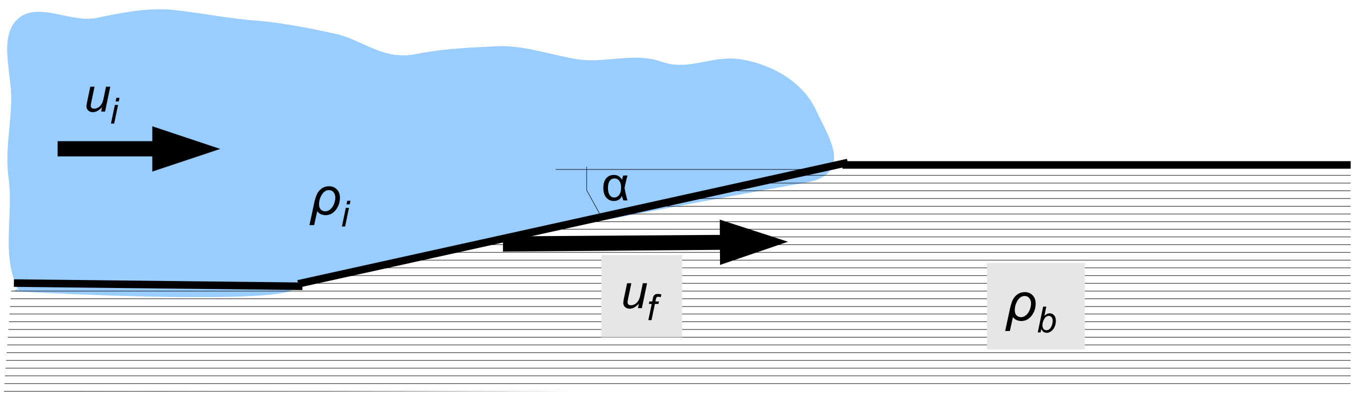
Front moves more rapidly than flowing material at front.

Main difficulties are:

- ➢ Entrainment depth is not determined dynamically.
- ➢ Unrealistically high front heights h_f in practical applications.

Modified model with inclined front surface (Grigorian and Ostroumov, Eglit et al.):

- Shock front is determined similar to supersonic flow.
- Overburden pressure and bed strength determine inclination α of erosion surface and entrainment rate.
- Requires $\rho_f > \rho_b$, but this may not always apply in nature!



Useful distinctions:

Erosion: Detaching bed particles (breaking bonds)

Entrainment: Accelerating/mixing particles into the flow

Erosion-limited flows: Bed shear strength larger than mean bed shear stress. Intermittent particle erosion by fluctuations (e.g. turbulent eddies).

Entrainment-limited flows: Bed shear strength less than bed shear stress. Erosion limited by capacity of the flow to accelerate the eroded particles.

Concentrate on *entrainment-limited* flows and assume *brittle fracture* behavior of bed material.

2. Entrainment in the flow equations

- Consider depth-integrated Eulerian models, described by balance equations for mass and linear momentum.

- Broad agreement on form of mass balance (1D case):

$$\partial_t h + \partial_x(h\bar{u}) = w_e$$

where h = flow depth,
$\bar{u}$ = depth-averaged flow velocity parallel to bed,
w_e = entrainment speed [m s^{-1}]

- Old and new controversy: *"entrainment force" – yes or no?*

$$\partial_t(h\bar{u}) + \partial_x\left(h\overline{u^2}\right) = g\sin\theta + \partial_x(h\hat{\bar{\sigma}}_{xx}) - \hat{\tau}_b - w_e\bar{u}$$

"Entrainment force" ???

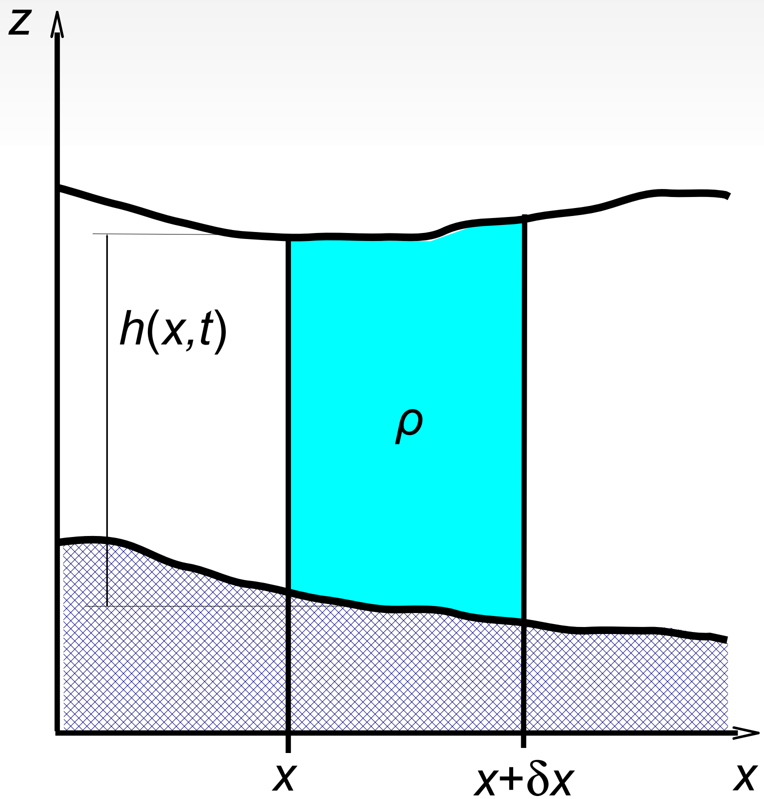
Gravitational acceleration

Slope angle

Bed shear stress

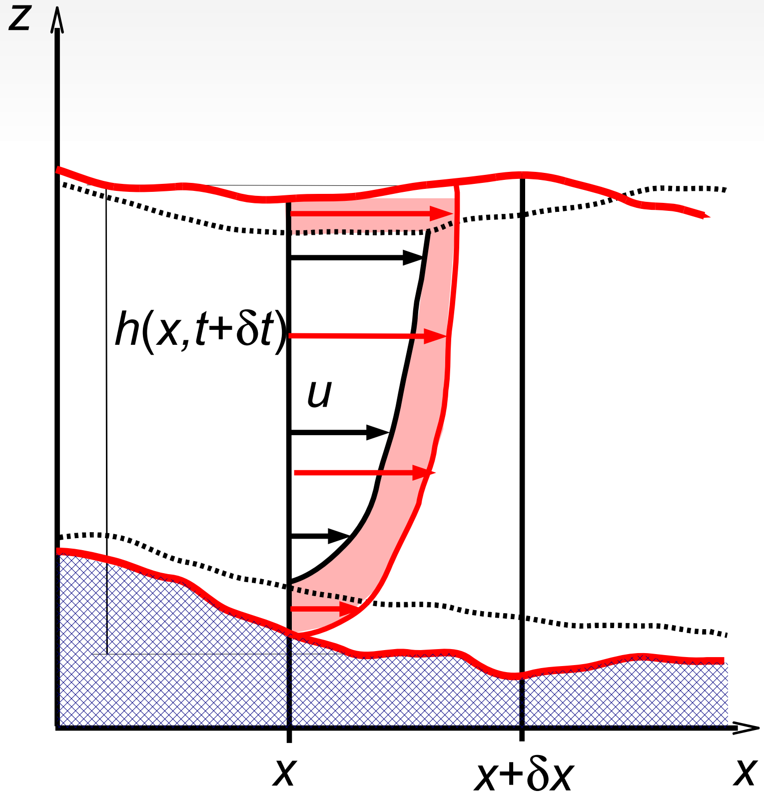
Avg. longitudinal stress

Write the mass and momentum balance for a thin flow slice, fixed between x and $x+\delta x$, but with variable bottom and top boundaries:



Momentum balance in the control volume:

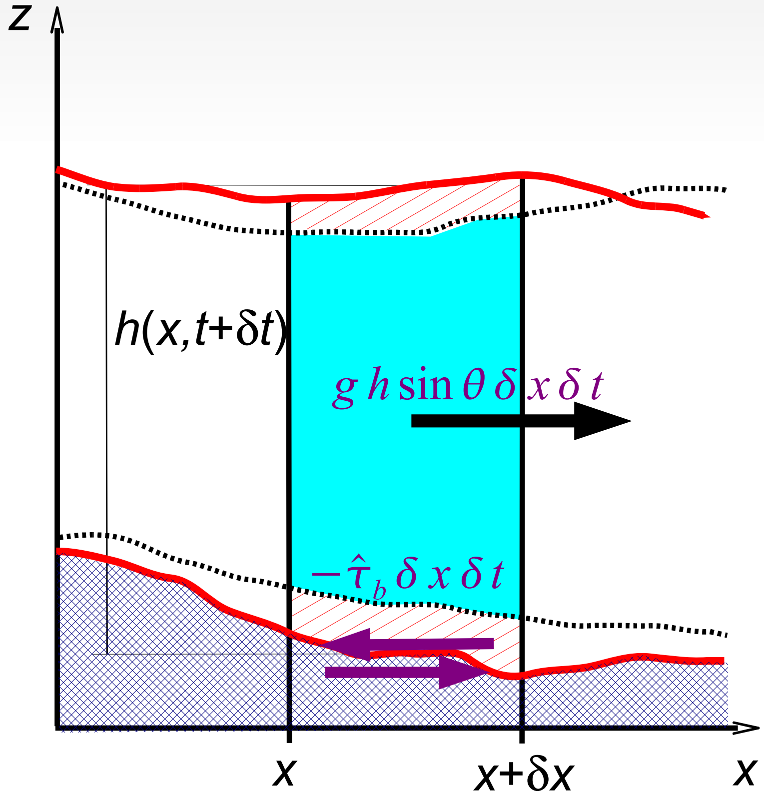
Change of momentum inside:



$$\left(\left[h\bar{u}\right](x,t+\delta t)-\left[h\bar{u}\right](x,t)\right)\delta x$$

Momentum balance in the control volume:

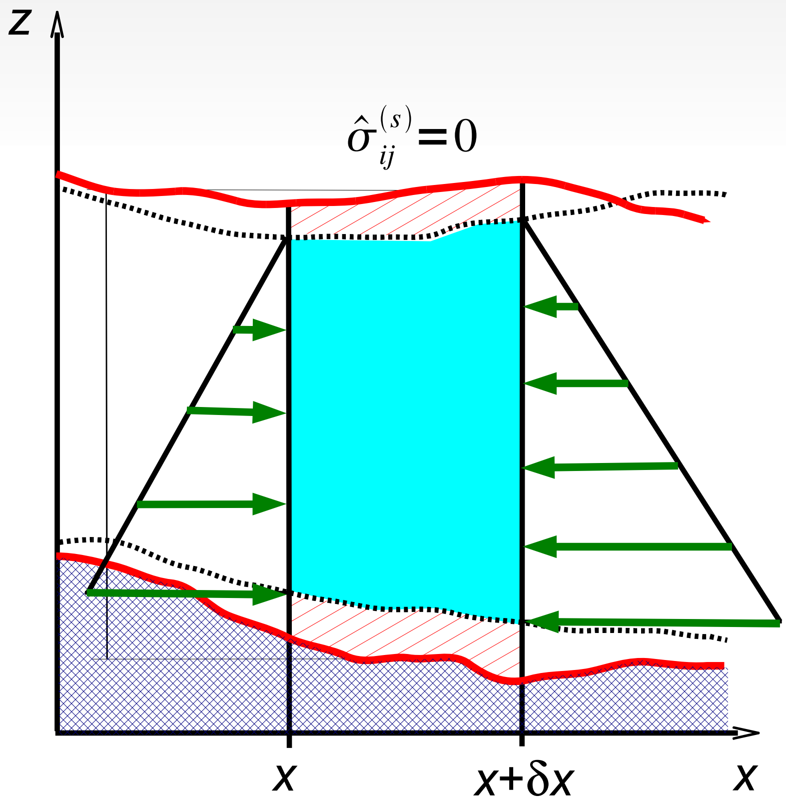
Change of momentum inside equals:
– body forces (gravity),
– bottom shear stress,



$$\left([h\bar{u}](x,t+\delta t)-[h\bar{u}](x,t)\right)\delta x$$
$$=\left([hg\sin\theta](x,t)-\hat{\tau}_b(x,t)\right)\delta x\,\delta t$$

Momentum balance in the control volume:

Change of momentum inside equals
- body forces (gravity),
- bottom shear stress,
- longitudinal stresses,

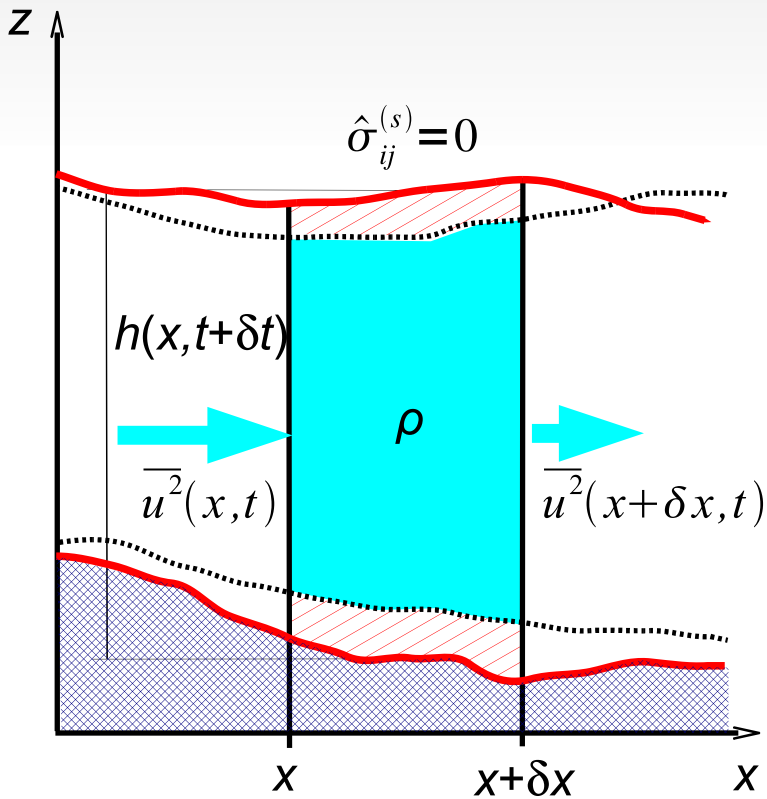


No contribution from material, stress-free top surface.

$$\left([h\bar{u}](x,t+\delta t)-[h\bar{u}](x,t)\right)\delta x$$
$$=\left([hg\sin\theta](x,t)-\hat{\tau}_b(x,t)\right)\delta x\delta t$$
$$+\left([h\hat{\bar{\sigma}}_{xx}](x+\delta x,t)-[h\hat{\bar{\sigma}}_{xx}](x,t)\right)\delta t$$

Momentum balance in the control volume:

Change of momentum inside equals
– body forces (gravity),
– bottom shear stress,
– longitudinal stresses,
– net advection through sides



No contribution from material, stress-free top surface.

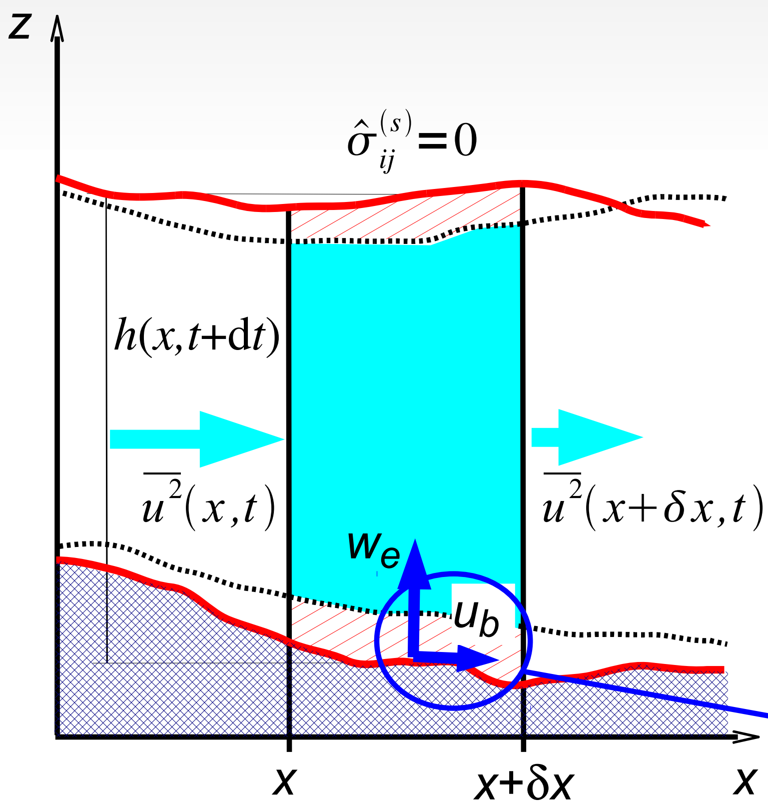
$$\left([h\bar{u}](x,t+\delta t)-[h\bar{u}](x,t)\right)\delta x$$
$$=\left([hg\sin\theta](x,t)-\hat{\tau}_b(x,t)\right)\delta x\delta t$$
$$+\left([h\hat{\bar{\sigma}}_{xx}](x+\delta x,t)-[h\hat{\bar{\sigma}}_{xx}](x,t)\right)\delta t$$
$$+\left([h\overline{u^2}](x,t)-[h\overline{u^2}](x+\delta x,t)\right)\delta t$$

Momentum balance in the control volume:

Change of momentum inside equals

- body forces (gravity),
- bottom shear stress,
- longitudinal stresses,
- net advection through sides,
- influx through bottom boundary with velocity w_e.

No contribution from material, stress-free top surface.



$$\big([h\bar{u}](x,t+\delta t) - [h\bar{u}](x,t)\big)\delta x$$

$$= \big([hg\sin\theta](x,t) - \hat{\tau}_b(x,t)\big)\delta x \delta t$$

$$+ \big([h\hat{\bar{\sigma}}_{xx}](x+\delta x,t) - [h\hat{\bar{\sigma}}_{xx}](x,t)\big)\delta t$$

$$+ \big([h\overline{u^2}](x,t) - [h\overline{u^2}](x+\delta x,t)\big)\delta t$$

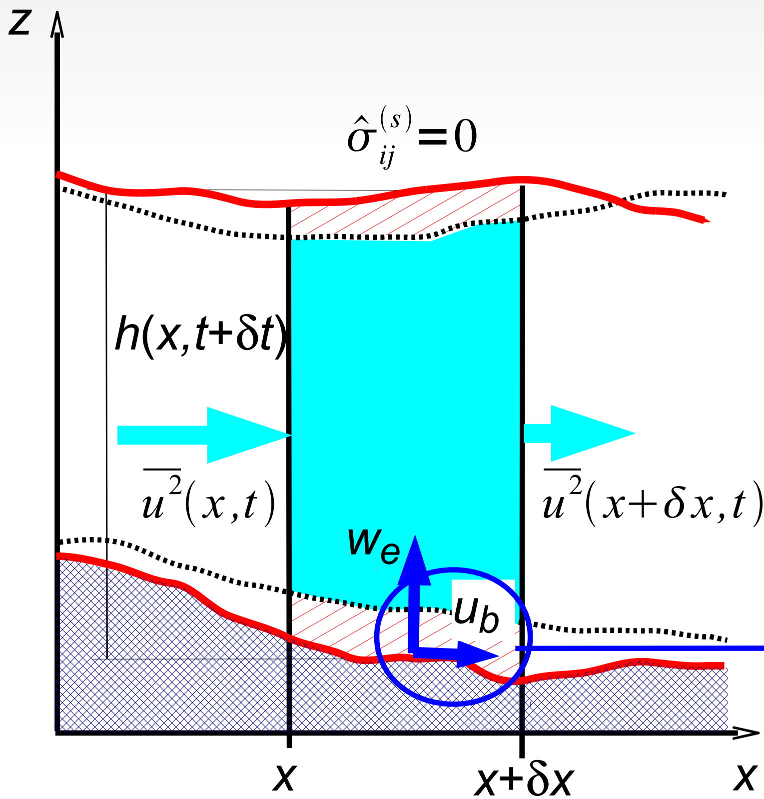
$$+ u_b w_e \delta x \delta t$$

Momentum balance in the control volume:

Change of momentum inside equals

– body forces (gravity),
– bottom shear stress,
– longitudinal stresses,
– net advection through sides,
– influx through bottom boundary with velocity w_e.

No contribution from material, stress-free top surface.



After division by δx, $\delta t \to 0$:

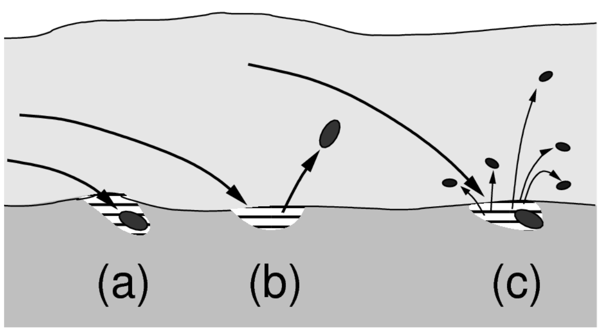
$$\partial_t(h\bar{u}) + \partial_x\left(h\overline{u^2}\right) = hg\sin\theta - \hat{\tau}_b + \partial_x(h\hat{\bar{\sigma}}_{xx}) + u_b w_e$$

Most common situation for GMFs:

Bed is at rest, $u_b = 0$

⇒ **No "entrainment force" term!**

However, other situations are possible (aeolian transport of sand or snow, or powder-snow avalanches):



(a) Saltating particles absorbed by bed correspond to $w_e < 0$, $u_b > 0 \;\Rightarrow\; w_e\, u_b < 0$.

(c) Bed particles are ejected with $u_b > 0$ due to impact, also $w_e > 0 \;\Rightarrow\; w_e\, u_b > 0$.

Equation of motion with entrainment?

- ☞ Go to the Lagrangean picture using $\frac{\mathrm{D} f}{\mathrm{D} t} = \partial_t f + \bar{u}\, \partial_x f$.
- ☞ Combine mass and momentum balance equations.

A little bit of algebra and the chain rule give

$$\frac{\mathrm{D}\bar{u}}{\mathrm{D}t} = g \sin\theta + \frac{1}{h}\left[\partial_x (h\hat{\bar{\sigma}}_{xx}) - \hat{\tau}_b - w_e(\bar{u} - u_b) - \partial_x\left(h\left(\overline{u^2} - \bar{u}^2\right)\right)\right]$$

"Entrainment force" term in the equation of motion (modified if particles carry momentum into the flow)!

Accelerating the entrained particles decelerates the flow (or reduces acceleration).

3. Differences between erosion and deposition?

- Consider simplified situation (~ block model, u_b = 0) on horizontal plane, concentrate on basal entrainment/deposition:

$$\frac{\mathrm{D}\,\overline{u}}{\mathrm{D}\,t} \approx -\frac{\tau_b}{h\,\overline{\rho}} - \frac{w_e\,\overline{u}}{h}$$

  Suppose there is deposition, i.e. $w_e < 0$:

  Acceleration like in a rocket if $w_e < -\dfrac{\tau_b}{\overline{\rho}\,\overline{u}}$ *!?!*

- Related questions:

  Under which conditions is (continuous) deposition possible?

  What is the difference between deceleration and deposition?

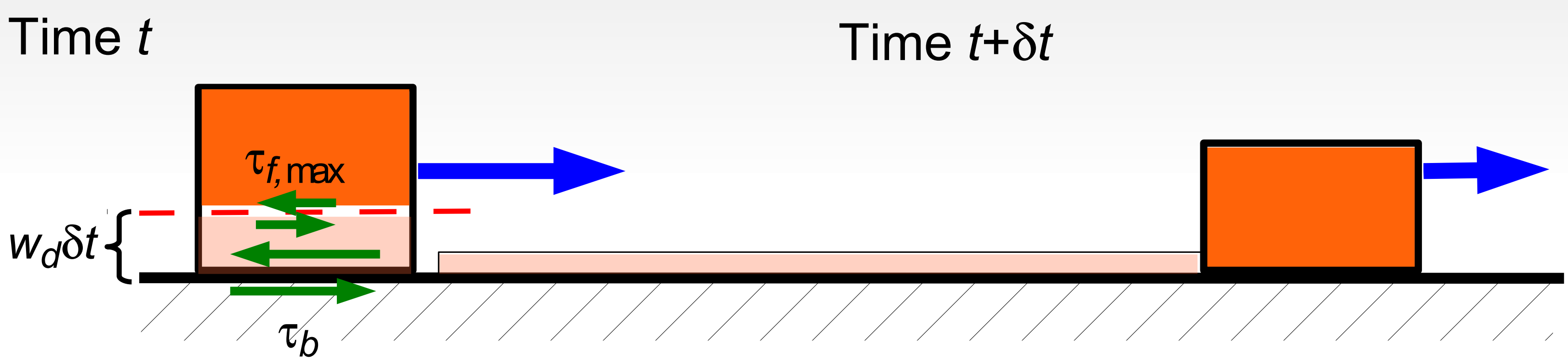
Is a deposition rate $w_d \equiv -w_e > \dfrac{\tau_b}{\bar{\rho}\,\bar{u}}$ **possible at all?**

Consider a simplified situation similar to abrasion of a solid:

Time t

Is a deposition rate $w_d \equiv -w_e > \frac{\tau_b}{\overline{\rho}\,\overline{u}}$ possible at all?

Consider a simplified situation similar to abrasion of a solid:



Momentum balance for abraded material:

$$\rho\, w_d \delta t\, u(t) - 0 = (\tau_b - \tau_{f,max})\delta t \quad \Rightarrow$$

Necessary condition $\tau_b > \tau_{f,max}$

$\rho w_d u > \tau_b$ leads to $\tau_{f,max} < 0$ and violates the 2nd Law!

Analysis carries over to flows with internal shear because shear stresses are dissipative.

☞ **GMFs do not accelerate by shedding mass!**

☞ However, a depositing GMF decelerates more slowly than the *bed* shear stress would dictate.
Deposition occurs because the GMF cannot internally sustain the bed shear stress.

Conditions for deposition along flow bottom:

0. *Bed shear strength* > *shear stress exerted by flow*,

   i.e. $\tau_c > \tau_b$.

1. Bottom boundary layer must decelerate more rapidly than layers above (otherwise bulk of the flow would simply stop).

2. Stopped particles must sinter or lock into bed very rapidly.

Only way to fulfill condition 2 seems to be if the bed exerts larger shear stress on the flow than the flow can sustain internally:

bed shear stress > *max. internal shear stress*,

i.e. $\tau_b > \tau_{f,max}$.

Fulfilled for granular materials (static friction > dynamic friction), but apparently not well understood!

4. How to estimate the entrainment rate?

Assumptions:

- Consider only entrainment along flow bottom.
- Assume brittle behavior of **bed material – breaks at stress τ_c.**

Physical consideration:

Entrainment rate must be determined by rheology of GMF and shear strength τ_c of bed material. ***No free parameters!***

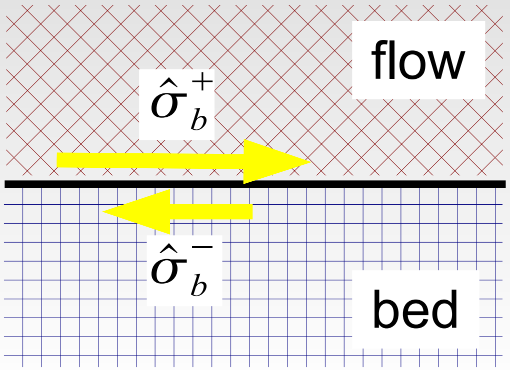
Approaches:

1. Solve special case of sliding block analytically.
2. Solve "toy" model analytically to study interplay between entrainment rate, shear stress and velocity profile.
3. Solve simplified 1D equation for advancement of entrainment front numerically.

4.1 Analytic solution for sliding blocks:

- Assume a bed (b) friction law of the form

$$\hat{\sigma}_b^+ \equiv \frac{\sigma_{xz}(z=b^+)}{\rho} = \hat{f}(\bar{u}, h, \ldots)$$



Shear stress at top of bed: $\hat{\sigma}_b^- = \hat{\tau}_c$.

- Jump condition for x-momentum across bed–flow interface:

$$w_e \cdot (u(z=b^+) - u(z=b^-)) = w_e \bar{u} = \hat{\sigma}_b^+ - \hat{\sigma}_b^- = \hat{f}(\bar{u}, h, \ldots) - \hat{\tau}_c$$

Now immediately find the entrainment rate:

$$\boxed{q_e = \rho w_e = \begin{cases} 0 & \text{if } f(\bar{u}, h, \ldots) \le \tau_c, \\ \dfrac{f(\bar{u}, h, \ldots) - \tau_c}{\bar{u}} & \text{else.} \end{cases}}$$

Some remarks:

- With the assumption of perfectly brittle fracture of the bed material, this seems to be the only physically consistent formula.
- However, material behavior may be more complicated if fracture propagation speed is not $\gg$ flow velocity or if it takes considerable time to reestablish local flow pattern after erosion of a particle.
- Concept of sliding slab entraining bed material is physically dubious: Eroded bed particles must instantaneously accelerate to slab velocity, immediately become part of the slab and then be able to sustain higher shear stress than before as bed particles...
- If shear stress is held *fixed*, entrainment rate drops $\propto u^{-1}$!

- Typical bed friction law: $\hat{\sigma}_b^+ = \operatorname{sgn}(\bar{u})\left(\hat{\sigma}_n \tan\delta + b|\bar{u}| + k\,\bar{u}^2\right)$

  with σ_n = normal stress on bed,

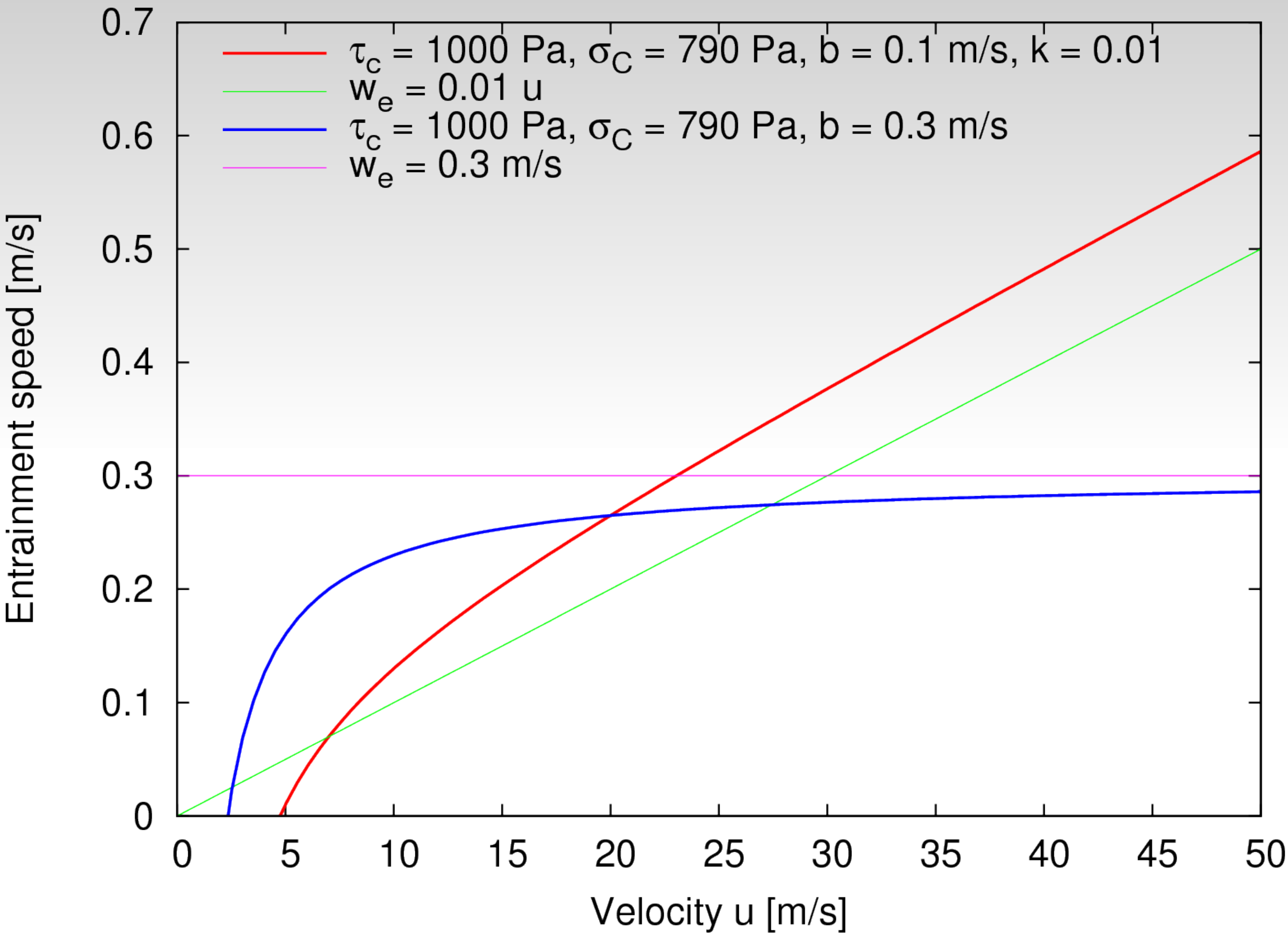
  δ = bed friction angle (assume $\tan\delta < \sin\theta$)

$$w_e = \begin{cases} 0 & \text{if } |\bar{u}| \le -\dfrac{b}{2\mathrm{k}} + \sqrt{\dfrac{b^2}{4k^2} + \dfrac{\hat{\tau}_c - \hat{\sigma}_n \tan\delta}{k}}, \\ \dfrac{\hat{\sigma}_n \tan\delta - \hat{\tau}_c}{\bar{u}} + b + k\,\bar{u} & \text{else.} \end{cases}$$

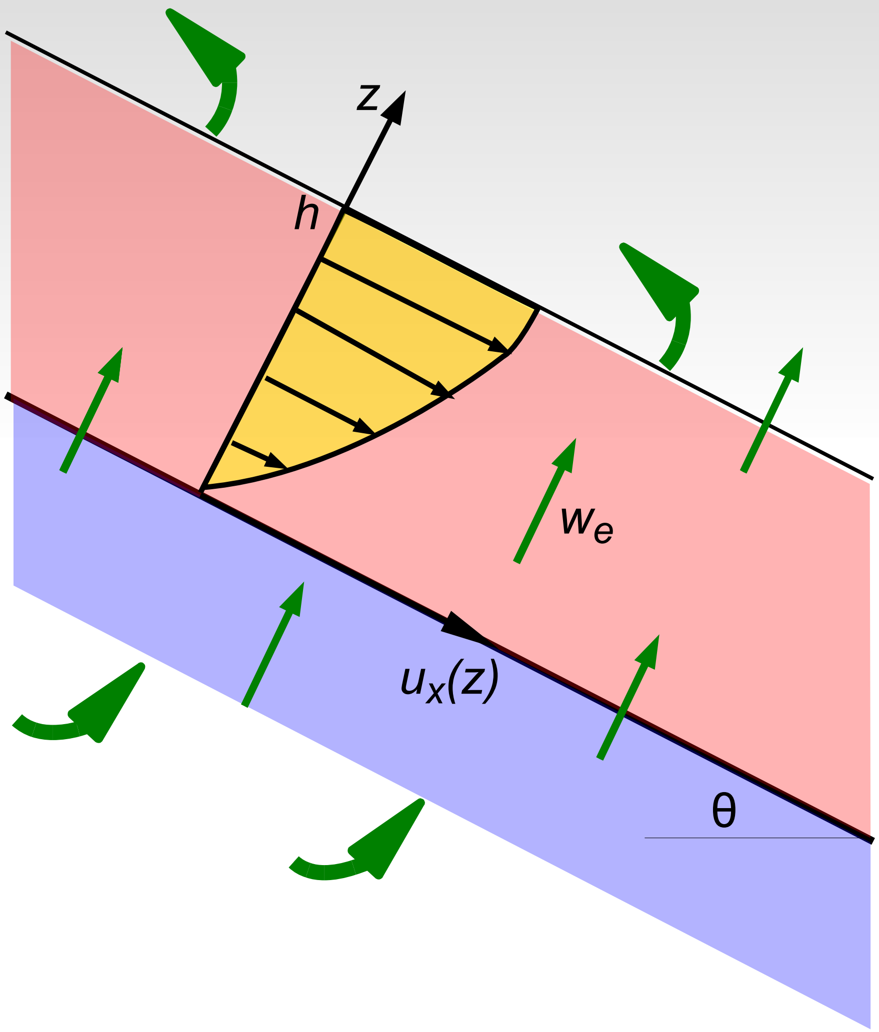
- Pure Coulomb friction law is problematic:

  (i) If $\tau_c < \sigma \tan\delta$, infinite entrainment as $\bar{u} \to 0$,

  (ii) otherwise no entrainment at all.


τ_c = 1000 Pa, σ_C = 790 Pa, b = 0.1 m/s, k = 0.01
w_e = 0.01 u
τ_c = 1000 Pa, σ_C = 790 Pa, b = 0.3 m/s
w_e = 0.3 m/s
Entrainment speed [m/s]
Velocity u [m/s]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0
5
10
15
20
25
30
35
40
45
50

4.2 A useful toy model



Infinitely long inclined plane

Steady flow conditions, flow height *h*

Assume flow is Newtonian fluid with kinematic viscosity ν, non-turbulent.

Erodible bed of brittle material with shear strength $\tau_c < \tau_0 \equiv \rho g h \sin\theta$.

Flow height held constant by replenishing bed at the entrainment rate and skimming flow at same rate.

Momentum balance simplifies to

$$w_e \dot{\gamma} = g \sin\theta + \frac{1}{\rho} \frac{\mathrm{d}\sigma_{xz}}{\mathrm{d}z}$$

$$= g \sin\theta + \nu \frac{\mathrm{d}\dot{\gamma}}{\mathrm{d}z}$$

where $\dot{\gamma} \equiv \mathrm{d}u/\mathrm{d}z$.

4. How can we estimate the entrainment rate? **(6)**

Procedure:

1. Assume entrainment velocity w_e to be given, solve ODE
$$\frac{\mathrm{d}\,\dot{\gamma}}{\mathrm{d}\,z} - \frac{w_e}{\nu}\,\dot{\gamma} = -\frac{g\sin\theta}{\nu}\;.$$
2. Find appropriate boundary condition, determine physically consistent entrainment rate.

First-order ODE easy to solve for Newtonian or Bingham fluid, most other rheologies lead to non-linear equations:

$$u(z) = \frac{g\sin\theta}{w_e}\left[z - \frac{\nu}{w_e}\,e^{-w_e h/\nu}\left(e^{w_e z/\nu} - 1\right)\right]$$

$$\tau(z) = \frac{\rho\,\nu\,g\sin\theta}{w_e}\left(1 - e^{-(h-z)w_e/\nu}\right)$$

Boundary condition for bottom shear stress τ_b:

- $\tau_b \geq \tau_c$ for erosion and entrainment to be possible.
- If $\tau_b < \tau_c$, erosion stops, τ_b rapidly increases to $\tau_0 = \rho\, g\, h \sin\theta > \tau_c$, erosion resumes.
- If $\tau_b > \tau_c$, more mass is eroded but less excess shear stress available to entrain the eroded mass, so τ_b must decrease again.

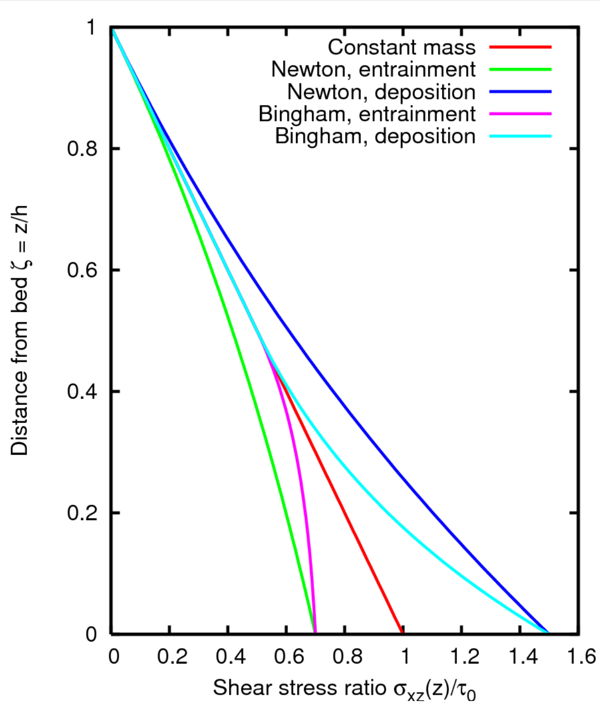
$\Rightarrow$ Equilibrium value for the bottom shear stress is $\sigma_b = \tau_c$.

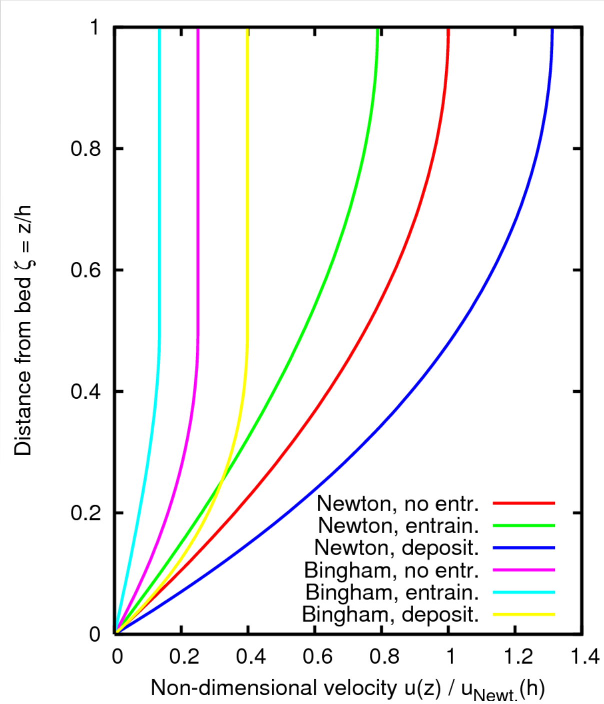
Entrainment rate can be determined (numerically) from

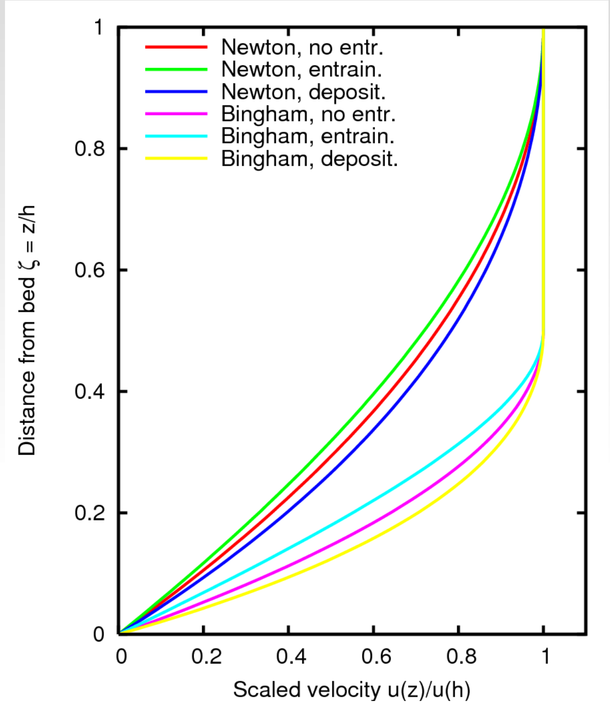
$$\boxed{\tau_c = \frac{\rho\, \nu\, g \sin\theta}{w_e}\left(1 - e^{-h\, w_e/\nu}\right)}$$

N.B. Similar b.c. proposed for aeolian transport by Owen (1964).

4. How can we estimate the entrainment rate? (8)







Excess shear stress is used for entraining the eroded material.

Entrainment reduces the equilibrium flow velocity, deposition increases it.

Shape of the velocity profiles is moderately modified by entrain-ment or deposition.

Does the model give realistic entrainment rates?

- Assume slope angle $\theta = 30°$
  flow height $h = 1\,\mathrm{m}$
  density $\rho = 200\,\mathrm{kg\,m^{-3}}$
  viscosity $\nu = 0.0556\,\mathrm{m^2\,s^{-1}}$
- This gives "gravitational traction" $\tau_0 = 1000\,\mathrm{Pa}$
  surface velocity $u_h = 45.0\,\mathrm{m\,s^{-1}}$ without entrainment
- Then the entrainment rates and velocities are

τ_c [Pa]	u_h [$\mathrm{m\,s^{-1}}$]	w_e [$\mathrm{m\,s^{-1}}$]	q_e [$\mathrm{kg\,m^{-2}\,s^{-1}}$]
500	28.2	0.089	17.8
700	35.5	0.042	8.4
900	41.9	0.012	2.4
1000	45.0	0.0	0.0

4.3 1D simulation of erosion front advancement and profile evolution

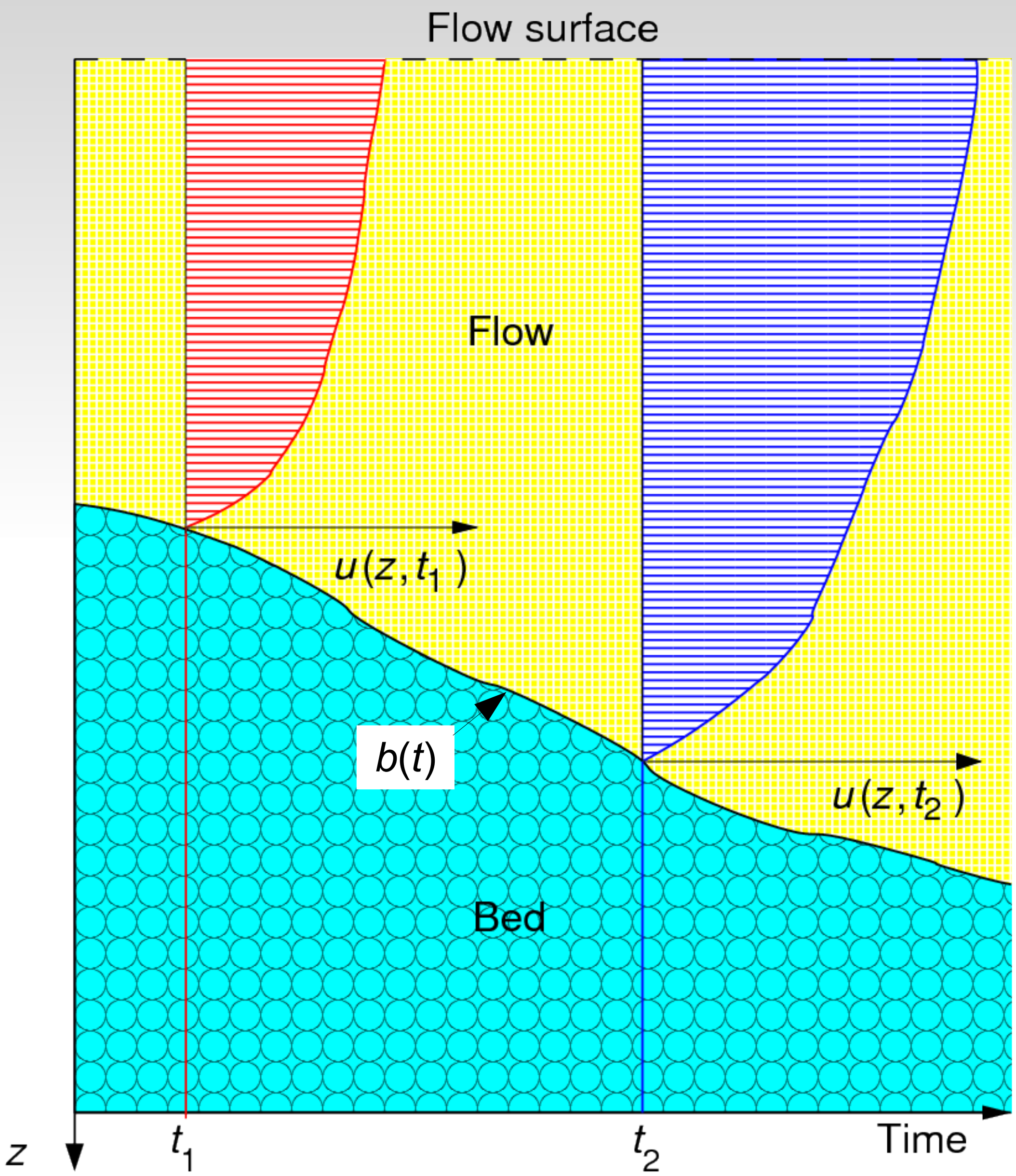
Toy model:

+ Useful because it is analytically solvable, helps in understanding interplay between erosion rate and velocity profile.
– Analytical solvability restricted to simple rheology.
– Stationarity essential, but unrealistic.

Develop a more advanced tool:

- Compute time evolution of velocity profile and advancement of entrainment front numerically.
- Neglect longitudinal stress gradients for the time being.
- Implement a rather general rheology covering most models of practical interest.

4. How can we estimate the entrainment rate? (11)



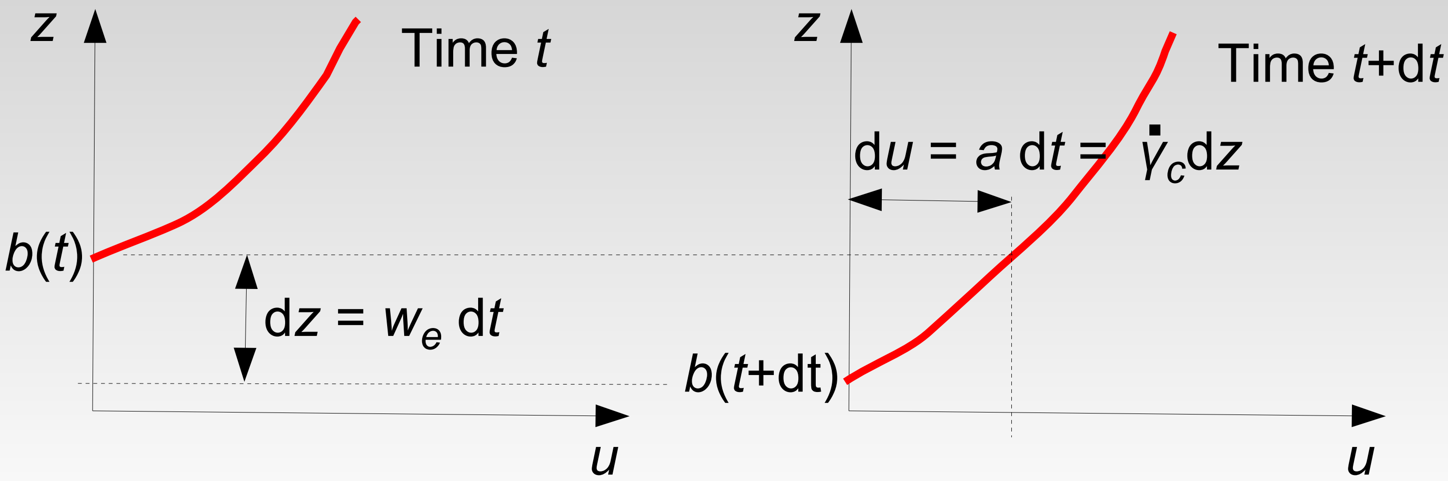
Governing equations and boundary conditions:

- Assume velocity $u(z,t)$ always parallel to bed. $\Rightarrow$ Mass balance OK.
- Assume constitutive equation $\hat{\sigma}_{xz}(z,t) \equiv \hat{\sigma}(z,t) = \hat{f}(\dot{\gamma}(z,t), h(t), \ldots)$.
- Momentum balance equation:

$$\boxed{\partial_t u = g \sin\theta + \partial_z \hat{\sigma}}$$

  in variable domain $0 \le z \le b(t)$.

- Initial condition: $b(t_0) = b_0$, $u(z,t_0) = u_0(z)$.
- Boundary conditions: $u(b(t),t) = 0$, $\hat{\sigma}(0,t) = 0$, $\sigma(b(t),t) = \tau_c$.
- Entrainment speed $w_e = \mathrm{d}b \,/\, \mathrm{d}t$ must be determined by *local* conditions at interface, i.e., by shear stress gradient. Rheology connects shear stress to shear rate gradient.



- Velocity at time $t + dt$ of particles eroded at time t:

$$u(b(t), t+dt) = 0 + (g \sin\theta + \partial_z \hat{\sigma})\, dt\ .$$

- Shear rate at erosion front must be critical shear rate:

$$\dot{\gamma}(b,t) = \frac{u(b,t+dt) - 0}{dz} = \frac{\left(g \sin\theta + \partial_z \hat{\sigma}(b,t)\right) dt}{w_e(t)\, dt} \overset{!}{=} \dot{\gamma}_c \quad \text{where} \quad f(\dot{\gamma}_c, b) = \tau_c$$

☞ $$\boxed{w_e(t) = \frac{g \sin\theta + \partial_z \hat{\sigma}(\dot{\gamma}_c, b, \ldots)}{\dot{\gamma}_c}\ .}$$

Numerical techniques:

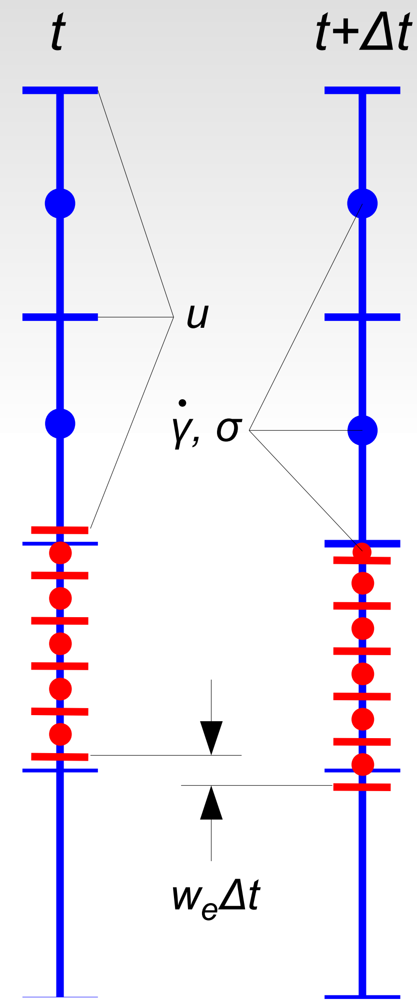
Explicit 1st order time stepping.

Finite differences on uniform Eulerian grid. Central differencing ➔ 2nd order in space.

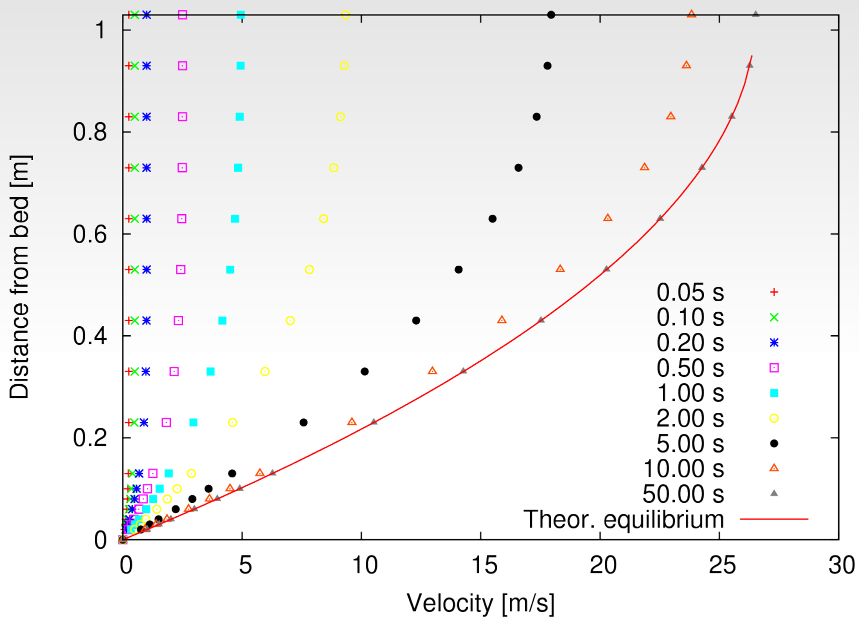
Staggered grid – shear rates and shear stresses evaluated at midpoints between nodes.

Front tracking and improved accuracy due to uniform fine mesh following the interface (shifted by $w_e \Delta t$ every timestep).

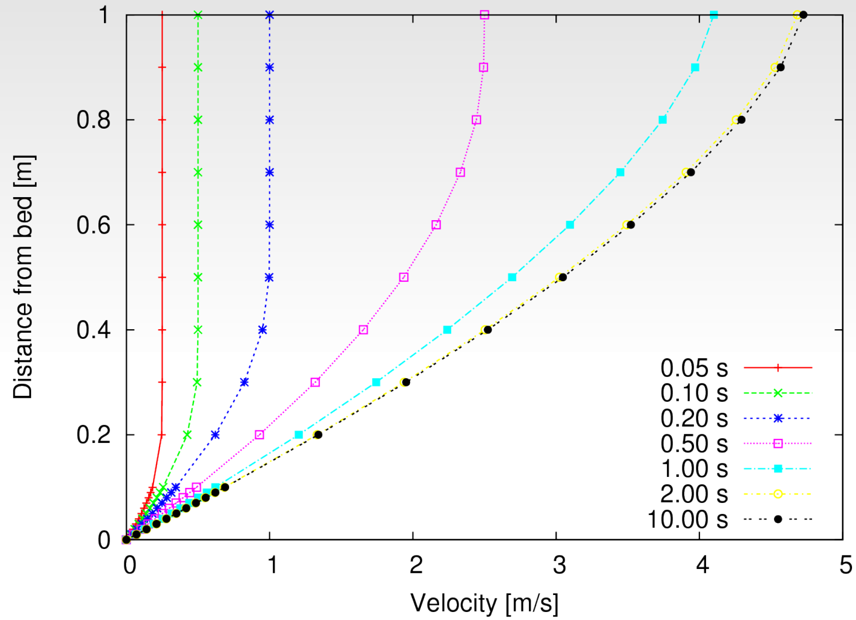
Quadratic interpolation/extrapolation at front and when needed at interface coarse/ fine grid.



Code validation – no entrainment

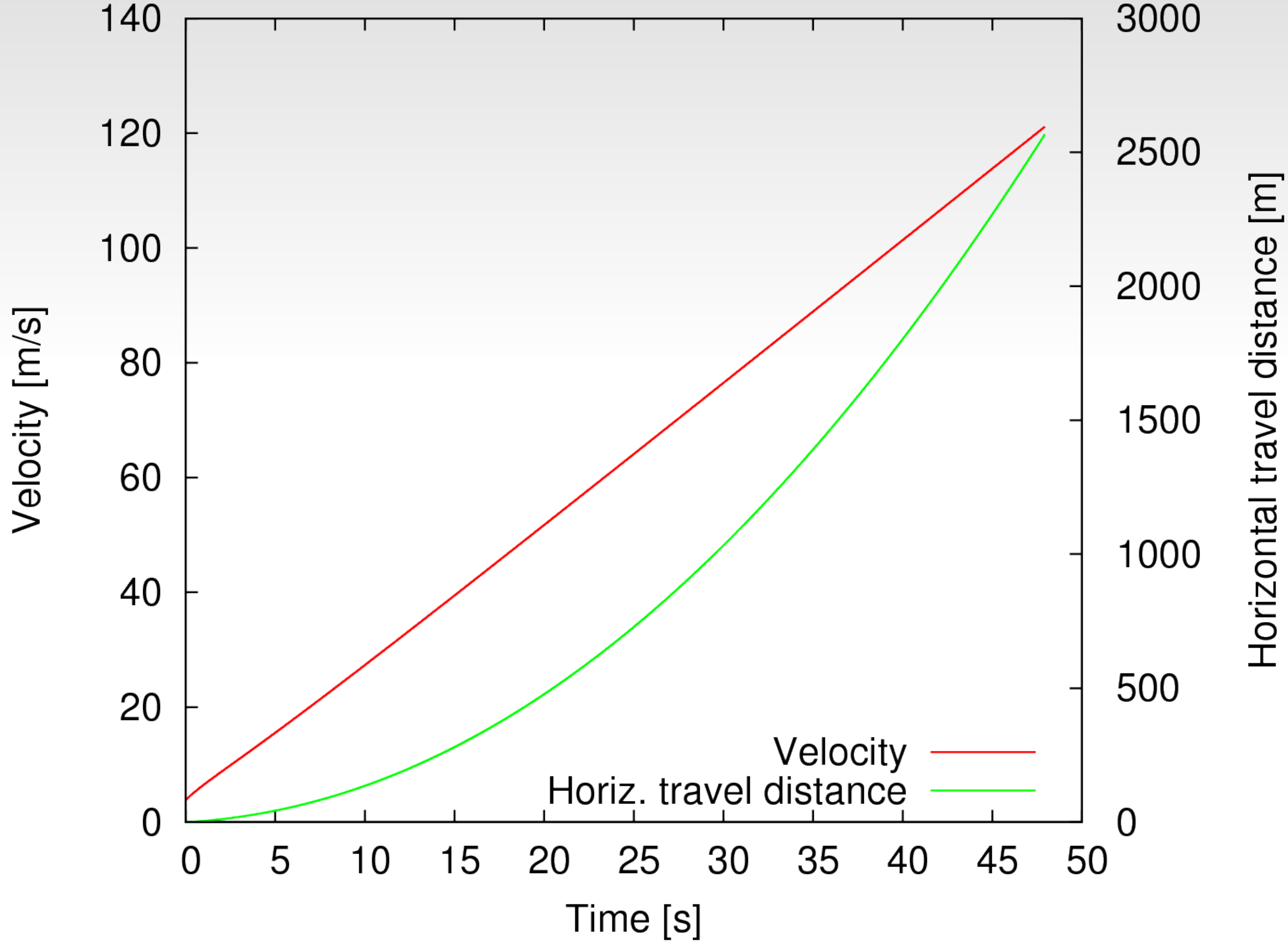


Time evolution of velocity profile for Newtonian fluid starting from rest

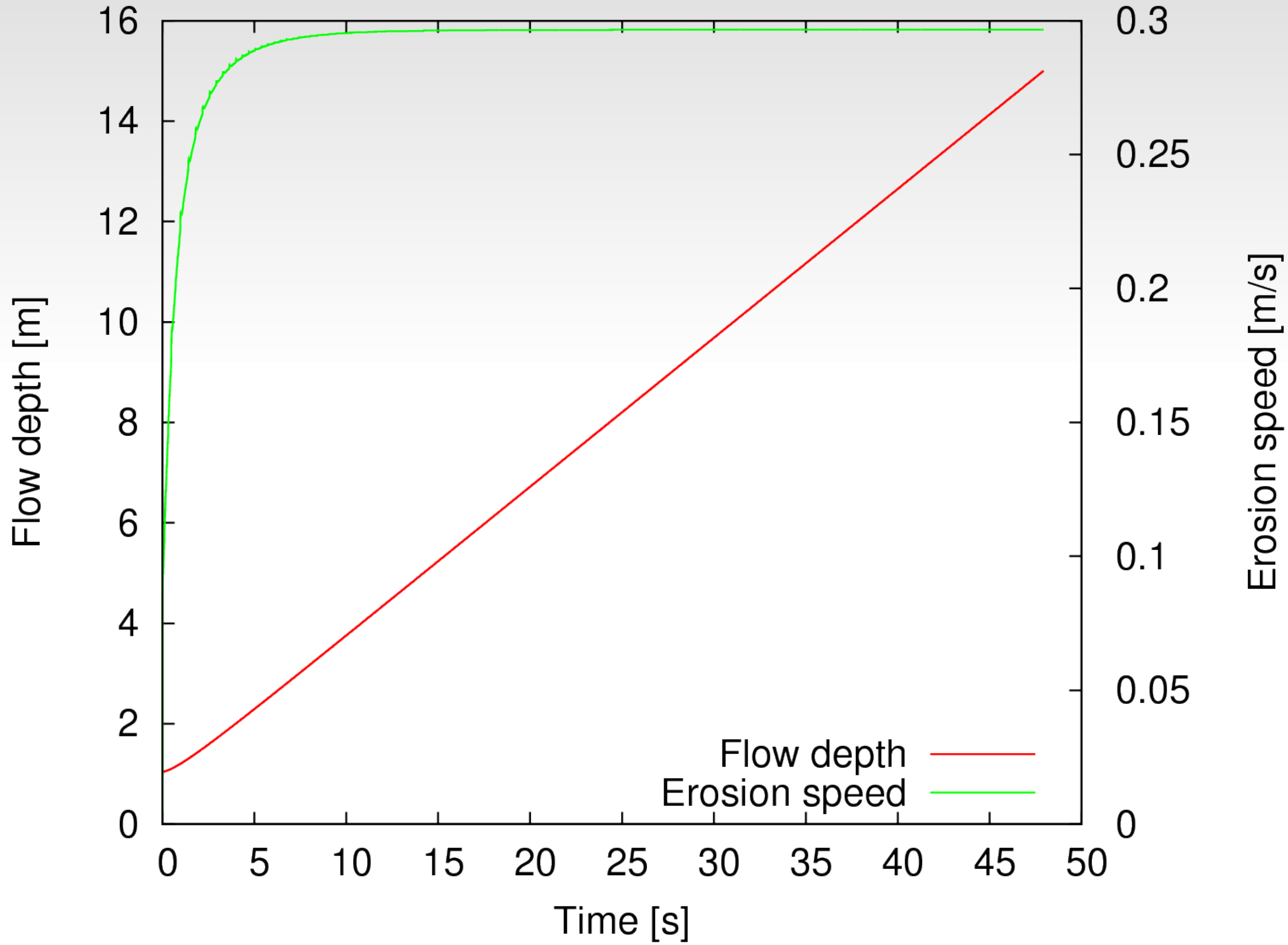


Time evolution of velocity profile for Bagnoldian fluid starting from rest

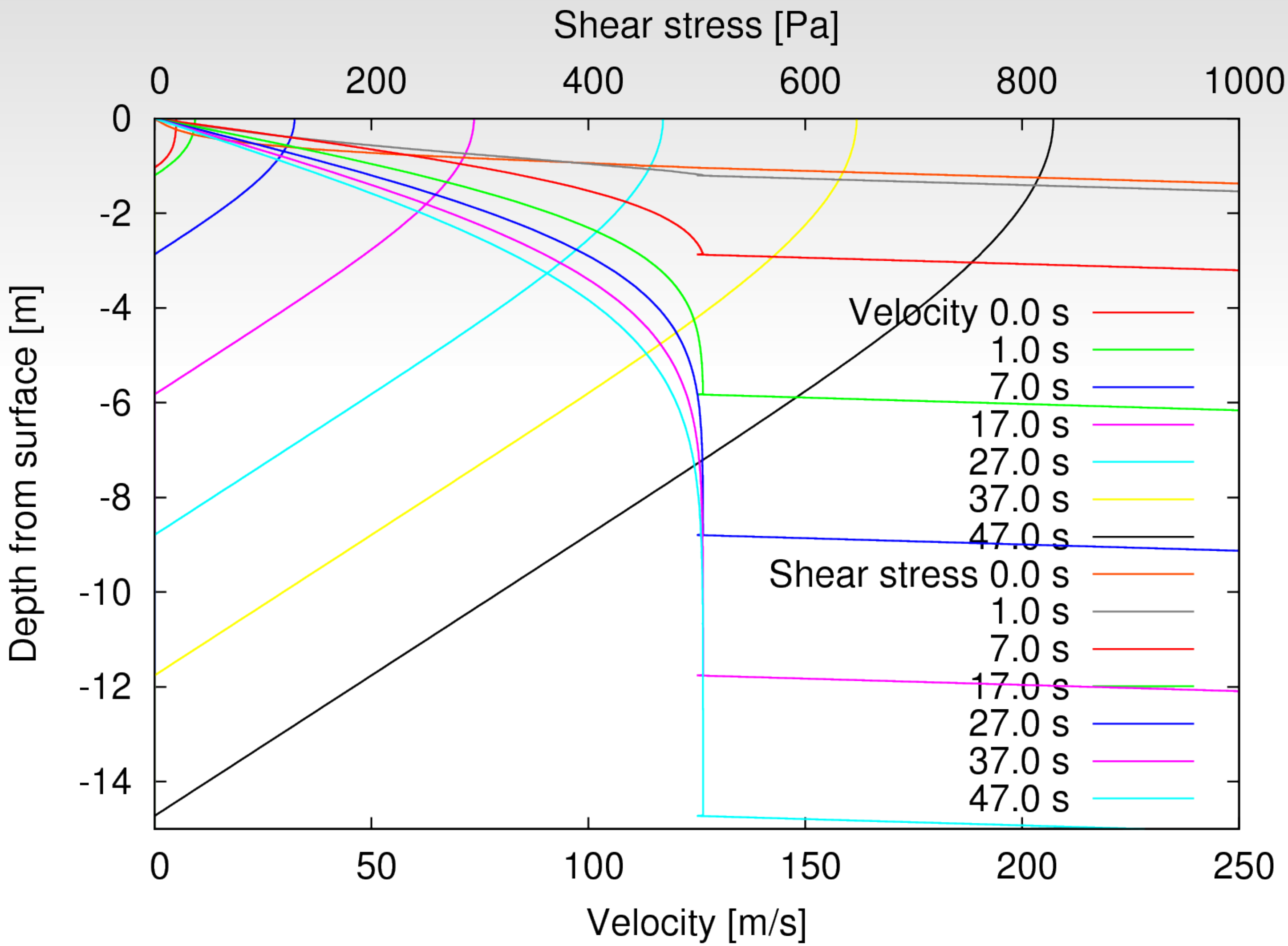
Newtonian fluid: Time evolution of distance and avg. velocity



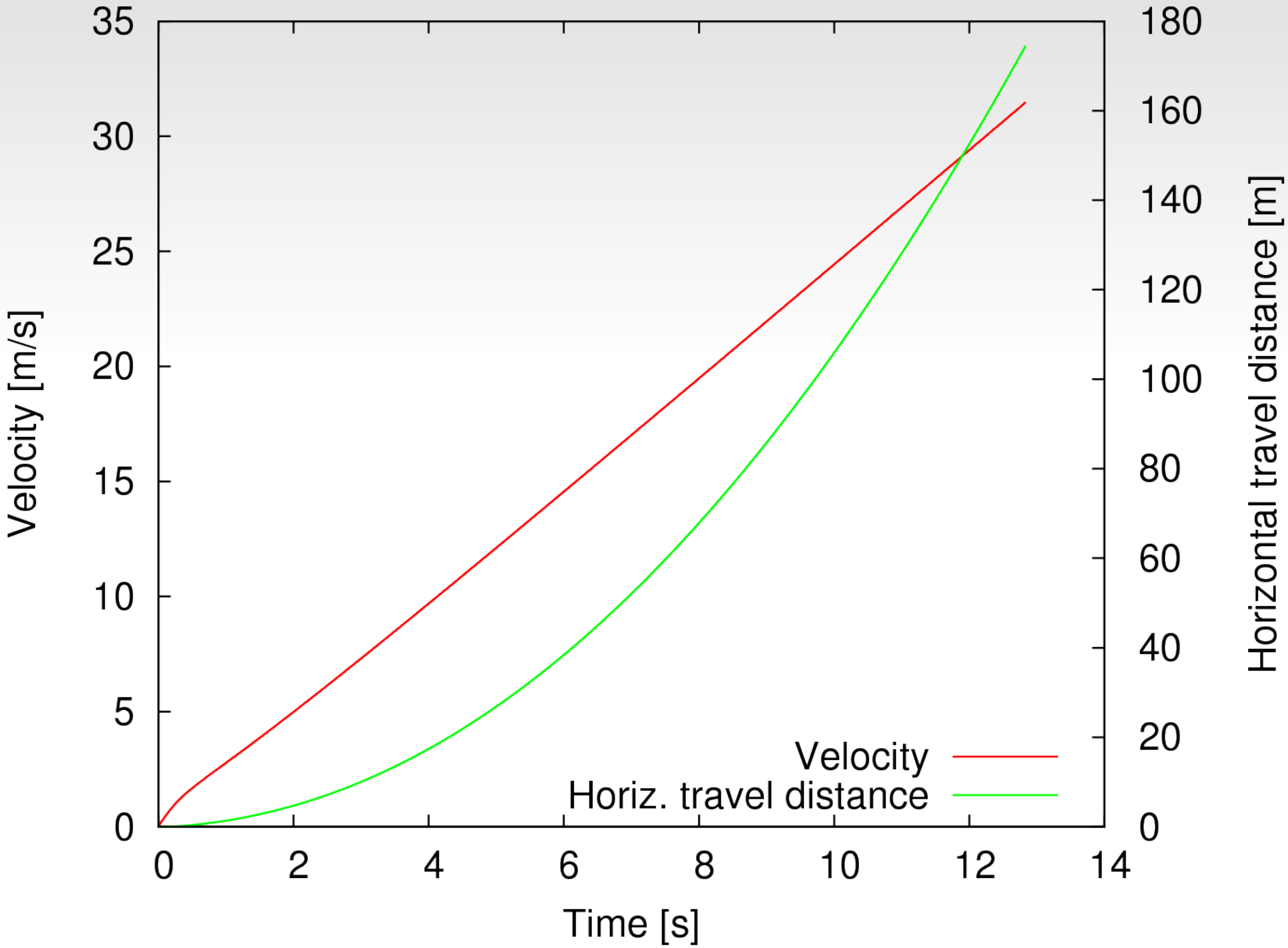
Newtonian fluid: Time evolution of flow depth and erosion rate



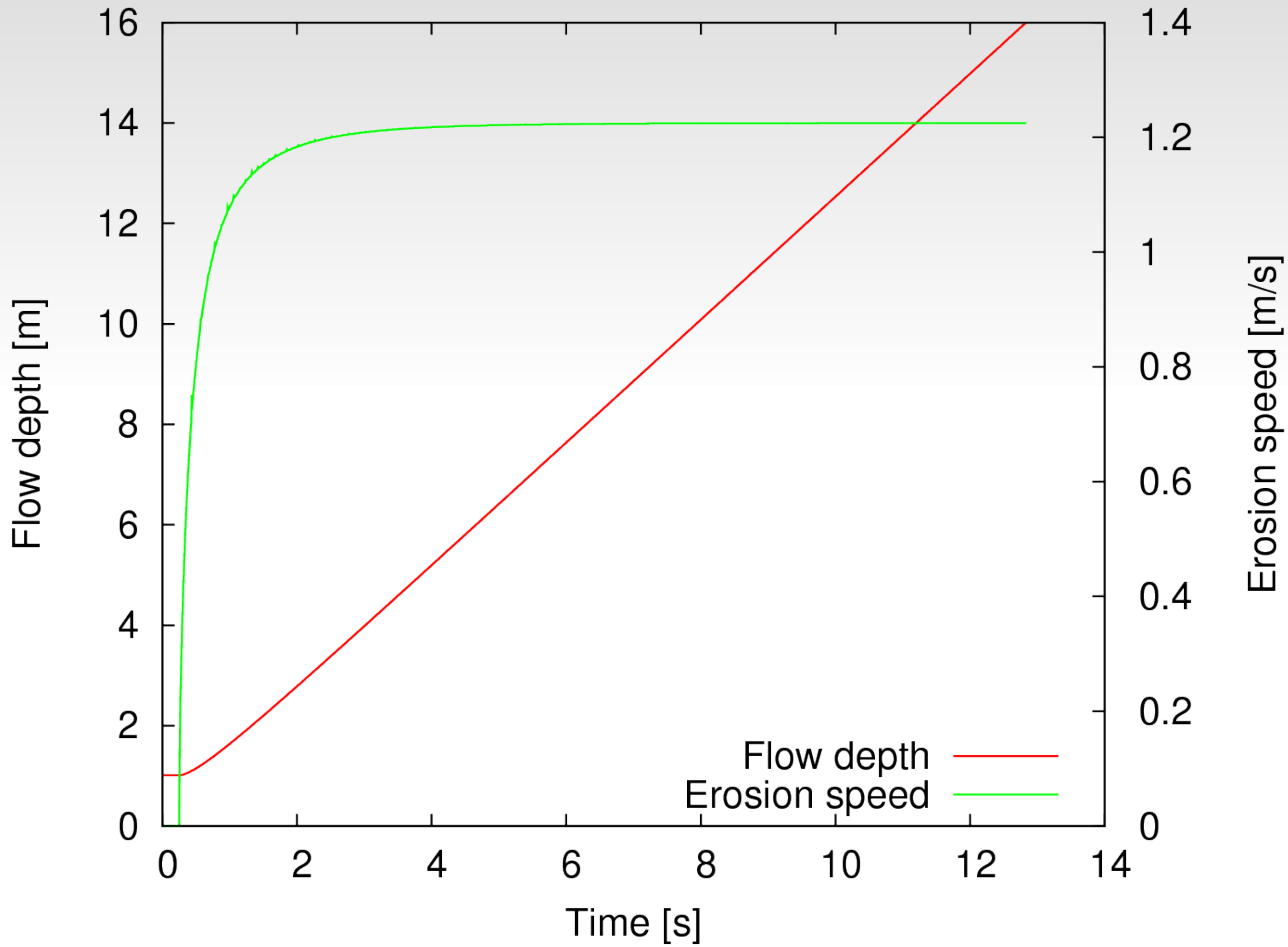
Newtonian fluid: Evolution of velocity and shear stress profiles



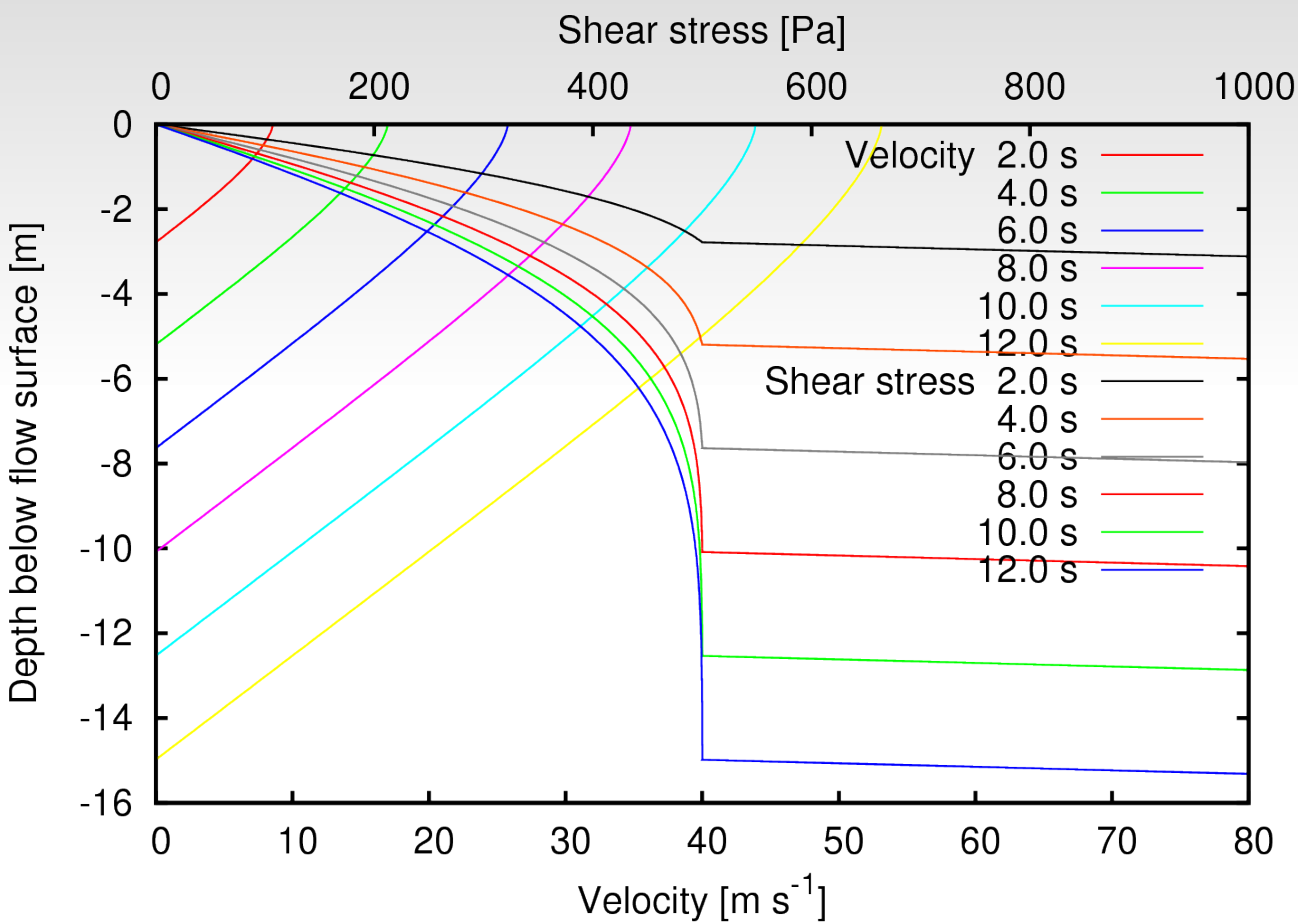
Bagnoldian fluid: Travel distance and average velocity vs. time



Bagnoldian fluid: Flow depth and erosion rate vs. time



Bagnoldian fluid: Evolution of velocity and shear stress profiles



4.4 Can we understand the numerical results?

Preliminary inferences from the numerical simulations:

After initial phase, *independent of rheology*,

- the flow accelerates uniformly at ≈ ½ g sin θ,
- the erosion rate is constant, the flow depth grows linearly,
- the velocity profiles are quite flat near the bed.

☞ *This looks almost like a granular flow with Coulomb friction!*

Recent statistical reanalysis of ~ 300 extreme dry-snow avalanches (Gauer et al., *Cold Regions Sci. Technol.,* in press) indicates

- ➢ Coulomb model with strongly slope-dependent friction coefficient gives best fit for both runout distance and front velocity,
- ➢ maximum velocity grows as ~ (drop height)^{½}.

Use paper, pencil and your head once more...

Seek asymptotic solution to depth-integrated equations with the following properties:

$$h(t) = h_0 + w_e t\,, \qquad \bar{u}(t) = \bar{u}_0 + \bar{a}\,t\,, \qquad w_e,\ \bar{a} = \text{cst.}$$

Then, equation of motion $d(h\bar{u})/dt = h\,g\sin\theta - \hat{\tau}_c$ transforms into

$$\dot{\bar{u}}(t) = g\sin\theta - \frac{\hat{\tau}_c}{h(t)} - \frac{\bar{u}(t)\,dh/dt}{h(t)} = g\sin\theta - \underbrace{\frac{(\hat{\tau}_c + w_e\bar{u}_0) + w_e\bar{a}\,t}{h_0 + w_e t}}_{\text{must be indep. of } t} \stackrel{!}{=} \bar{a}$$

Simple algebra yields

$$\boxed{\bar{a} = \frac{g}{2}\,, \qquad w_e = \frac{1}{\bar{u}_0}\left(\frac{1}{2}\,g\,h_0 - \hat{\tau}_c\right)}$$

Somewhat surprising...

... *but in nearly perfect agreement with the simulations!*

(Less than 1% discrepancy – due to setting entrainment rate to 99% of theoretical value to avoid oscillations.)

N.B. Both relations are independent of rheology! However, rheology determines relation between h_0 and $\bar{u}_0$ as well as velocity profile.

Open question: Need better understanding of stability and domain of attraction of this solution. (It appears to be rather stable!)

5. Conclusions

- Correct form of momentum balance equation / equation of motion unambiguously determined, depends on properties of mass exchange processes.
- In continuum models, deposition occurs if

  bed shear stress > max. shear strength of flow

  Deposition cannot accelerate flow, but reduces deceleration.

- If bed shows perfectly brittle behavior with shear strength τ_c, simple formulas for entrainment rate are found for rigid-plug models with slip condition, stationary flow of Bingham fluid with constant flow height and asymptotic solutions for a wide class of rheologies.

- More work is needed on alternative entrainment mechanisms, in particular frontal entrainment.