Free Topological Groups

Mikhail Tkachenko

Minicourse, Universidad Complutense de Madrid

Introduction

Let $(G, \cdot)$ be an abstract group with multiplication

$$m\colon G \times G \to G,\ m(x, y) = x \cdot y.$$

Suppose that G carries a Hausdorff topology τ such that m is jointly continuous as a mapping of $G \times G$ to G, and that the inversion

$$Inv\colon G \to G,\ Inv(x) = x^{-1},$$

is also continuous. Than the triple $(G, \cdot, \tau)$ is called a **topological group** while τ is said to be a **topological group topology**. Usually, we will omit the symbols designated to the multiplication and topology on G and will say that G is a topological group, if it is not ambiguous. By Pontryagin's theorem, every topological group G is a Tychonoff space, i.e., the topology of G is generated by the family of all continuous real-valued functions on X. Therefore, **every subspace of a topological group is Tychonoff.**

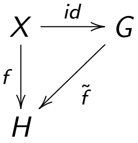
Definitions. Graev's extension of pseudometrics

By several reasons, the *free topological groups* constitute a very important and interesting subclass of topological groups.

Definition (1.1)

Suppose that X is a subspace of a topological group G. We say that G is a **free topological group** over X if the following hold:

1. X generates algebraically a dense subgroup of G;
2. for every continuous mapping $f\colon X \to H$ to an arbitrary topological group H, there exists an extension of f to a continuous homomorphism $\tilde{f}\colon G \to H$.



Unicity Theorem

It turns out that a free topological group over a space X is, in a sense, unique (we postpone the discussion of the existence of free topological groups):

Theorem (1.2)

Suppose that G_1 and G_2 are free topological groups over a Tychonoff space X. Then there exists a topological isomorphism $\varphi\colon G_1 \to G_2$ such that $\varphi(x) = x$, for each $x \in X$. Furthermore, both groups G_1 and G_2 are algebraically generated by X, i.e., X is a set of generators for G_1 and G_2.

Proof.

By Definition 1.1, there exist continuous homomorphisms $\varphi_1 \colon G_1 \to G_2$ and $\varphi_2 \colon G_2 \to G_1$ which extend the identity mapping of X onto itself. Let $\psi_1 = \varphi_1 \circ \varphi_2$ and $\psi_2 = \varphi_2 \circ \varphi_1$. Then ψ_1 is a continuous homomorphism of G_1 to itself whose restriction to X is the identity mapping of X. Since X generates a dense subgroup of G_1 (and G_1 is Hausdorff), it follows that ψ_1 is the identity automorphism of G_1. Similarly, ψ_2 is the identity automorphism of G_2. Hence, φ_1 and φ_2 are topological isomorphisms that do not move the points of X.

Finally, it is clear that the dense subgroup $\langle X \rangle$ of G_1 generated by X satisfies all conditions of Definition 1.1. Hence, it follows from the first claim of the theorem that $\langle X \rangle = G_1$. □

The uniqueness of a free topological group over a Tychonoff space X enables us to introduce a name for this object, say $F(X)$.

If all groups in Definition 1.1 are assumed Abelian, we obtain the definition of the *free Abelian group over* X which is denoted by $A(X)$.

Let us now explain the meaning of the word "**free**" in the name of the group $F(X)$.
Since, by Theorem 1.2, X algebraically generates the group $F(X)$, every element of $g \in F(X)$ can be written in the form

$$g = x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n},$$

where $x_1, \ldots, x_n \in X$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$. The expression $x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n}$ is called a *word.* A word $x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n}$ is said to be *reduced* if it contains no pair of consecutive symbols of the form $x \cdot x^{-1}$ or $x^{-1} \cdot x$. It turns out that if a word is reduced and non-empty, then it is different from the identity of $F(X)$. Using the language of the theory of groups, we say that X is a set of *free generators* for $F(X)$. Equivalently, from the algebraic point of view, $F(X)$ is a *free group* over the set X.

In the Abelian case, the meaning of the word "free" is, of course, slightly different. Again, one can show that if $x_1, \ldots, x_n$ are pairwise distinct elements of X and $k_1, \ldots, k_n$ are arbitrary integers, then the equality

$$k_1 x_1 + k_2 x_2 + \cdots + k_n x_n = 0_{A(X)}$$

implies that $k_1 = k_2 = \cdots = k_n = 0$. Therefore, the group $A(X)$ is torsion-free and, again, X is a set of *free generators* for $A(X)$. Equivalently, $A(X)$ is a *free Abelian group* over X.

This is a good point to turn back to the problem of the existence of free topological groups. Let us consider the non-Abelian case first. It follows from Definition 1.1 that the topology of the group $F(X)$ (when the latter exists) is maximal in some sense. Here is the exact mathematical formulation of this fact.

Theorem (1.3)

The topology of $F(X)$ is maximal among all topological group topologies on $F(X)$ that induce on X its original topology.

We now explain Graev's approach to the existence proof. Suppose that $\mathcal{G}$ is a family of topological group topologies (not necessarily Hausdorff) on an abstract group G. Then the upper bound $\bigvee \mathcal{G}$ of the topologies in $\mathcal{G}$ is again a topological group topology on G. This simple fact tells us that it suffices to find **any Hausdorff topological group topology on $F(X)$ that agrees with the original topology of X**. Then, by Theorem 1.3, the upper bound of all these topologies will be the topology of the free topological group $F(X)$.
To find such a topology on $F(X)$, Graev proved the following remarkable theorem. We need just one definition prior to Graev's theorem.

Definition (1.4)

A pseudometric ϱ on a group G is called *invariant* if $\varrho(xa, xb) = \varrho(a, b) = \varrho(ax, bx)$, for all $a, b, x \in G$.

Theorem (1.5)

Every pseudometric ϱ on a Tychonoff space X can be extended to an invariant pseudometric $\widehat{\varrho}$ on $F(X)$.

Here are some lines explaining Graev's extension procedure.
Suppose that ϱ is a pseudometric on a non-empty set X. First, we extend ϱ to a pseudometric ϱ^* onto $X^* = X \cup \{e\} \cup X^{-1} \subseteq F(X)$, where e is the identity of $F(X)$. Clearly, we write $F(X)$ in place of the *free Abelian group* on X.
Choose a point $x_0 \in X$ and for every $x \in X$, put

$$\varrho^*(e, x) = \varrho^*(e, x^{-1}) = 1 + \varrho(x_0, x).$$

Then for $x, y \in X$, define the distances $\varrho^*(x^{-1}, y^{-1})$, $\varrho^*(x^{-1}, y)$ and $\varrho^*(x, y^{-1})$ by

$$\varrho^*(x^{-1}, y^{-1}) = \varrho^*(x, y) = \varrho(x, y),$$
$$\varrho^*(x^{-1}, y) = \varrho^*(x, y^{-1}) = \varrho^*(x, e) + \varrho^*(e, y).$$

From our definition it follows immediately that ϱ^* extends ϱ and $\varrho^*(z, t) = \varrho^*(t, z) \geq 0$, for all $z, t \in \tilde{X}$. It is easy to verify that ϱ^* satisfies the triangle inequality, i.e., it is a pseudometric on X^*.

Let g, h be arbitrary elements of $F(X)$. We write these elements as words, not necessarily reduced, of one the same length, in all possible forms:

$$g = x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n}$$

and

$$h = y_1^{\delta_1} \cdot \ldots \cdot y_n^{\delta_n},$$

where $x_i, y_i \in X \cup \{e\}$ and $\epsilon_i, \delta_i \in \{-1, 1\}$. Given a couple of words representing g and h, as above, we calculate the number

$$D = \sum_{i=1}^{n} \varrho^*(x_i^{\epsilon_i}, y_i^{\delta_i}).$$

The upper lower bound of the numbers D is denoted by $\widehat{\varrho}(g, h)$. Clearly, $\widehat{\varrho}(g, h) \geq 0$ and $\widehat{\varrho}(h, g) = \widehat{\varrho}(g, h)$, for all $g, h \in F(X)$. The most difficult part of the job is to verify that $\widehat{\varrho}$ satisfies the triangle inequality, which we must skip by obvious reasons. Thus, $\widehat{\varrho}$ is a pseudometric. Our definition of $\widehat{\varrho}$ implies almost immediately that $\widehat{\varrho}$ is invariant.

The final part of the existence proof goes as follows. For every continuous pseudometric ϱ on X, consider the topology τ_ϱ on the abstract free group $F(X)$ generated by $\widehat{\varrho}$. Since the pseudometric $\widehat{\varrho}$ is invariant, one can easily verify that τ_ϱ is a topological group topology on $F(X)$, not necessarily Hausdorff. Since ϱ is continuous and $\widehat{\varrho}$ extends ϱ, the restriction of the topology τ_ϱ to X is coarser than the original topology of X. Therefore, the upper bound $\tau = \bigvee_\varrho \tau_\varrho$ of these topologies is again a topological group topology on $F(X)$. The advantage of τ, compared to the topologies τ_ϱ, is that this topology is **Hausdorff** and **generates on X its original topology.** Taking the upper bound of the topological group topologies on $F(X)$ with the same properties, we get the required "free" topological group topology on $F(X)$. This finishes the existence proof in the non-Abelian case.

A similar argument, with obvious modifications, implies the existence of the free Abelian topological group $A(X)$ over a Tychonoff space X.

Once the existence and the uniqueness theorems are established, one can wonder what the free topological groups serve for and what topological properties they possess. At this point we come to the origins of this topic. In the early thirties of the twentieth century L. S. Pontryagin showed that every Hausdorff topological group is completely regular, i.e., Tychonoff. This gave rise to the problem whether every Hausdorff topological group is a normal space. To solve the problem, A. A. Markov invented the theory of free topological groups. His idea was to embed every Tychonoff space X as a **closed** subspace of an appropriate topological group G_X, which he constructed as the **free topological group** over X. So, he proved the following:

Theorem (1.6)

Every Tychonoff space X is closed in the free topological group $F(X)$ and in the free Abelian topological group $A(X)$.

This fact enabled Markov to give an elegant solution to the normality problem.

Indeed, take any Tychonoff space X which is not normal, for example, the Tychonoff plank $(\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$. Then the free topological group $F(X)$ contains X as a closed subset and, therefore, $F(X)$ cannot be a normal space — otherwise X would be normal.

Theorem 1.6 can be given a sharper form in the sense that the group "envelope" of a space can be chosen σ-compact. This requires a small modification in the original Markov's argument.

Theorem (1.7)

Every Tychonoff space X can be embedded as a closed subspace into a topological group which is a dense subgroup of a σ-compact topological (Abelian) group.

Proof.

Let bX be any compactification of the space X. Then $A(bX)$ is a topological Abelian group which, by Theorem 1.3, contains bX as a closed subspace. Let $G = \langle X \rangle$ be a subgroup of $A(bX)$ generated by the set X. Clearly, G is dense in $A(bX)$ since X is dense in bX, and the group $A(bX)$ is σ-compact. In addition, we have that

$$X = G \cap bX.$$

It follows that X is closed in G. □

A non-compact Tychonoff space X cannot be closed in a compact topological group. However, we can refine Theorem 1.7 as well. We recall that a topological group is **precompact** if it is a (dense) subgroup of a compact topological group.

Theorem (1.8)

Every Tychonoff space X can be embedded as a closed subspace into a precompact topological (Abelian) group.

Proof.

Let $Y = C(X)$ be the family of all continuous real-valued functions on X, and φ the diagonal product of the functions in $C(X)$, $\varphi\colon X \to \mathbb{T}^Y$. Then φ is a topological embedding of X into the compact topological Abelian group $\mathbb{T}^Y$, and one can verify that the subspace $\varphi(X)$ of the precompact topological group $G = \langle\varphi(X)\rangle$ generated by $\varphi(X)$ is closed in G. □

Surprisingly, Theorem 1.8 is not the strongest result about closed embeddings:

Theorem (1.9)

Every Tychonoff space X can be embedded as a closed subspace into a pseudocompact Abelian topological group.

Proof.

Let X be a closed subspace of a precompact topological Abelian group H (Theorem 1.8). By a theorem of M. Ursul, H is a **closed** topological subgroup of a pseudocompact topological Abelian group G. Hence, X is a closed subspace of G. □

Lecture 2.

1. **The direct limit property in free topological groups.**
2. **Completeness of free topological groups.**
3. **Bounded subsets of free topological groups.**

The main difficulty in working with free topological groups is a non-constructive way we introduced them. It is almost impossible to describe constructively open subsets of a free topological group $F(X)$ for a non-discrete space X.

Theorem (2.1)

The following conditions are equivalent for a Tychonoff space X:

(a) $F(X)$ *is locally compact;*

(b) $F(X)$ *is first countable;*

(c) $F(X)$ *is discrete;*

(d) X *is discrete.*

In particular, $F(X)$ is metrizable if and only if X is discrete.
In the case of a compact space X, however, there is a good chance to understand the "shape" of open subsets of $F(X)$. This requires one useful concept.

Definition (2.2)

Suppose that a space X is the union of a family $\xi = \{X_n : n \in \omega\}$ of its subspaces, where $X_n \subseteq X_{n+1}$ for each $n \in \omega$. We say that X is the **inductive limit** of the sequence ξ provided that a set $F \subseteq X$ is closed in X if and only if $F \cap X_n$ is closed in X_n, for each $n \in \omega$. If a space X is the inductive limit of a sequence ξ of compact subsets, then X is called a k_ω-**space.** The sequence ξ is called a k_ω-**decomposition** of X.

Every k_ω-space is σ-compact. Every locally compact σ-compact space is a k_ω-space. However, the space $\mathbb{Q}$ of rationals with the topology inherited from $\mathbb{R}$ is evidently σ-compact but is not a k_ω-space.

The following important result has to be attributed to Graev and to Mack–Morris–Ordman.

For a given integer $n \geq 0$, we denote by $F_n(X)$ the subspace of the free topological group $F(X)$ consisting of all elements g of the form

$$g = x_1^{\epsilon_1} \cdot \ldots \cdot x_n^{\epsilon_n},$$

where $x_1, \ldots, x_n \in X$ and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

Theorem (2.3)

If X is a k_ω-space with a k_ω-decomposition $\{X_n : n \in \omega\}$, then the free topological group $F(X)$ is the inductive limit of the sequence $\{F_n(X_n) : n \in \omega\}$, i.e., the latter sequence is a k_ω-decomposition for $F(X)$.

Note that each set $F_n(X_n)$ in the above theorem is a compact subset of $F(X)$, since the sets X_n are compact.

Sketch of the proof. Denote by τ the family of all sets $O \subseteq F(X)$ such that $O \cap F_n(X_n)$ is open in $F_n(X_n)$, for each $n \in \omega$. It is clear that τ is a topology for $F(X)$ and that τ induces the original topology on X. It is also clear that $O^{-1} \in \tau$, for each $O \in \tau$. One can also verify that if $g \cdot h \in U \in \tau$, then there are $V \in \tau$ and $W \in \tau$ such that $g \in V$, $h \in W$, and $VW \subseteq U$. In other words, the multiplication in $(F(X), \tau)$ is continuous and, therefore, $F(X)$ endowed with the topology τ is a (Hausdorff) topological group. Obviously, τ is finer than the topology $\mathcal{T}$ of the free topological group $F(X)$. However, since τ is a topological group topology on $F(X)$ and it induces the original topology on X, we conclude, by Theorem 1.3, that $\tau = \mathcal{T}$. $\square$

Corollary (2.4)

If X is a compact space, then the free topological group $F(X)$ is the inductive limit of the sequence $\{F_n(X) : n \in \omega\}$ of its compact subspaces.

Corollary (2.5)

If X is a Tychonoff space and a set $K \subseteq F(X)$ satisfies $|K \cap F_n(X)| < \omega$ for each $n \in \omega$, then K is closed and discrete in $F(X)$.

Proof.

Let bX be a compactification of X. The identity embedding $X \hookrightarrow bX$ extends to a continuous monomorphism $f: F(X) \to F(bX)$. Clearly, $|f(K) \cap F_n(bX)| < \omega$ for each $n \in \omega$. Since, by Corollary 2.4, $F(bX)$ is the inductive limit of the subspaces $F_n(bX)$, it follows that $f(K)$ is closed in $F(bX)$, and the same argument implies that every subset of $f(K)$ is closed in $F(bX)$. Hence $f(K)$ is discrete. Since f is a continuous monomorphism, K is closed and discrete in $F(X)$. □

Corollary (2.6)

Let X be an arbitrary Tychonoff space. If C is a compact subset of $F(X)$, then $C \subseteq F_n(X)$, for some $n \in \omega$.

Here are more applications of Theorem 2.3.

Theorem (2.7)

Let Y be a closed subset of a Tychonoff space X. Then the subgroup $F(Y, X) = \langle Y \rangle$ of $F(X)$ generated by Y is closed in $F(X)$.

Sketch of the proof. Modify the argument given in the proof of Corollary 2.5. □

After Theorem 2.7 we come to the following:

Problem

If Y is a closed subspace of a Tychonoff space X, is then $F(Y) \cong F(Y, X)$?

Corollary (2.8)

If K is a compact subset of a Tychonoff space X, then $F(K) \cong F(K, X)$.

Proof.

Denote by bX an arbitrary compactification of X and let $f\colon K \hookrightarrow X$ and $g\colon X \hookrightarrow bX$ be the natural embeddings. Extend f and g to continuous monomorphisms $\hat{f}\colon F(K) \to F(X)$ and $\hat{g}\colon F(X) \to F(bX)$, respectively. By Corollary 2.4, $F(K)$ and $F(bX)$ are k_ω-spaces with k_ω-decompositions $F(K) = \bigcup_{n=0}^{\infty} F_n(K)$ and $F(bX) = \bigcup_{n=0}^{\infty} F_n(bX)$.

Clearly, $\varphi = \hat{g} \circ \hat{f}$ is a continuous monomorphism of $F(K)$ to $F(bX)$. Let C be an arbitrary closed set in $F(K)$. Then $C_n = C \cap F_n(K)$ is compact and, hence, $\varphi(C) \cap F_n(bX) = \varphi(C_n)$ is closed in $F_n(bX)$, for each $n \in \omega$. We conclude, therefore, that $\varphi(C)$ is closed in $F(bX)$. This proves that φ is a closed mapping, whence it follows that φ is a topological isomorphism between $F(K)$ and $F(K, bX)$. Finally, the equality $\varphi = \hat{g} \circ \hat{f}$ implies that $\hat{f}$ is a topological isomorphism between $F(K)$ and $F(K, X)$. $\square$

Definition (2.9)

We say that a topological group G is **complete** (meaning **Raĭkov complete**) if G is closed in any topological group that contains G as a topological subgroup. Equivalently, G is complete if every Cauchy filter in G converges.

Here is yet another application of Theorem 2.3 given by Graev and Hunt–Morris.

Theorem (2.10)

If X is a k_ω-space, then the group $F(X)$ is complete.

Corollary (2.11)

The free topological group $F(X)$ is complete, for every compact space X.

Problem

Characterize the spaces X such that the free Abelian topological group $A(X)$ or the free topological group $F(X)$ is complete.

The solution to this problem is very far from obvious.

Definition (2.12)

A space X is called **Dieudonné complete** if X is homeomorphic to a closed subspace of a product of metrizable spaces.

All metrizable spaces and arbitrary products of metrizable spaces are Dieudonné complete. Less trivially: If a Tychonoff space X admits a continuous one-to-one mapping onto a metrizable space, then X is Dieudonné complete (such a space X is called **submetrizable**).

Theorem (2.13)

The free Abelian topological group $A(X)$ is complete if and only if the space X is Dieudonné complete. Furthermore, the equality $\varrho A(X) \cong A(\mu X)$ holds for every Tychonoff space X.

Here ϱ stands for the **Raĭkov completion** of a topological group, while μX is the **Dieudonné completion** of the space X. Therefore, $\varrho \circ A \cong A \circ \mu$, in the language of functors.

Corollary (2.14)

The free Abelian topological group $A(X)$ is complete for any submetrizable space X.

Every paracompact space is Dieudonné complete, so we have:

Corollary (2.15)

The free Abelian topological group $A(X)$ is complete for any paracompact space X.

The non-Abelian case requires a very different technique, but the results are the same in nature:

Theorem (2.16)

The free topological group $F(X)$ is complete if and only if the space X is Dieudonné complete. In addition, $\varrho F(X) \cong F(\mu X)$, for an arbitrary Tychonoff space X.

We now deduce the following result (reformulation of Theorem 2.16):

Corollary (2.17)

If X is a closed subspace of a product of metrizable spaces, then the free topological group $F(X)$ is complete.

Bounded subsets of free topological groups

Definition (2.18)

A subset B of a Tychonoff space X ia called **bounded** in X provided every continuous real-valued function defined on X is bounded on B.

Every compact subset of a space X is bounded in X, but not vice versa. In fact, there exist bounded, closed, discrete, infinite subsets (a closed copy of $\mathbb{N}$) of Tychonoff spaces.

Lemma (2.19)

If B is a bounded subset of a Dieudonné complete space X, then the closure of B in X is compact.

We now turn back to topological groups:

Proposition (2.20)

Every bounded subset B of a topological group G is precompact in G, i.e., for every neighbourhood U of the identity in G, there exists a finite set $F \subseteq G$ such that $B \subseteq FU$ and $B \subseteq UF$.

Here is another useful fact about bounded subsets of topological groups:

Theorem (2.21)

If A and B are bounded subsets of a topological group G, then $A \cdot B$ is a bounded subset of G.

Problem

Does Theorem 2.21 remain valid for paratopological groups?

Precompact subsets of free topological groups are, in a sense, very special.
For an element $g \in F(X)$, let

$$g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$$

be the irreducible representation of g, where $x_1, \ldots, x_n \in X$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. Then we put

$$\mathrm{s}upp\,(g) = \{x_1, \ldots, x_n\} \text{ and } \mathrm{s}upp\,(B) = \bigcup_{g \in B} \mathrm{s}upp\,(g).$$

Lemma (2.22)

If B is a precompact subset of a free topological group $F(X)$, then $Y = \mathrm{s}upp\,(B)$ is bounded in X, and $B \subseteq F_n(Y, X)$, for some $n \in \mathbb{N}$. In other words, the lengthes of the elements in B with respect to the basis X are uniformly bounded.

Finally, we can characterize bounded subsets of free topological groups:

Theorem (2.23)

The following condition are equivalent for a subset B of a free topological group $F(X)$:

(a) *B is bounded in $F(X)$;*

(b) *B is precompact in $F(X)$;*

(c) *there exist a bounded subset Y of X and an integer $n \in \mathbb{N}$ such that $B \subseteq F_n(Y, X)$.*

Proof.

The implication (a) $\Rightarrow$ (b) follows from Proposition 2.20, while (b) $\Rightarrow$ (c) is Lemma 2.22. Finally, (c) $\Rightarrow$ (a) follows from Theorem 2.21. □

Lecture 3.

1. **Free groups over metrizable spaces.**
2. **Quotients of zero-dimensional groups.**
3. **Commercial propaganda.**

We know that for every Tychonoff space X and any integer $n \in \mathbb{N}$, the subset $F_n(X)$ of $F(X)$ consisting of all words of reduced length $\leq n$ is closed. Here is a stronger result in the case when the space X is metric (essentially due to V. K. Bel'nov):

Proposition (3.1)

Let (X, d) be a metric space, and $\mathcal{T}_d$ be the topology on $F(X)$ generated by the Graev extension $\widehat{d}$ of d over $F(X)$. Then $F_n(X)$ is closed in $(F(X), \mathcal{T}_d)$ for each integer $n \in \mathbb{N}$.

Proof.

Apply the definition of Graev's extension $\widehat{d}$ of the metric d. □

We have to complement Proposition 3.1, which requires some simple definitions.

Given an integer $n \in \mathbb{N}$ and a space X, we put

$$C_n(X) = F_n(X) \setminus F_{n-1}(X).$$

Let also $\widetilde{X} = X \cup \{e\} \cup X^{-1}$ and consider the multiplication mapping

$$i_n \colon \widetilde{X}^n \to F(X), \quad i_n(y_1, y_2, \ldots, y_n) = y_1 \cdot y_2 \cdots y_n.$$

Theorem (3.2)

Let (X, d) be a metric space and $\mathcal{T}_d$ the topology on $F(X)$ generated by the Graev extension $\widehat{d}$ of d. Then the mapping i_n is a homeomorphism of $C_n^*(X) = i_n^{-1}(C_n) \subseteq \widetilde{X}^n$ onto the subspace $C_n(X)$ of $(F(X), \mathcal{T}_d)$. In particular, $C_n(X)$ is metrizable as a subspace of $(F(X), \mathcal{T}_d)$.

Corollary (3.3)

Let (X, d) be a metric space. Then the topology on $C_n(X)$ inherited from $F(X)$ coincides with the one induced by the Graev extension $\widehat{d}$ of the metric d.

Our aim now is to establish that a free topological group over a metrizable space is paracompact (Arhangel'skii's theorem). To this end, we need the definition of an *s*-approximation.

Definition (3.4)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X such that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$. If the space $(X, \mathcal{T}_2)$ has a σ-discrete family of subsets which is a network for $(X, \mathcal{T}_1)$, then the topology $\mathcal{T}_2$ is called an *s*-**approximation** for $\mathcal{T}_1$.

Lemma (3.5)

Suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X such that $\mathcal{T}_2$ is an s-approximation for the topology $\mathcal{T}_1$. Then the following hold:

(a) *for every subset Y of X, $\mathcal{T}_2 \restriction Y$ is an s-approximation for $\mathcal{T}_1 \restriction Y$;*

(b) *for every integer $n > 0$, the topology of the product $(X, \mathcal{T}_2)^n$ is an s-approximation for the topology of $(X, \mathcal{T}_1)^n$.*

Sometimes the existence of an s-approximation for a given topology $\mathcal{T}$ on a set X implies that the space $(X, \mathcal{T})$ is paracompact.

Lemma (3.6)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be regular topologies on a set X such that $\mathcal{T}_2$ is an s-approximation for $\mathcal{T}_1$ and the space $(X, \mathcal{T}_2)$ is collectionwise normal. Then the space $(X, \mathcal{T}_1)$ is paracompact.

A paracompact space with a σ-discrete network is said to be a **paracompact σ-space.** All metric spaces, as well as their arbitrary images under closed continuous mappings, are paracompact σ-spaces. Paracompact σ-spaces can be characterized in terms of s-approximations.

Theorem (3.7)

A regular T_1-space $(X, \mathcal{T})$ is a paracompact σ-space iff $\mathcal{T}$ admits a metrizable s-approximation.

Here is the promised Arhangel'skii's theorem, given in a slightly more general form:

Theorem (3.8)

The free topological group $F(X)$ is a paracompact σ-space iff X is a paracompact σ-space.

Proof.

If $F(X)$ is a paracompact σ-space, then X is closed in $F(X)$ and, hence, X is also a paracompact σ-space.

Conversely, if X is a paracompact σ-space, then Theorem 3.7 implies that the topology τ of X admits a metrizable s-approximation τ_1. Choose a metric ϱ on X which generates the topology τ_1 and consider the topology $\mathcal{T}_\varrho$ on $F(X)$ generated by the Graev extension $\widehat{\varrho}$ of ϱ to $F(X)$. Let $F_\varrho(X) = (F(X), \mathcal{T}_\varrho)$. By Proposition 3.1, the sets $F_n(X)$ are closed in $F_\varrho(X)$, so each set $C_n(X) = F_n(X) \setminus F_{n-1}(X)$ is open in the subspace $F_n(X)$ of $F_\varrho(X)$. Therefore, $C_n(X) = \bigcup_{k\in\omega} C_{n,k}$, where each $C_{n,k}$ is closed in $F_\varrho(X)$. For all $n, k \in \omega$, choose a σ-discrete network $\gamma_{n,k}$ for $(C_{n,k}, \widehat{\varrho})$. By Corollary 3.3, the topology on $C_n(X)$ inherited from $F(X)$ coincides with the one inherited from $F_\varrho(X)$, so the family $\gamma = \bigcup_{n,k\in\omega} \gamma_{n,k}$ is a network for $F(X)$. Since γ is σ-discrete in $F_\varrho(X)$, we conclude that $\mathcal{T}_\varrho$ is a metrizable s-approximation for the original topology $\mathcal{T}$ of $F(X)$. Therefore, Theorem 3.7 implies that $F(X)$ is a paracompact σ-space. □

We say that a Tychonoff space X is σ-**closed-metrizable** if it can be represented as the union of countably many closed metrizable subspaces. The next result complements Theorem 3.7.

Theorem (3.8)

The free topological group $F(X)$ is σ-closed-metrizable and paracompact if and only if the space X is σ-closed-metrizable and paracompact.

Corollary (3.9)

The free topological group $F(X)$ on a metrizable space X is σ-closed-metrizable and paracompact.

Problem

When is the group $F(X)$ zero-dimensional?

Definition (3.10)

A continuous mapping $f\colon X \to Y$ will be called **gentle** if there is a network $\mathcal{S}$ in the space X such that its image $\{f(P) : P \in \mathcal{S}\}$ is a σ-discrete family of sets in Y.

We also recall that $\dim X$ denotes the **covering dimension** of a Tychonoff space X defined in terms of finite cozero coverings of X, while $\operatorname{ind} X$ stands for the small inductive dimension of X.

Theorem (3.11)

Let X be a paracompact σ-space such that $\operatorname{ind} X = 0$. Then $\dim X = 0$ if and only if there is a one-to-one gentle mapping of X onto a metrizable space Y such that $\dim Y = 0$.

The next fact is slightly more general than the corresponding part of Theorem 3.11:

Theorem (3.12)

Suppose that g is a one-to-one gentle mapping of a space X with $\operatorname{ind} X = 0$ *onto a metrizable space Y satisfying* $\dim Y = 0$*. Then* $\dim X = 0$ *and X is paracompact.*

Note that the paracompactness of X in Theorem 3.12 follows even without assumption that $\operatorname{ind} X = 0$ or $\dim X = 0$.

The following lemma is evident:

Lemma (3.13)

The product of any countable family of (one-to-one) gentle mappings is a (one-to-one) gentle mapping.

Proposition (3.14)

Let X_n be a paracompact σ-space such that $\dim X_n = 0$, for each $n \in \omega$. Then the product space $X = \prod_{n\in\omega} X_n$ satisfies $\dim X = 0$, and X is paracompact.

Proof.

By Theorem 3.11, we can fix, for each $n \in \omega$, a one-to-one gentle mapping g_n of X onto a metrizable space Y_n such that $\dim Y_n = 0$. Then $\operatorname{Ind} Y_n = 0$, by Theorem 7.1.10 in [Engelking]. For the product space $Y = \prod_{n\in\omega} Y_n$, we have $\dim Y = \operatorname{Ind} Y = 0$, since each Y_n is metrizable. Obviously, $\operatorname{ind} X = 0$. It remains to refer to Lemma 3.13 and Theorem 3.12. □

Theorem (3.15)

Let X be a non-empty paracompact σ-space. Then $\dim F(X) = 0$ *if and only if* $\dim X = 0$.

Proof.

Since every continuous function $f\colon X \to \mathbb{R}$ extends to a continuous homomorphism $\tilde{f}\colon F(X) \to \mathbb{R}$, the set X is C-embedded in $F(X)$. Therefore, $\dim X \le \dim F(X)$ by Theorem 7.1.8 of [Engelking]. Now, suppose that $\dim X = 0$. Since $\widetilde{X} = X \oplus \{e\} \oplus X^{-1}$ is a paracompact σ-space, Theorem 3.2 and Lemma 3.5 imply that $C_n(X) = F_n(X) \setminus F_{n-1}(X)$ is a paracompact σ-space. As in the proof of Theorem 3.8, one can represent every $C_n(X)$ as the union $C_n(X) = \bigcup_{k\in\omega} C_{n,k}$, where each $C_{n,k}$ is closed in $F_n(X)$ and in $F(X)$. Then $C_{n,k}$ is homeomorphic to a closed subspace of $\widetilde{X}^n$. Hence, by Proposition 3.14,

$$\dim C_{n,k} \le \dim \widetilde{X}^n = \dim \widetilde{X} = 0.$$

Since $F(X) = \bigcup_{n,k\in\omega} C_{n,k}$, the countable sum theorem for the dimension dim implies that $\dim F(X) = 0$.

Here is the main result of the lecture.

Theorem (3.16)

Every topological group G is a quotient of a topological group H satisfying the following conditions:

(a) $\dim H = 0$ *and* $\dim Y = 0$ *for every* $Y \subseteq H$*;*

(b) *H is a paracompact σ-space hereditarily;*

(c) *H admits a continuous isomorphism onto a metrizable topological group.*

All we need now is the following lemma:

Lemma (3.17)

Every T_1-space X is an image under a quotient mapping of a topological sum $Z = \bigoplus_{\alpha \in A} Z_\alpha$, where each Z_α is a normal space of countable pseudocharacter with at most one non-isolated point.

TO APPEAR SOON:

TOPOLOGICAL GROUPS AND RELATED STRUCTURES

by

A. V. ARHANGEL'SKII and M. G. TKACHENKO

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